SAMPLING IN PALEY-WIENER SPACES, UNCERTAINTY AND THE PROLATE SPHEROIDAL WAVEFUNCTIONS

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Abstract. After an introduction to the properties of prolate spheroidal wavefunctions and schemes for their computation, we investigate their role in the sampling theory of band limited signals. We then give a result on the ℓ^2-energy of the tail of sequences generated by their uniform samples, and as a corollary, a local approximate sampling formula for bandlimited functions which are well-concentrated on an interval.

1. Introduction

The prolate spheroidal wavefunctions (PSWF's or *prolates*) have long been studied in the context of solving the wave equation in prolate spheroidal coordinates. This incarnation of the prolates is studied thoroughly in the monographs of Flammer [10], Meixner and Schäfke [17], Stratton et al [21] and Morse and Feshbach [16] among others. The prolates experienced a "second coming" in the work of Landau, Pollak and Slepian published in a remarkable series of papers in the Bell Labs Technical Journal in the 1960's. Now the prolates are undergoing another revival of interest as they are being exploited in telecommunications applications. Here we review some of the most important literature on prolate functions, making connections as we go with sampling theory and Fourier uncertainty.

This paper is organized as follows. Section 2 gives background material on sampling and Fourier uncertainty, including a discussion of the Classical Sampling Theorem, oversampling, and the difficulties encountered when one tries to implement oversampling algorithms. Several examples of Fourier uncertainty principles are described and their consequences for the construction of Fourier bump functions (for oversampling) are explored. In section 3 we explore the way the prolate functions arise in the context of spectral concentration as eigenfunctions of a Hilbert-Schmidt integral operator, describe the basic properties of the prolates and their associated eigenvalues, discuss modern methods for computing the prolates through the "lucky accident" that the prolates are also eigenfunctions of a well-known differential operator, and further consequences of this fact. In section 4 the role of the prolates in the sampling theory of bandlimited functions is explored. We discuss localized sampling results due to Xiao, Rokhlin and Yarvin [25] and a result of the authors on the decay of the ℓ^2-energy of the tails of sequences generated through uniformly sampling prolate functions which has an application to localized sampling of bandlimited functions.

2010 *Mathematics Subject Classification.* Primary 42C10; Secondary 94A20.

Key words and phrases. sampling, uncertainty, prolate spheroidal wavefunctions, quadrature.

2. Sampling and Fourier Uncerainty

2.1. **Classical sampling and oversampling.** To explain the appearance of the prolates in modern signal processing, we go back to the 1961 paper of Slepian and Pollak [1] in which an orthogonal basis for the Paley-Wiener space of functions band-limited to an interval is produced, consisting of eigenfunctions of an integral operator defined in terms of time- and band-limiting operators. Some authors named these functions *Slepians.* It had long been known that functions in the Paley-Wiener spaces are determined (and may be reconstructed from) their (sufficiently dense) uniformly sampled values. To fix notation, let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transform normalized so that on L^1 functions it acts via

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i t\xi}\, dt$$

and let $\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(t)|^2\, dt\right)^{1/2}$. The Paley-Wiener space PW_Ω of bandlimited signals is the collection of those $f \in L^2(\mathbb{R})$ which are band-limited to the interval $[-\Omega/2, \Omega/2]$, i.e.,

$$PW_\Omega = \{f \in L^2(\mathbb{R});\ \hat{f}(\xi) = 0 \text{ for } |\xi| > \Omega/2\}.$$

Signals in PW_Ω may be recovered from their uniform samples $\{f(k/\Omega)\}_{k=-\infty}^{\infty}$. In fact, the celebrated Classical Sampling Theorem states that it $f \in PW_\Omega$, then

$$f(t) = \frac{1}{\Omega} \sum_{k=-\infty}^{\infty} f(k/\Omega)\, \mathrm{sinc}(\Omega t - k) \tag{1}$$

where $\mathrm{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$ is the cardinal sine function. In (1), convergence is in L^2. The history of this result is long and intricate and the interested reader should consult the book by Higgins [11] and references therein.

The sampling theorem (1) allows for the storage of an analog signal f as a digital signal $\{f(k/\Omega)\}_{k=-\infty}^{\infty}$ (or at least a quantized, truncated version of this sequence) and subsequent reconstruction of f via (1). The difficulties encountered by practitioners in the application of (1) arise largely from the slow decay of the cardinal sine, which is $O(1/|t|)$ as $|t| \to \infty$. This means that knowledge of f at a point t_0 requires samples of f taken at points k/Ω far from t_0 since the series decays slowly. One way to overcome this is through *oversampling.*

A *Fourier bump* on $[-\Omega/2, \Omega/2]$ is a function $\varphi \in L^2(\mathbb{R})$ such that

$$\hat{\varphi}(\xi) = \begin{cases} 1 & \text{if } |\xi| < \Omega/2 \\ 0 & \text{if } |\xi| > \Omega. \end{cases}$$

An example is $\varphi(t) = \mathrm{sinc}(\Omega t) = \dfrac{\sin(\pi\Omega t)}{\pi\Omega t}$, whose Fourier transform is the characteristic function of $[-\Omega/2, \Omega/2]$, but here we seek examples with faster decay in time.

Theorem 1 (Oversampling theorem)**.** *Let $f \in PW_\Omega$ and φ be a Fourier bump on $[-\Omega/2, \Omega/2]$. Then*

$$f(t) = \frac{1}{2\Omega} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\Omega}\right) \varphi\left(t - \frac{k}{2\Omega}\right) \tag{2}$$

with convergence in $L^2(\mathbb{R})$.

The reason for (2) is that for all $f \in PW_\Omega$, $\hat{f}\hat{\varphi} = \hat{f}$. The advantage of the reconstruction (2) over (1) is that the Fourier bump of (2) can decay quite rapidly. Hence if the values of f are required on some interval I, samples of f taken far from I do not contribute significantly to the sum in (2) and therefore the sum may be truncated with minimal error. The question that arises here is: How fast can a Fourier bump decay?

2.2. **Fourier bumps.** The simplest examples of Fourier bumps are those generated by splines. For example, the Fourier bump φ whose Fourier transform is the piecewise linear function

$$\hat{\varphi}(\xi) = \frac{2}{\Omega} \mathbf{1}_{[-\Omega/4, \Omega/4]} * \mathbf{1}_{[-3\Omega/4, 3\Omega/4]}(\xi),$$

is the product of scaled cardinal sine functions and has decay of the form $|\varphi(t)| \leq C|t|^{-2}$. This construction may be extended to N-fold convolutions, giving Fourier bumps which are products of scaled cardinal sines with polynomial decay.

Another example may be generated as follows. Define $\psi \in L^2(\mathbb{R})$ via its Fourier transform:

$$\hat{\psi}(\xi) = \begin{cases} \exp\left(\frac{\xi^2}{\xi^2 - 1}\right) & \text{if } |\xi| < 1 \\ 0 & \text{if } |\xi| \geq 1 \end{cases}$$

and let $\psi_\Omega(t) = \psi(4t/\Omega)$. Define φ by

$$\hat{\varphi}(\xi) = c\mathbf{1}_{[-3\Omega/4, 3\Omega/4]} * \widehat{\psi_\Omega}(\xi)$$

where c is a normalizing constant chosen so that $\hat{\varphi}(0) = 1$. Then φ is a Fourier bump on $[-\Omega/2, \Omega/2]$ with *root-exponential* decay: $|\varphi(t)| \leq Ce^{-\sqrt{2\pi|t|}}$.

Sub-exponential decay is also possible. For example, given $0 < \sigma < 1$, there exists $\psi \in PW_1$ with $\psi(0) = 1$ and with decay

$$|\psi(t)| \leq C2^{1/(1-\sigma)} e^{-|\pi t|^\sigma}. \tag{3}$$

With $\varphi(t) = \dfrac{3\Omega}{2} \operatorname{sinc}\left(\dfrac{3\Omega t}{2}\right) \psi\left(\dfrac{\Omega t}{2}\right)$, then φ is a Fourier bump on $[-\Omega/2, \Omega/2]$ with sub-exponential decay (3).

There are obvious barriers to the decay of Fourier bumps. For example, no such function could have Gaussian decay ($|\varphi(t)| \leq ce^{-\alpha t^2}$ with $c, \alpha > 0$) since such a function would violate Hardy's theorem (see section 2.3).

Suppose φ has exponential decay of the form

$$|\varphi(t)| \leq Ce^{-\beta|t|}$$

for some positive constants C and β. Then $\hat{\varphi}$ admits an analytic extension off the real line defined by $\hat{\varphi}(z) = \int_{-\infty}^{\infty} \varphi(t) e^{-2\pi z i t}\, dt$ ($z \in \mathbb{C}$). Since the integral converges on the strip $|\mathrm{Im}(z)| \leq \beta/(2\pi)$, we see that $\hat{\varphi}$ must be analytic on the line. Hence $\hat{\varphi}$ cannot be compactly supported and therefore cannot be a Fourier bump.

Given that we've seen that sub-exponential decay (3) is possible for a Fourier bump, but exponential decay is not, we might wonder whether decay of the form

$$|\varphi(t)| \leq C \exp(-|t|/\log^{\gamma}(|t|)) \tag{4}$$

is achievable. The smaller the value of γ in (4), the faster the decay of φ, and we know that $\gamma = 0$ is not achievable by Fourier bumps. So the natural question is: can a Fourier bump have decay of the form (4) for any $\gamma > 0$? In 1933, Ingham [9], provided the answer.

Theorem 2. *Suppose $\varepsilon(t) \geq 0$, $\varepsilon(t) \downarrow 0$ as $|t| \to \infty$ and $0 \not\equiv \varphi \in L^2(\mathbb{R})$ is such that $\hat{\varphi}(\xi) = 0$ for $|\xi| > \Omega$ and $|\varphi(t)| \leq Ce^{-\varepsilon(|t|)|t|}$. Then*

$$\int_R^{\infty} \varepsilon(t) \frac{dt}{t} < \infty \tag{5}$$

for all $R > 0$.

As a corollary of Ingham's result we see that if a Fourier bump decays as in (4), then $\gamma > 1$. The question of whether such decay is achievable was settled by Güntürk and DeVore (unpublished manuscript). Their construction mimics the construction of spline-type Fourier bumps with polynomial decay given above and starts with a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n < \Omega/4$. Let $\mathbf{1}_n = \mathbf{1}_{[-a_n, a_n]}$ and define $\varphi \in L^2(\mathbb{R})$ by

$$\hat{\varphi} = c\mathbf{1}_{[-3\Omega/4, 3\Omega/4]} * \mathbf{1}_1 * \mathbf{1}_2 * \cdots * \mathbf{1}_n * \cdots$$

with c a suitable normalizing constant. Then φ is a Fourier bump on $[-\Omega/2, \Omega/2]$ with

$$|\varphi(t)| \leq C_{\gamma} \exp\left(\frac{-|t|}{2\log^{\gamma}(|t|)}\right) \tag{6}$$

with $\gamma > 1$ and $C_{\gamma} \to \infty$ as $\gamma \downarrow 1$.

While these constructions are impressive and the implications for the efficiency of oversampling are significant, the difficulty we encounter in using such Fourier bumps in (2) is that (apart from the spline-based Fourier bumps which have relatively slow decay) the exact form of φ is unknown, and computation of the right-hand side of (2) in these examples is not possible.

2.3. **Fourier Uncertainty.** The situation described in the previous section is an example of Fourier uncertainty in action. By Fourier uncertainty we mean the general principle that a function and its Fourier transform cannot both be highly concentrated – high concentrations of f mean low concentration of $\hat{f}$ (and similarly when the roles of f and $\hat{f}$ are reversed). The various incarnations of Fourier uncertainty are essentially described by differing measures of concentration.

The most celebrated of the Fourier uncertainty principles is the Heisenberg inequality, which states that if $f \in L^2(\mathbb{R})$ and $t_0, \xi_0 \in \mathbb{R}$ then

$$\|(t - t_0)f\|_2 \|\|(\xi - \xi_0)\hat{f}\|_2 \geq \frac{\|f\|_2^2}{4\pi} \tag{7}$$

with equality attained in (7) by shifted, modulated Gaussians.

Hardy's theorem takes a different view of concentration: if $|f(t)| \leq Ce^{-\pi\alpha t^2}$ and $|\hat{f}(\xi)| \leq Ce^{-\pi\beta\xi^2}$, then

(i) if $\alpha\beta > 1$ then $f \equiv 0$; and
(ii) if $\alpha\beta = 1$ then f is a Gaussian.

In equation (7), concentrations of f and $\hat{f}$ are measured by the "variances" $\|(t-t_0)f\|_2$ and $\|(\xi-\xi_0)\hat{f}\|_2$. We could, instead, measure the concentration of a function on a set $E \subset \mathbb{R}$ by $\|\mathbf{1}_E f\|_2$. Cowling and Price [8] showed that for all measurable $E \subset \mathbb{R}$ and all $0 < \theta < 1$, there exists a constant $C > 0$ such that

$$\int_E |f(t)|^2\, dt \leq C|E|^\theta \||t|^\theta f\|_2^2$$

where $|E|$ is the Lebesgue measure of E. This may be interpreted as meaning that if the θ-variance $\||t|^\theta f\|_2$ of f is small, then $\hat{f}$ cannot be highly concentrated on a set E of small Lebesgue measure. This is an example of a *local* uncertainty inequality.

Nazarov showed that for all measurable $S \subset \mathbb{R}$ and $\Sigma \subset \hat{\mathbb{R}}$, there exist constants C and $A > 0$ and independent of f such that

$$\|f\|_2^2 \leq Ce^{2A|S||\Sigma|}(\|\mathbf{1}_{S'} f\|_2^2 + \|\mathbf{1}_{\Sigma'}\hat{f}\|_2^2) \tag{8}$$

where S', Σ' are the complements of S and Σ respectively. Details of the proof takes up large part of the text [18]. One way to interpret this result is to say that if $f \in L^2(\mathbb{R})$ is highly concentrated on S and $\hat{f}$ is highly concentrated on Σ (so that $\|\mathbf{1}_S f\|_2/\|f\|_2$ and $\|\mathbf{1}_\Sigma \hat{f}\|_2/\|\hat{f}\|_2$ are large), then the product $|S||\Sigma|$ must also be large.

Beurling's theorem [3] is a joint localization inequality: if $f, \hat{f} \in L^1(\mathbb{R})$ and

$$\iint_{\mathbb{R}^2} |f(t)\hat{f}(\xi)|e^{2\pi|t\xi|}\, dt\, d\xi < \infty$$

then $f = 0$ a.e.

It is clear from analyticity considerations that a function and its Fourier transform cannot both be compactly supported. Benedicks' theorem [2] rules out the possibility of a function and its Fourier transform both being supported on sets of finite measure: if $f \in L^2(\mathbb{R})$, let

$$A = \{t \in \mathbb{R};\, f(t) \neq 0\}, \quad B = \{\xi \in \hat{\mathbb{R}};\, \hat{f}(\xi) \neq 0\}.$$

Then $|A||B| < \infty \Rightarrow f = 0$ a.e.

Several of these results have far-reaching generalizations to locally compact abelian groups, compact groups and semi-simple Lie groups. See [12] and references therein.

3. Prolates and uncertainty

3.1. **The spectral concentration problem.**

Another view of time-frequency localization was studied extensively by Slepian, Pollak and Landau in a series of papers which appeared in the Bell Labs Technical Journal in the early 1960's.

Let $T, \Omega > 0$. Functions in PW_Ω are analytic, hence not supported on $[-T, T]$. Slepian, Pollak and Landau considered the supremum

$$\gamma = \sup_{f \in PW_\Omega} \frac{\int_{-T}^{T} |f(t)|^2 \, dt}{\int_{-\infty}^{\infty} |f(t)|^2 \, dt}$$

and asked the question: can we have $\gamma = 1$? To answer the question, they considered the time and frequency localization operators Q_T and P_Ω defined as follows. Given intervals $[-T, T] \subset \mathbb{R}$ and $[-\Omega/2, \Omega/2] \subset \hat{\mathbb{R}}$, the time-limiting operator Q_T and band-limiting operator P_Ω are defined by

$$Q_T f = \mathbf{1}_{[-T,T]} f; \qquad P_\Omega f(t) = \mathcal{F}^{-1} Q_{\Omega/2} \mathcal{F} f. \tag{9}$$

Here $f \in L^2(\mathbb{R})$ and if $I \subset \mathbb{R}$ then $\mathbf{1}_I$ is the characteristic function of I. The operator Q_T is the orthogonal projection onto the space of time-limited signals and P_Ω is the orthogonal projection onto the space of band-limited signals. The operator $P_\Omega Q_T : PW_\Omega \to PW_\Omega$ is self-adjoint with kernel

$$K(x,t) = \Omega \mathbf{1}_{[-T,T]}(t) \operatorname{sinc}(\Omega(x-t)).$$

Since $\iint_{\mathbb{R}^2} |K(x,t)|^2 \, dx \, dt < \infty$, $P_\Omega Q_T$ is Hilbert-Schmidt, hence compact.

The eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ of $P_\Omega Q_T$ are distinct, positive, and we order them so that $\lambda_0 > \lambda_1 > \cdots > \lambda_n > \cdots$. The corresponding eigenfunctions $\{\psi_n\}_{n=0}^{\infty}$ will be referred to (temporarily) as *Slepians*. When suitably normalized, the Slepians form an orthonormal basis for PW_Ω. Surprisingly, they have another (local) orthogonality property:

$$\int_{-T}^{T} \psi_n(t) \psi_m(t) \, dt = \lambda_n \delta_{nm}$$

and, in fact, the collection $\{\tilde{\psi}_n = \sqrt{\lambda_n} Q_T \psi_n\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2[-T, T]$. Furthermore,

$$\lambda_0 = \|P_\Omega Q_T\| = \sup_{f \in PW_\Omega} \frac{|\langle P_\Omega Q_T f, f \rangle|}{\|f\|_2^2} = \sup_{f \in PW_\Omega} \frac{\|Q_T f\|_2^2}{\|f\|_2^2} \tag{10}$$

so that the top eigenvalue is the maximum concentration on $[-T, T]$ of any function $f \in PW_\Omega$ of norm equal to 1. By the compactness of $P_\Omega Q_T$, the supremum in (10) is attained, i.e., there exists $f \in PW_\Omega$ with $\|f\|_2 = 1$ and $\|Q_T f\|_2^2 = \lambda_0$. By Benedick's theorem we see that $\lambda_0 < 1$.

The behaviour of the eigenvalues of $P_\Omega Q_T$ were studied by Landau and Widom [15] who found that with $c = 2\Omega T$ (the *time-bandwidth product*), $0 < \alpha < 1$ and $N(\alpha)$ the number of eigenvalues λ_n larger than α, we have

$$N(\alpha) = c + \frac{1}{\pi^2} \log\left(\frac{1-\alpha}{\alpha}\right) \log c + o(\log c) \tag{11}$$

as $c \to \infty$. This explains the observed behaviour of the eigenvalues:

(i) the first approximately c eigenvalues are bunched near 1;
(ii) the next approximately $\log c$ eigenvalues plunge towards 0;
(iii) the remaining eigenvalues are very small and decay rapidly.

In fact, it was shown by Widom [23] that after the plunge of the eigenvalues, they decay super-exponentially:

$$\lambda_n \leq C e^{-\alpha n \log n}.$$

The Slepians have many other curious properties beyond their double orthogonality. For example, their Fourier transforms satisfy

$$\hat{\psi}_n(\xi) = \pm i^n \sqrt{\frac{2T}{\Omega\lambda_n}} Q_T \psi_n\left(\frac{2T\xi}{\Omega}\right), \tag{12}$$

i.e., the Slepians are locally self-similar under the Fourier transform.

3.2. **Computing the Slepians.** Galerkin methods for the numerical solution of the eigenvalue problem $P_\Omega Q_T \psi = \lambda\psi$ fail for large c due to the fact that the first (approximately c) eigenvalues are bunched near 1 and, after the plunge region of the eigenvalues, they are bunched near 0. Hence the eigenspaces with eigenvalues near 1 or near 0 cannot be resolved.

It can be shown [14] that the vectors $(\psi_n(k/\Omega))_{k=-\infty}^{\infty}$ are eigenvectors of the doubly-infinite matrix A with (k,ℓ)-the entry

$$A_{k\ell} = \int_{-T}^{T} S(\Omega t - k) S(\Omega t - \ell)\, dt.$$

With appropriate truncation of the matrix A, approximations of the eigenvectors $(\psi_n(k/\Omega))_{k=-\infty}^{\infty}$ may be computed. It is possible to interpolate these values via the classical sampling theorem:

$$\psi_n(t) = \sum_{k=-\infty}^{\infty} \psi_n(k/\Omega) S(\Omega t - k). \tag{13}$$

This algorithm, however, suffers from the slow decay of the entries of A away from the diagonal and, in (13), the slow decay of the cardinal sine, meaning that large truncations of A are required.

3.3. **The "lucky accident".** When solving the Helmholtz equation in $\mathbb{R}^3$ using separation of variables in prolate spheroidal coordinates, the eigenvalue problem

$$\mathcal{P}_c^{(m)} \varphi = \chi\varphi \tag{14}$$

arises where $\mathcal{P}_c^{(m)}$ is the self-adjoint differential operator

$$\mathcal{P}_c^{(m)} = (1-t^2)\frac{d^2}{dt^2} - 2t\frac{d}{dt} - \left(\frac{m}{1-t^2} + c^2 t^2\right) \tag{15}$$

Eigenfunctions of $\mathcal{P}_c^{(m)}$ are known as *prolate spheroidal wave functions* and are denoted $\{\varphi_{m,n}^{(c)}\}_{m,n=0}^{\infty}$.

Slepian and Pollak [1] observed that $\mathcal{P}_c^{(0)}$ commutes with $P_c Q_1$ – the so-called "lucky accident". This means that $\mathcal{P}_c^{(0)}$ and $P_c Q_1$ share their eigenfunctions, so that the zero-th order prolates correspond to the Slepians at least in the case $T = 1$, i.e., ψ_n is a constant multiple of $\varphi_{0,n}^{(c)}$. If the Slepians associated with the intervals $[-T,T]$ and $[-\Omega/2, \Omega/2]$ are denoted $\psi_n^{(T,\Omega)}$ then we have

$$\sqrt{a}\psi_n^{(T,\Omega)}(at) = \psi_n^{(T/a, a\Omega)}(t)$$

so consideration of the Slepians $\psi_n^{(1,c)}$ is sufficient. From now on we will call the Slepian functions *prolates*.

Since the prolates $\psi_n = \psi_n^{(1,c)}$ have now been realized as eigenfunctions of a Sturm-Liouville system, a number of remarkable properties follow:

1. Each ψ_n is real-valued.
2. ψ_n is even when n is even and odd when n is odd.
3. ψ_n has precisely n zeroes in $[-1, 1]$, each of which are nodes (ψ changes sign at each of its zeroes).
4. The zeroes of ψ_n and ψ_{n+1} are interlaced.
5. $\psi_n(\pm 1) \neq 0$.
6. $\{\psi_n\}_{n=0}^{\infty}$ is a *Markov system* on $[-1, 1]$. This means that for all $N \geq 0$, any non-trivial linear combination $\sum_{n=0}^{N} c_n \psi_n$ vanishes at most $N - 1$ times on $[-1, 1]$.
7. $\{\psi_n\}_{n=0}^{\infty}$ has a unique *Gaussian quadrature*.
8. The eigenvalues $\{\chi_n\}_{n=0}^{\infty}$ of $\mathcal{P}_c^{(0)}$ are simple and well-separated:

$$0 < \chi_0 < \chi_1 < \cdots < \chi_n < \cdots \uparrow \infty.$$

In fact, $\chi_n(c) \approx n(n+1)\left[1 + O\left(\dfrac{c^2}{2n^2}\right)\right]$ and as $c \to \infty$, we have $\chi_n(c) \approx c(2n+1) - \dfrac{n^2 + n + 3/2}{2} + O(1/c)$ [6].

Property 7 will be explored further in section 4. Property 8 leads to the feasibility of efficient and accurate Galerkin methods for computing the prolates – even those of small or large order whose computation from the integral operator $P_c Q_1$ is unfeasible due to the bunching of the eigenvalues of that operator.

Bouwkamp [4] developed a method for the construction of the prolates which exploited the Sturm-Liouville property. Separately, Xiao, Rokhlin and Yarvin [25] and Boyd [5] developed a Galerkin approach in which the eigenproblem (14) is reduced to an eigenproblem for a tri-diagonal matrix. Since $\psi_n \in L^2[-1, 1]$ we have the expansion

$$\psi_n = \sum_{j=0}^{\infty} \beta_{nj} P_j$$

where $\{P_j\}_{j=0}^{\infty}$ are the Legendre polynomials. We then have

$$\chi_n \psi_n = \mathcal{P}_c^{(0)} \psi_n = \sum_{j=0}^{\infty} \beta_{nj} \mathcal{P}_c^{(0)} P_j.$$

However, the Legendre polynomials satisfy

$$(1 - t^2) P_n''(t) - 2t P_n'(t) = -n(n+1) P_n(t), \quad \text{and}$$

$$t P_n(t) = \frac{1}{2n+1} [(n+1) P_{n+1}(t) + n P_{n-1}(t)]$$

so that $\mathcal{P}_c^{(0)} P_j = \sum_{k=0}^{\infty} A_{jk} P_k$ with $A = (A_{jk})_{j,k=0}^{\infty}$ a doubly-infinite tri-diagonal matrix. We therefore have

$$\chi_n \psi_n = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} A_{jk} \beta_{nj} \right) P_k = \chi_n \sum_{j=0}^{\infty} \beta_{nj} P_j.$$

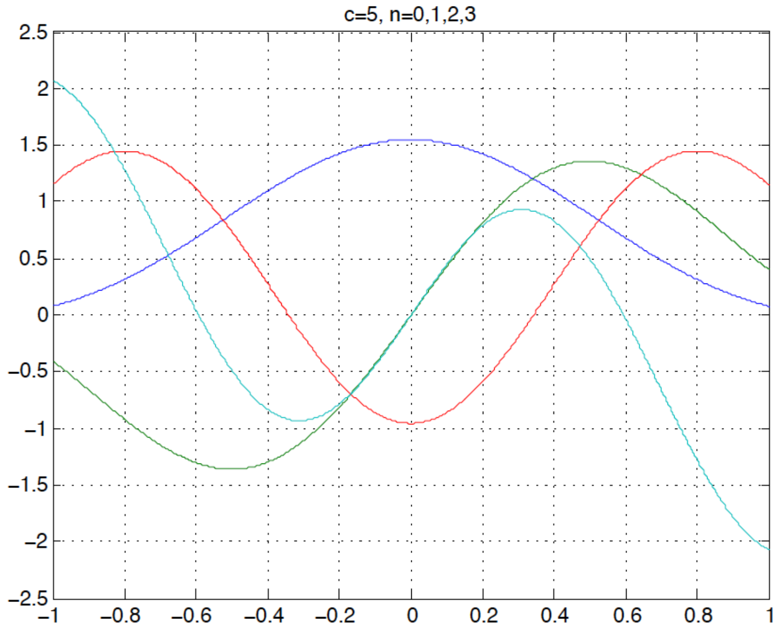

FIGURE 1. The prolates ψ_0 (blue), ψ_1 (green), ψ_2 (red), ψ_3 (azure) associated with the time limit $T = 1$ and frequency limit $\Omega = 5/2$ so that $c = 5$.

With $\mathbf{b}_n = (\beta_{n0}, \beta_{n1}, \dots, \beta_{nj}, \dots)^T$, we have

$$A^T \mathbf{b}_n = \chi_n \mathbf{b}_n,$$

a matrix eigenproblem for the Legendre coefficients of the prolates on $[-1, 1]$ and associated eigenvalues of $\mathcal{P}_c^{(0)}$. This problem may be accurately truncated and solved numerically for the eigenvectors $\mathbf{b}_n$ and eigenvalues χ_n (see [5], [25], [13], [19] for further details).

This procedure works well when c is small. For large values of c, it is best to use Hermite functions in place of Legendre polynomials. See [13], [6] and [24] for details. The first 4 prolate functions are plotted in figure 1 for the case $c = 5$.

4. PROLATES AND SAMPLING

4.1. **Sampling properties of the prolates.** Since PW_Ω is translation-invariant, $S(\Omega t) = \operatorname{sinc}(\Omega t) \in PW_\Omega$, and the prolates form an orthonormal basis for PW_Ω, we have

$$S(\Omega t - k) = \sum_{n=0}^{\infty} \left(\int_{-\infty}^{\infty} S(\Omega(s - k/\Omega))\psi_n(s)\, ds \right) \psi_n(t) = \sum_{n=0}^{\infty} \psi_n(k/\Omega)\psi_n(t).$$

As a consequence of the cardinality of the cardinal sine, we have the discrete orthogonality properties of the prolates:

$$\sum_{n=0}^{\infty} \psi_n(j/\Omega)\psi_n(k/\Omega) = \delta_{jk} \quad (j,k \in \mathbb{Z})$$
$$\sum_{k=-\infty}^{\infty} \psi_n(k/\Omega)\psi_m(k/\Omega) = \delta_{nm} \qquad (m,n \in \mathbb{Z}).$$

and hence the sampling expansion

$$\begin{aligned} f(t) &= \frac{1}{\Omega}\sum_{k=-\infty} f(k/\Omega)S(\Omega t - k) \\ &= \frac{1}{\Omega}\sum_{k=-\infty}^{\infty} f(k/\Omega)\sum_{n=0}^{\infty}\psi_n(k/\Omega)\psi_n(t) \\ &= \frac{1}{\Omega}\sum_{n=0}^{\infty}\Big(\sum_{k=-\infty}^{\infty} f(k/\Omega)\psi_n(k/\Omega)\Big)\psi_n(t). \end{aligned} \tag{16}$$

The question we will try to answer in section 4.2 is whether the double sum above can be accurately truncated to give an implementable, approximate sampling formula.

4.2. **Localised sampling.** As mentioned in section 3, the prolates $\{\psi_n\}_{n=0}^{\infty}$ have unique Gaussian quadratures. This means that for each fixed integer $N \geq 1$, there are unique nodes $\{t_i\}_{i=1}^{N} \subset [-1,1]$ and positive weights $\{w_i\}_{i=1}^{N}$ such that

$$\int_{-1}^{1} \psi_n(t)\,dt = \sum_{i=1}^{N} w_i\psi_n(t_i) \tag{17}$$

for all $0 \leq n \leq 2N-1$. Equation (17) involves $2N$ equations in $2N$ unknowns w_i and t_i $(1 \leq i \leq N)$. These equations are non-linear in the t_i variables. To compute approximate Gaussian quadrature nodes $\{t_i\}_{i=1}^{N}$, one starts with an initial approximation $\{t_i^*\}_{i=1}^{N}$ (which might be, for example, the well-known Legendre quadrate nodes) and then applies Newton iteration. The process is outlined in [13] and [5].

Xiao, Rokhlin and Yarvin [25] investigate the use of generalized Gaussian quadratures in localized sampling reconstructions of functions in PW_c. Their work relies on the following generalization of the division algorithm for polynomials.

Theorem 3. *Suppose $f \in PW_{2c}$ and $p \in PW_c$. Then there exists $q, r \in PW_c$ with $\|p\|_2, \|r\|_2 \leq C\|f\|_2$ such that*

$$f = pq + r.$$

Let $\{t_i\}_{i=1}^{N}$ be the N zeroes of $\psi_N^{(1,c)}$ on $[-1,1]$ and let $\{w_i\}_{i=1}^{N}$ be weights for which (17) is valid for $0 \leq n \leq N-1$. The existence of such weights is guaranteed by the fact that the prolates form a Markov system on $[-1,1]$. Then we have

Theorem 4. *Let $g \in PW_{2c}$ and $\{t_i\}_{i=1}^N$, $\{w_i\}_{i=1}^N$ be as above. Then*

$$\left| \int_{-1}^{1} g(t)\,dt - \sum_{i=1}^{N} w_i g(t_i) \right| \le C\lambda_N \|f\|_2. \tag{18}$$

When applied to $g = f\psi_n$ with $f \in PW_c$, Theorem 4 leads to the following local sampling result.

Theorem 5. *Let $\{t_i\}_{i=1}^N$ and $\{w_i\}_{i=1}^N$ be as above and*

$$s_n(t) = w_n \sum_{k=0}^{N-1} \psi_k(t_n)\psi_k(t) \qquad (1 \le n \le N).$$

Then for all $f \in PW_c$, and $|t| \le 1$,

$$\left| f(t) - \sum_{n=1}^{N} f(t_n) s_n(t) \right| \le C\lambda_N \|f\|_2. \tag{19}$$

This (and similar) results aim to accurately recover the values of $f \in PW_c$ on $[-1,1]$ from values $\{f(t_i)\}_{i=1}^N$ at specialized quadrature nodes $\{t_i\}_{i=1}^N$. We ask what happens when samples are taken uniformly, as in the classical sampling formula (1), but only finitely many values are taken. Since PW_c is translation-invariant, we cannot expect finitely many uniform samples to accurately recover arbitrary functions in PW_c. We can, however, ask if functions which are well-localized on $[-1,1]$ (e.g., the prolates) can be accurately recovered from samples taken on (or near) $[-1,1]$.

The starting point is the sampling formula (16). We now change our point of view by dilating the prolates so that $\Omega = 1$ and $T = c$. Then (16) takes the form

$$f(t) = \sum_{n=0}^{\infty} \Big(\sum_{k=-\infty}^{\infty} f(k)\psi_n(k) \Big) \psi_n(t)$$

for $f \in PW_1$. The projection of f onto $\mathrm{sp}\{\psi_n\}_{n=0}^{N-1}$ is

$$f_N(t) = \sum_{n=0}^{N-1} \Big(\sum_{k=-\infty}^{\infty} f(k)\psi_n(k) \Big) \psi_n(t)$$

and the local approximation is obtained by truncating the infinite sum over k:

$$f_{N,K}(t) = \sum_{n=0}^{N-1} \Big(\sum_{|k|\le K} f(k)\psi_n(k) \Big) \psi_n(t).$$

Lemma 6. *Let $f \in PW_1$ and $f_{N,K}$ be as above, Then*

$$\|Q_T(f - f_{N,K})\|_2^2 \le [(\lambda_N + \varepsilon)T + \sum_{n=0}^{N-1} \lambda_n \alpha_{K,n}] \|f\|_2^2 \tag{20}$$

where $\alpha_{K,n} = \sum_{|k|\ge K} \psi_n(k)^2$.

Proof. Suppose f is well-localized on $[-T,T]$, i.e., $\int_{|t|>T}|f(t)|^2\,dt \le \varepsilon\|f\|_2^2$ for some small ε. With $Q'_T = I - Q_T$, we have

$$
\begin{aligned}
\|Q_T(f-f_N)\|_2^2 &= \int_{-T}^{T}\left|\sum_{n=N}^{\infty}\langle f,\psi_n\rangle\psi_n(t)\right|^2 dt\\
&= \sum_{n=N}^{\infty}|\langle f,\psi_n\rangle|^2\lambda_n\\
&= \sum_{n=N}^{\infty}|\langle Q_T f,\psi_n\rangle + \langle Q'_T f,\psi_n\rangle|^2\lambda_n\\
&\le \sum_{n=N}^{\infty}[\|Q_T f\|_2\sqrt{\lambda_n} + \|Q'_T f\|_2]^2\lambda_n\\
&\le \sum_{n=N}^{\infty}(\sqrt{\lambda_N}+\sqrt{\varepsilon})^2\lambda_n\|f\|_2^2\\
&\le 2\sum_{n=N}^{\infty}(\lambda_n+\varepsilon)\lambda_n\|f\|_2^2 \le C(\lambda_N+\varepsilon)c\|f\|_2^2 \qquad (21)
\end{aligned}
$$

where in the last line we have used the fact that $\sum_{n=0}^{\infty}\lambda_n = c(=2T)$, a consequence of the fact that P_1Q_T is a trace class operator with eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. Also,

$$
\begin{aligned}
f_N(t) - f_{N,K}(t) &= \sum_{n=0}^{N-1}\langle f,\psi_n\rangle\psi_n(t) - \sum_{n=0}^{N-1}\Bigg(\sum_{|k|<K} f(k)\psi_n(k)\Bigg)\psi_n(t)\\
&= \sum_{n=0}^{N-1}\Bigg(\sum_{|k|\ge K} f(k)\psi_n(k)\Bigg)\psi_n(t)
\end{aligned}
$$

so that, with an application of Cauchy-Schwarz and the equality $\sum_k |f(k)|^2 = \|f\|_2^2$, we have

$$
\begin{aligned}
\|Q_T(f_N - f_{N,K})\|_2^2 &= \sum_{n=0}^{N-1}\lambda_n\left|\sum_{|k|\ge K} f(k)\psi_n(k)\right|^2\\
&\le \sum_{n=0}^{N-1}\lambda_n\|f\|_2^2\sum_{|k|\ge K}\psi_n(k)^2 = \|f\|_2^2\sum_{n=0}^{N-1}\lambda_n\alpha_{K,n} \qquad (22)
\end{aligned}
$$

where $\alpha_{K,n} = \sum_{|k|\ge K}\psi_n(k)^2$. Combining (21) and (22) gives (20). The proof is complete. □

Note that equation (20) represents an accurate, localized sampling expansion of f on $[-T,T]$ provided $\alpha_{K,n}$ can be made small by choosing K sufficiently large.

In [22], Walter and Shen show that if $K = T$, then $\alpha_{T,n} \le CT\sqrt{1-\lambda_n}$. In [14], the authors show that if $K = M(T) = CT\log^{\gamma} T$ with $\gamma > 1$, then

$$\alpha_{M(T),n} \le C(1-\lambda_n).$$

Note the independence of the right-hand side of this inequality on T. The restriction $\gamma > 1$ comes from the use of a Fourier bump in the proof (see section 2.2). Note also that $M(T)/T \to \infty$ as $T \to \infty$. An improved result is as follows.

Theorem 7. *Let ψ_n denote the n-th prolate band-limited to $[-1/2, 1/2]$, time-concentrated on $[-T, T]$ and normalized so that $\int_{\infty}^{\infty} |\psi(t)|^2\, dt = 1$. Then there are constants C_1 and C_2 such that if $M = T(1 + C_1 \log T)$, then*

$$\sum_{|k|>M} \psi_n(k)^2 \leq C_2(1 - \lambda_n). \tag{23}$$

Proof. For simplicity we write $\psi = \psi_n$. Let $g(t) = e^{-\pi t^2}$ and $g_\varepsilon(t) = \varepsilon^{-1} g(t/\varepsilon)$ be the L^1-normalised dilate of g. Let

$$\hat{\Phi}(\xi) = \frac{1}{2}\mathbf{1}_{[-2,2]} * \mathbf{1}_{[-1,1]}(\xi)$$

so that $\hat{\Phi}(\xi) = 1$ if $|\xi| \leq 1$, $\hat{\Phi}(\xi) = 0$ if $|\xi| \geq 3$ and $0 \leq \hat{\Phi}(\xi) \leq 1$ for all ξ. Note that

$$|\Phi(t)| = \left| \frac{\sin(4\pi t)\sin(2\pi t)}{2\pi^2 t^2} \right| \leq \frac{C}{1+t^2}.$$

Hence $\Phi \in L^1$ and the majorant $\Phi^{\#}$ given by $\Phi^{\#}(t) = \sup_{|t-x|\leq 1} |\Phi(x)|$ satisfies $\Phi^{\#}(t) \leq \dfrac{C}{1+t^2}$. Let $\hat{\Gamma} = \hat{\Phi} * g_\varepsilon$. Then $\Gamma(t) = \Phi(t) e^{-\pi\varepsilon^2 t^2} \leq e^{-\pi^2\varepsilon^2 t^2}/t^2$. Also, $\hat{\Gamma}$ is non-negative and

$$\hat{\Gamma}(\xi) = \int_{-\infty}^{\infty} \hat{\Phi}(\eta) g_\varepsilon(\xi - \eta)\, d\eta \leq \int_{-\infty}^{\infty} g_\varepsilon(\xi - \eta)\, d\eta = 1.$$

On the other hand, for $|\xi| \leq 1/2$,

$$\hat{\Gamma}(\xi) \geq \int_{-1}^{1} g_\varepsilon(\eta - \xi)\, d\eta = \int_{-1-\xi}^{1-\xi} g_\varepsilon(u)\, du \geq \int_{-1/2\varepsilon}^{1/2\varepsilon} g(t)\, dt = 1 - \operatorname{erfc}(\sqrt{\pi}/2\varepsilon)$$

where erfc is the complementary error function $\operatorname{erfc}(t) = \dfrac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2}\, dx$. For sufficiently large t, $\operatorname{erfc}(t) \leq C\dfrac{e^{-t^2}}{t}$. Hence, for $|\xi| < 1/2$, $\hat{\Gamma}(\xi) \geq 1 - C\varepsilon e^{-\pi/(4\varepsilon^2)}$ and consequently, for $|\xi| < 1/2$, $|1 - \hat{\Gamma}(\xi)| \leq C\varepsilon e^{-\pi/(4\varepsilon^2)}$. Now we write

$$\psi = \psi * \Gamma + (\psi - \psi * \Gamma). \tag{24}$$

Given $M > 0$, we want to estimate $\sum_{|k|>M} \psi(k)^2$. From (24) we see that

$$\sum_{|k|>M} \psi(k)^2 \leq C \sum_{|k|>M} \psi * \Gamma(k)^2 + C \sum_{|k|>M} (\psi - \psi * \Gamma)(k)^2 = A + B. \tag{25}$$

Since $\hat\psi$ and $\hat\psi\hat\Gamma$ are supported on $[-1/2,1/2]$, an application of the Poisson summation formula yields

$$\begin{aligned}
B &\le C\sum_{k=-\infty}^{\infty}|\psi(k)-\psi*\Gamma(k)|^2\\
&= C\int_{-1/2}^{1/2}\left|\sum_{k=-\infty}^{\infty}(\psi(k)-(\psi*\Gamma)(k))e^{2\pi ik\xi}\right|^2 d\xi\\
&= C\int_{-1/2}^{1/2}\sum_{\ell=-\infty}^{\infty}|\hat\psi(\xi+\ell)-\hat\psi(\xi+\ell)\hat\Gamma(\xi+\ell)|^2\,d\xi\\
&= C\int_{-1/2}^{1/2}|\hat\psi(\xi)(1-\hat\Gamma(\xi))|^2\,d\xi \le C\varepsilon^2e^{-\pi/(2\varepsilon^2)}\|\psi\|_2^2 = C\varepsilon^2e^{-\pi/(2\varepsilon^2)}. \qquad (26)
\end{aligned}$$

We now want to estimate $A=\sum_{|k|>M}\psi*\Gamma(k)^2$ and we write $\psi*\Gamma=(Q_T\psi)*\Gamma+(Q_T')\psi*\Gamma$ so that

$$A\le C\sum_{|k|>M}Q_T\psi*\Gamma(k)^2+C\sum_{|k|>M}Q_T'\psi*\Gamma(k)^2=E+F. \qquad (27)$$

For all $s\in[k-1,k+1]$, $|\Gamma*(Q_T'\psi)(k)|\le|(\Gamma*(Q_T'\psi))^\#(s)|$ so that

$$|\Gamma*(Q_T'\psi)(k)|^2\le\int_{k-1/2}^{k+1/2}|(\Gamma*(Q_T'\psi))^\#(s)|^2\,ds$$

and

$$\begin{aligned}
F &\le C\sum_k|\Gamma*(Q_T'\psi)(k)|^2\le C\int_{-\infty}^{\infty}|(\Gamma*(Q_T'\psi))^\#(s)|^2\,ds\\
&\le C\int_{-\infty}^{\infty}|Q_T'\psi|*\Gamma^\#(s)^2\,ds\le C_\Gamma\int_{-\infty}^{\infty}\mathcal{M}(|Q_T'\psi|)(s)^2\,ds\\
&\le C_\Gamma\|Q_T'\psi\|_2^2=C_\Gamma(1-\lambda_n) \qquad (28)
\end{aligned}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function. We have used the boundedness of $\mathcal{M}$ on $L^2(\mathbb{R})$ and also the fact that $|\Gamma^\#*f(x)|\le C_\Gamma\mathcal{M}f(x)$ where C_Γ is constant that depends only on Γ (see [20]). On the other hand, since $(f*g)^\#\le f^\#*|g|$, we have

$$\begin{aligned}
E=C\sum_{|k|>M}(\Gamma*Q_T\psi)(k)^2 &\le C\sum_{|k|>M}\int_{k-1/2}^{k+1/2}(\Gamma*Q_T\psi)^\#(s)^2\,ds\\
&= C\int_{|s|>M-1/2}(\Gamma*Q_T\psi)^\#(s)^2\,ds\\
&\le C\int_{|s|>M-1/2}\Gamma^\#*|Q_T\psi|(s)^2\,ds\\
&= C\int_{|s|>M-1/2}\left(\int_{|t|<T}|Q_T\psi(t)|\Gamma^\#(s-t)\,dt\right)^2ds. \qquad (29)
\end{aligned}$$

But $|s| > M - 1/2$ and $|t| < T$ implies that $|s - t| \geq M - T - 1/2$, so from (29) we have

$$\begin{aligned} E &= C \sum_{|k|>M} (\Gamma * Q_T\psi)(k)^2 \\ &\leq C \int_{|s|>M-1/2} \left(\int_{|t|<T} |Q_T\psi(t)| Q'_{M-T-1/2}\Gamma^{\#}(s-t)\,dt \right)^2 ds \\ &\leq C \| |Q_T\psi| * Q'_{M-T-1/2}\Gamma^{\#} \|_2^2 \\ &\leq C \|Q_T\psi\|_2^2 \|Q'_{M-T-1/2}\Gamma^{\#}\|_1^2 = C\lambda_n \|Q'_{M-T-1/2}\Gamma^{\#}\|_1^2. \end{aligned} \tag{30}$$

However, $\Gamma^{\#}(t) \leq Ce^{-\pi\varepsilon^2 t^2}$, so for each $R > 1$,

$$\|Q'_R\Gamma^{\#}\|_1 \leq C \int_{|t|>R} e^{-\pi\varepsilon^2 t^2} \frac{dt}{t^2} \leq Ce^{-\pi\varepsilon^2 R^2}$$

and applying this to (30) gives

$$E \leq C\lambda_n e^{-\pi\varepsilon^2 R^2}. \tag{31}$$

with $R = M - T - 1/2$. From (24)–(28) and (31) we see that it's enough to satisfy simultaneously the requirements

$$C\varepsilon^2 e^{-\pi/(2\varepsilon^2)} \leq 1 - \lambda_n; \qquad \text{and} \tag{32}$$

$$Ce^{-2\pi\varepsilon^2 R^2} \leq 1 - \lambda_n. \tag{33}$$

However, $\lambda_n \leq \lambda_0$ so it's enough to have (32) and (33) with λ_n replaced by λ_0. From (11) we have

$$\lambda_0 \exp(-\pi^2(C + c/\log c) \leq 1 - \lambda_0 \leq \lambda_0 \exp(\pi^2(C - c/\log c)) \tag{34}$$

for some constant C and sufficiently large time-bandwidth product $c = 2T$. Hence, for (32) to be satisfied, the inequality

$$C\varepsilon^2 e^{-\pi/(2\varepsilon^2)} \leq \lambda_0 e^{-(C+T/\log T)}$$

is sufficient, and for this we see that

$$\varepsilon = C\sqrt{\frac{\log T}{T}} \tag{35}$$

is sufficient. With ε thus defined, we now choose R so that (33) is satisfied. By (34), the condition

$$Ce^{-2\pi\varepsilon^2 R^2} \leq e^{-\pi^2(C+T/\log T)}$$

is sufficient, which is in turn implied by $2\varepsilon^2 R^2 \geq CT/\log T$, or equivalently, $R = CT/\log T$. Hence

$$M = R + T + 1/2 = C\frac{T}{\log T} + T.$$

This completes the proof. □

Acknowledgements

Hogan gratefully acknowledges the support of the University of Newcastle's Centre for Computer Assisted Research in Mathematics and its Applications (CARMA). The authors are indebted to the referee, whose meticulous work helped improve the readability of the paper. Thanks Roy. Thanks HG.

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