SOME PROBLEMS OF SPECTRAL THEORY

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An operator T, in a finite-dimensional space, is of scalar type - its matrix is diagonal in suitable coordinates - if and only if there exist numbers λ_j and pairwise disjoint projections P_j such that

(1)
$$T = \sum_j \lambda_j P_j .$$

Also compact operators of scalar type in any Banach can be so expressed, provided, the sum (1) is allowed to be countably infinite.

It is perhaps less often noted that any operator of scalar type can be expressed in the form (1), with the index j running over all positive integers. But, in general, the projections P_j cannot be chosen pair-wise disjoint, that is, the product of any distinct pair of them might not be equal to the zero-operator.

In fact, an operator T in a space E is of scalar type if and only if, there exists an abstract space Ω, a σ-algebra S of its subsets, a σ-additive and multiplicative measure P : $S \rightarrow L(E)$ such that $P(\Omega) = I$, the identity operator, and a P-integrable function f such that

(2)
$$T = \int f dP .$$

It is then easy to see that there exist sets $X_j \in S$ and numbers λ_j such that (1) holds with $P_j = P(X_j)$, j = 1,2,....

Now, it might very well happen that an operator T can be expressed in the form (1) but it is not possible to guarantee that the projections P_j are values of a spectral measure P such that (2) holds. To be sure, the expression (1) is of little value if nothing is known about the operators P_j save that they are projections; some additional structure

underneath is necessary. For example, one should expect at least that the operators P_j commute.

The following situation occurs quite often. A projection operator $P(X)$ can be constructed for every set X belonging to a semiring Q generating the σ-algebra S. Furthermore, the context might suggest that if the set function $P : Q \rightarrow L(E)$ could be extended to become an L(E)-valued σ-additive measure on the whole of S, then this measure would be multiplicative and the equality (2) would hold. However, as it not infrequently turns out, P may not be σ-additive on Q. Modulo some kind of completeness of the space E, this is the same as saying that, for every vector x ∈ E, there exists a linear functional x' ∈ E' such that the set function

(3) $X \mapsto \langle P(X)x, x' \rangle$, $X \in Q$,

is not σ-additive.

If there exists a separating family Γ of linear functionals x' ∈ E', invariant with respect to the adjoint operator T', such that the scalar set function (3) is σ-additive for every x ∈ E and x' ∈ Γ, then the procedure described by Werner Ricker can save the situation. For, there exists then a space F, continuously containing a copy of the space E, and a σ-additive and multiplicative measure $P : S \rightarrow L(F)$, extending in an obvious sense the set function $P : Q \rightarrow L(E)$, such that (2) holds for some P-integrable function f. I would refer to Werner Ricker's talk and his papers [6] and [7] for further details.

However, if the set of linear functionals x' ∈ E', such that the set function (3) is σ-additive for every x ∈ E, is not separating, then the situation cannot be saved by weakening the topology of the space E and by its subsequent extension. In that case, the set function $P : Q \rightarrow L(E)$ quite radically fails to be σ-additive. Nevertheless, the operator T can

still have many properties in common with scalar operators. Indeed, it can have all the properties which do matter from some point of view. Actually some authors study quite intensively certain classes of operators which can be expressed in the form (2), using some variants of the Rieman – Stieltjes integral, where P is an L(E)-valued set function on a semiring, such as the semirings of intervals on the real-line. For the present purpose, a reference to the book [1] of H.R. Dowson would perhaps suffice.

Now I am coming to the main point I wish to make. An additive set function can fail to be σ-additive for two very different reasons. These are illustrated by the following examples in which $\Omega = (0,1]$ and Q is the semiring of intervals (u,v] such that $0 \leq u \leq v \leq 1$.

Example A. Let P(X) = 1 for every set X $\in$ Q such that there exists a $\delta > 0$ with $(0,\delta] \subset X$, and let P(X) = 0 for every other set X $\in$ Q.

Example B. Let $\phi(t) = t \sin t^{-1}$, if $t \neq 0$, and $\phi(0) = 0$. Let $P((u,v]) = \phi(v) - \phi(u)$ for any u and v such that $0 \leq u \leq v \leq 1$.

In Example A, the sequence $\{P(X_n)\}_{n=1}^{\infty}$ is summable for any pair-wise disjoint sets $X_n \in Q$, n = 1,2,..., whose union is a set X $\in$ Q. So, P fails to be σ-additive because, for some such sets, the sum of the sequence $\{P(X_n)\}_{n=1}^{\infty}$ is not equal to P(X). We can say that P, or, rather, the space Ω, is deficient in some way. This deficiency can be removed by adding some extra points to the space. In this case, a single point, 0, would do.

By contrast, the set function P of Example B fails to be σ-additive because there are pair-wise disjoint sets $X_n \in Q$, n = 1,2,..., whose union belongs to Q such that the sequence of values $\{P(X_n)\}_{n=1}^{\infty}$ is either not summable at all or it is only conditionally summable so that its summability and sum depend on the order of its terms. In this case, σ-additivity

cannot be gained by adding extra points to Ω. There is nothing deficient about P or Ω. We can think of a real-valued function of this kind as describing a distribution of signed mass (such as electric charge) in the space Ω. There is an infinite amount of positive and also an infinite amount of negative mass present but they are distributed so that the net difference in any set from Q is finite.

The theory, including integration, of a set function failing to be σ-additive solely for the first reason can be based on its σ-additive extension on the completed space - a suitable compactification, say. Such an approach seems to have been originated by E. Hewitt and K. Yosida [8]. However, if P fails to be σ-additive due to the second cause, this possibility does not arise; the theory of integration with respect to set functions of this type apparently has to be developed from the beginning. Nevertheless, the existing literature seems to avoid it to the extent of discouraging any attempts. (See, for example, the discussion in Section 28 of [2].)

Let us call set functions which possibly fail to be σ-additive, but solely for the reason illustrated by Example B, indeficient. Integration with respect to an indeficient set function can be introduced by a method closely resembling the Archimedes method of exhaution. This method has been discussed by J.L. Kelley and T.P. Srinivasan [3]. It was used by J. Mikusiński for the definition and study of the Bochner integral. Susumu Okada has used the same method for the construction (of a concrete representation) of the completion of the space of Pettis integrable functions [5].

Specifically, if P is an indeficient additive set function on a semiring Q of subsets of a space Ω, a real or complex valued function f is said to be (Archimedes) integrable with respect to P, if there exist numbers λ_j and sets $X_j \in Q$, with characteristic functions f_j, $j = 1,2,\ldots,$ such that

the sequence $\{\lambda_j P(Y_j)\}_{j=1}^{\infty}$ is summable for any choice of sets $Y_j \in Q$, $Y_j \subset X_j$, $j = 1,2,\ldots$, and

$$f(w) = \sum_{j=1}^{\infty} \lambda_j f_j(w)$$

for every $w \in \Omega$ for which

$$\sum_{j=1}^{\infty} |\lambda_j| f_j(w) < \infty.$$

The integral of the function f is then defined by the formula

$$\int f dP = \sum_{j=1}^{\infty} \lambda_j P(X_j).$$

The indeficiency of P is formally defined as precisely that property which guarantees that the so defined integral is independent of any particular choice of such numbers λ_j and sets X_j, $j = 1,2,\ldots$.

By (a vector version of) the Beppo Levi theorem, if Q is a σ-algebra (in fact, just a δ-ring), then a σ-additive set function on Q is indeficient. Hence, operators T having a representation on (2), where f is a scalar valued function integrable with respect to an indeficient additive and multiplicative set function $P : Q \rightarrow L(E)$, are natural generalisations of scalar type operators. Actually, the theory of scalar operators generalized in this way seems more promising that the theory of operators whose resolution of identity is a spectral distribution rather than a spectral measure.

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