THE EXISTENCE OF MAXIMAL SURFACES

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The minimal surface (Plateau) problem is well-known - one seeks a surface with minimal area amongst all surfaces spanning a given boundary. Instead we ask the analogous question in a Lorentzian space, so in the simplest case we are considering spacelike surfaces in flat Minkowski space $\mathbb{R}^{3.1}$ which maximise area. Recall that $\mathbb{R}^{3.1}$ is the 4-dimensional Euclidean space with metric $\sum_1^3 dx^{i^2} - dt^2$ and that a vector (x,t) is spacelike/timelike/null according as $|x|^2 - t^2 > 0 \;/ < 0 \;/ = 0$ respectively. A surface $M = \text{graph}_\Omega u$, $u \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^3$ is spacelike if all its tangent vectors are spacelike. This means that the induced metric g_{ij} is Riemannian,

(1) $$g_{ij} = \delta_{ij} - u_i u_j > 0,$$

where $u_i = \frac{\partial u}{\partial x^i}$, and hence $|Du| < 1$. The maximal surface equation is the Euler-Lagrange equation arising from the induced area functional:

(2) $$\text{Area}(M) = \int_\Omega \sqrt{\det g_{ij}}\, dx = \int_\Omega \sqrt{1-|Du|^2}\, dx ,$$

and in Minkowski space can be written

(3) $$\frac{1}{\sqrt{1-|Du|^2}} \left(\delta_{ij} - \frac{D_i u\, D_j u}{1-|Du|^2}\right) D_{ij}^2 u = 0.$$

Like the minimal surface equation, this is a nonlinear, non-uniformly elliptic equation and apriori estimates for $|Du|$ are needed in order to prove existence theorems. Quite good estimates are now available [BS], [G], [B], but before describing these I'll discuss some applications.

Lorentzian manifolds are of interest because of general relativity, where they are called space-times. They have quite different properties from the familiar Riemannian manifolds, owing to the non-compactness of the Lorentz group. It is known [HE] that even physically reasonable spacetimes can have singularities and these can have quite unexpected properties. One promising approach to exploring the properties of a spacetime is to decompose it into "space + time". Now, the geometric generalization of (3) is that the surface M is a spacelike submanifold of the spacetime with constant (zero) mean extrinsic curvature, and the constant mean curvature (CMC) surfaces provide a natural space + time decomposition. For example, it is conjectured [ES] that CMC surfaces avoid singularities in phsyically reasonable spacetimes. These surfaces have already been used to study the space of solutions to Einstein's equations, and more importantly, maximal slicing conditions have proved very useful in numerical studies of colliding black holes and other physically interesting situations. A more geometric application was the positive mass conjecture [SY] which used the fact that a maximal surface in a spacetime satisfying the weak energy condition has positive scalar curvature.

These applications all assume that CMC surfaces are smooth spacelike submanifolds, but is is only very recently that this has been proved in any generality [G], [B]. The method is to prove apriori gradient and height bounds and then apply standard nonlinear elliptic theory. The basic assumption is that the spacetime admits a time function: - this allows us to define the height function u of a surface M, and provides a reference timelike vector field T. The basic equations are then

(4) $$\alpha\Delta_M u = \nu H_M + \mathrm{div}_M T$$

(5) $$\Delta_M \nu = \nu(|A|^2 + \mathrm{Ric}(N,N)) + T(H_T) - \langle T, \nabla^M H_M \rangle ,$$

where α is the lapse function of the coordinates, N is the unit normal vector to M, $|A|$ is the length of the second fundamental form and $\nu = -\langle N,T\rangle \sim (1-|Du|^2)^{-\frac{1}{2}}$ measures $|Du|$. A complete derivation and explanation of these formulae is given in [B]: - note that (4) is the nonflat generalization of (3), and (5) follows from the shift-lapse equation/second variation formula.

Using these equations and a maximum principle argument it is possible ([B] theorem 3,1) to estimate ν in terms of $\sup|u|$ and the mean curvature of the boundary of M. Previous estimates [BS],[G] depended on $|\Omega|$ and boundary gradient estimates, which were difficult to obtain. In many cases an estimate for $\sup|u|$ follows from compactness assumptions and this leads immediately to existence theorems, for the Dirichlet problem ([B] theorem 4.2), and the cosmological problem:

THEOREM [G] *Let* V *be a* C^∞ *cosmological spacetime* (i.e. $V \simeq S \times \mathbb{R}$ *where* S *is a compact, boundaryless* 3-*manifold*), *with past and future crushing singularities* [ES]. *Then there is a Cauchy surface* $M_\Lambda \subset V$ *such that* $H_{M_\Lambda} = \Lambda$ *for any* $\Lambda \in \mathbb{R}$.

The maximal surface problem considers zero mean curvature surfaces in asymptotically flat spacetimes. Because the domain is unbounded and estimates for $\sup|u|$ do not follow from natural conditions, this is a more difficult problem. However, a test function argument based on (4) and using the asymptotic flatness conditions very strongly gives the required estimate. This leads to the main theorem:

THEOREM [B] Existence of maximal surfaces

Let V *be an asymptotically flat spacetime with uniform interior* (see [B]§5 for a precise definition). *Then there is a maximal surface asymptotic to every level set of the time function.*

The conditions are satisfied by a wide class of spacetimes, one easy example being asymptotically simple spacetimes. These are topologically $\mathbb{R}^4$ and have metric $g_{\lambda\mu}$ satisfying

$$r\,|g_{\lambda\mu} - \eta_{\lambda\mu}| + r^2|\partial_k g_{\lambda\mu}| + r^3|\partial_k\partial_\lambda g_{\eta\nu}| \leqq C$$

$$r^3|H^o| \leqq C$$

where $r = \left(\sum_1^3 {x^i}^2\right)^{\frac{1}{2}}$ and H^o is the mean curvature of the slices t = constant. The uniform interior condition will be satisfied if there is a constant K such that the vector $(\xi, K|\xi|)$, $\xi \in \mathbb{R}^3$ is timelike (with respect to $g_{\lambda\mu}$) for all $\xi \in \mathbb{R}^3$, throughout V. Heuristically, this says that the light cones of V don't tip over. A complete statement with proofs is given in [B].

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