A MAXIMAL FUNCTION APPROACH TO OPERATOR TRACES

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Abstract. A trace class operator $T : L^2(\mu) \to L^2(\mu)$ has the property that for a distinguished kernel k representing T, the formula $\mathrm{trace}(T) = \int_\Sigma k(x,x)\,d\mu(x)$ is valid. The Hardy-Littlewood maximal function is used to establish an analogous formula $\int_\Sigma \langle T, dm\rangle = \int_\Sigma k(x,x)\,d\mu(x)$ for a class of integral operators T wider than trace class operators, including the Volterra integral operator. The 'generalised trace' $\int_\Sigma \langle T, dm\rangle$ surfaces in the Cwikel-Lieb-Rosenbljum inequality for dominated semigroups on $L^2(\mu)$.

1. Introduction

The *singular values* $\{\lambda_j\}_{j=1}^\infty$ of a compact linear operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ are the eigenvalues of the compact selfadjoint operator $(T^*T)^{\frac{1}{2}}$. The operator T is called *trace class* if $\sum_{j=1}^\infty \lambda_j < \infty$, or equivalently, $\sum_{j=1}^\infty |(Th_j, h_j)| < \infty$ for any orthonormal set $\{h_j\}_{j=1}^\infty$ in $\mathcal{H}$. Many facts about trace class operators are collected in [13, 30]. In the case where the Hilbert space $\mathcal{H}$ has finite dimension $n = 1, 2, \dots$, the trace of T is the number $\sum_{j=1}^n a_{jj}$ for any matrix representation $\{a_{jk}\}_{j,k=1}^n$ of the linear map T with respect to a basis of $\mathcal{H}$. In the case of an infinite dimensional separable Hilbert space $\mathcal{H}$, the trace is

$$\mathrm{trace}(T) = \sum_{j=1}^\infty (Th_j, h_j)$$

with respect to any orthonormal basis $\{h_j\}_{j=1}^\infty$ of $\mathcal{H}$ [30, Theorem 3.1]. By analogy with the finite dimensional case, if $T_k : L^2([0,1]) \to L^2([0,1])$ is a trace class linear operator with an integral kernel k, then one might hope that

$$\mathrm{trace}(T_k) = \int_0^1 k(x,x)\,dx. \tag{1.1}$$

However, $\{(x,x) : x \in [0,1]\}$ is a set of measure zero in $[0,1]^2$ and if $k = k_1$ almost everywhere on $[0,1]^2$, then $T_k = T_{k_1}$, so the right hand side of equation (1.1) is not well defined. Nevertheless, there does exist a *distinguished* kernel k for which (1.1) is valid, because any trace class operator

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$T : L^2([0,1]) \to L^2([0,1])$ has a representation

$$T : h \longmapsto \sum_{j=1}^{\infty} \lambda_j \phi_j (h, \psi_j), \quad h \in L^2([0,1]), \tag{1.2}$$

with respect to the L^2-inner product $(f,g) = \int_0^1 f(x)\overline{g(x)}\,dx$, $f,g \in L^2([0,1])$, and the singular values $\{\lambda_j\}_{j=1}^{\infty}$ of T. The sets $\{\phi_j\}$ and $\{\psi_j\}$ of vectors are orthonormal in $L^2([0,1])$, so the representation (1.1) is valid for the distinguished integral kernel k defined by

$$k(x,y) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)\overline{\psi_j(y)}$$

for all $x, y \in [0,1]$ for which the right hand side is absolutely convergent. In particular, if $T_k : L^2([0,1]) \to L^2([0,1])$ is a trace class linear operator with a continuous integral kernel k on $[0,1]^2$, then (1.1) holds [30, Theorem 3.9].

C. Brislawn observed in [4] that if $T : L^2([0,1]) \to L^2([0,1])$ is a trace class linear operator and k_0 is any integral kernel of T, then the kernel $k = \lim_{\epsilon \to 0+} \varphi_\epsilon * k_0$ has the property that $T = T_k$ and equation (1.1) holds. Here $\varphi_\epsilon(x) = \epsilon^{-2}\varphi(x/\epsilon)$, $x \in \mathbb{R}^2$, $\epsilon > 0$, for some nonnegative function φ on $\mathbb{R}^2$ that is is zero outside $[-1,1]^2$ and with the property that $\int_{\mathbb{R}^2} \varphi(x)\,dx = 1$ and φ has an integrable radially decreasing majorant: the characteristic function $\varphi = \chi_{[-\frac{1}{2},\frac{1}{2}]^2}$ will do. The convolution $u * v$ of $u, v \in L^1(\mathbb{R}^n)$ is defined for almost all $x \in \mathbb{R}^n$ by the formula

$$u * v(x) = \int_{\mathbb{R}^n} u(x-y)v(y)\,dy.$$

The convolution $u*v$ of a function $u \in L^1(\mathbb{R}^2)$ with $v \in L^1([0,1]^2)$ is defined almost everywhere in $[0,1]^2$ by setting v equal to zero outside $[0,1]^2$.

Then the map $k_0 \longmapsto \lim_{\epsilon \to 0+} \varphi_\epsilon * k_0$ is a *smoothing operator* for which the value of $k = \lim_{\epsilon \to 0+} \varphi_\epsilon * k_0$ at a point (x,x) of the diagonal is defined by averages in $[0,1]^2$ about (x,x) for almost every $x \in [0,1]$. A related approach appears on [13, Theorem 8.4].

It is clear that this idea need not be confined to trace class operators.

Example 1.1 ([4, Example 3.2])**.** The Volterra operator T is defined by

$$(Tf)(x) = \int_0^x f(y)\,dy, \quad x \in [0,1],$$

for $f \in L^2([0,1])$. Then T is also defined by the integral kernel $k_0 = \chi_{\{y<x\}}$. The (lattice) positive linear map $T : L^2([0,1]) \to L^2([0,1])$ is a Hilbert-Schmidt operator but not trace class: it has singular values $\lambda_n = 2/(\pi(2n+1))$, $n = 1, 2, \ldots$. For the regularised kernel $k = \lim_{\epsilon \to 0+} \varphi_\epsilon * k_0$ defined above, we have $k = k_0$ off the diagonal in $]0,1[^2$ and $k(x,x) = 1/2$ for all $x \in]0,1[$, so $\int_0^1 k(x,x)\,dx = \frac{1}{2}$. The operator T is *not* hermitian positive on the complex Hilbert space $L^2([0,1])$, that is, we don't have $(Tu,u) \geq 0$ for every $u \in L^2([0,1])$.

Suppose that the finite rank operator $T \in \mathcal{L}(L^2([0,1]))$ has an integral kernel $k = \sum_{j=1}^n f_j \otimes \chi_{A_j}$ with $\mu(A_j) < \infty$ and $f_j \in L^2([0,1])$ for $j = 1, \ldots, n$.

Then it is natural to view

$$\int_\Sigma \langle T, dm\rangle := \sum_{j=1}^n \int_{A_j} f_j\, d\mu = \int_\Sigma k(x,x)\, d\mu(x)$$

as a bilinear integral of the operator T with respect to the $L^2([0,1])$-valued measure $m : A \longmapsto \chi_A$, $A \in \mathcal{B}([0,1])$. The point of view adopted in this note is to use Brislawn's averaging process to extend the bilinear integral $\int_\Sigma \langle T, dm\rangle$ to a wider class of absolute integral operators T acting on $L^2([0,1])$ or a general Banach function space X, so that $\int_\Sigma \langle T, dm\rangle$ is actually the trace of T in the case where $T \in \mathcal{L}(L^2([0,1]))$ is a trace class operator. The bilinear integral $\int_\Sigma \langle T, dm\rangle$ features in the recent proof of the Cwikel-Lieb-Rosenbljum inequality for dominated semigroups in [15].

Some basic facts about the Hardy-Littlewood maximal operator in the unit square are gathered in Section 2 and these are applied in Section 3 to integrable functions to produce a Banach function space $L^1(\rho)$ embedded in $L^1([0,1]^2)$. Functions belonging to a certain closed subspace of $L^1(\rho)$ have the property that the set of its Lebesgue points has full linear measure on the diagonal of $[0,1]^2$ and in this sense, they are *traceable*. For an operator T whose integral kernel is of this class, the bilinear integral $\int_\Sigma \langle T, dm\rangle$ converges unequivocally. In Section 4, absolute integral operators $T : X \to X$ acting on a Banach function space X over a σ-finite measure space are considered. Now that convolution is unavailable, the same idea with respect to the *martingale maximal function* is applied. In the case that $X = L^2([0,1])$, the dyadic martingale on $[0,1]$ yields the same results as in Section 3. The standard facts we need about complex Banach lattices are laid out in the monograph [22].

In Theorem 4.1, a *lattice ideal* $\mathfrak{C}_1(\mathcal{E}, X)$ in the space or regular operators on a Banach function space X is constructed. The Banach lattice $\mathfrak{C}_1(\mathcal{E}, X)$ depends on the given filtration $\mathcal{E}$. By contrast, the collection of trace class operators on a Hilbert space $\mathcal{H}$ is an *operator ideal* in the space $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. Happily, if μ is a σ-finite measure, then a hermitian positive bounded linear operator on the complex Hilbert space $L^2(\mu)$ is an element of $\mathfrak{C}_1(\mathcal{E}, X)$ if and only if it is trace class, no matter what filtration $\mathcal{E}$ is given [16]. We end with a short discussion of other 'generalised traces' in recent literature.

2. The Hardy-Littlewood Maximal Operator

The Lebesgue measure on $\mathbb{R}$ is denoted by λ. The Lebesgue measure of a Borel subset B of $\mathbb{R}^n$ is sometimes written as $|B|$ and it will be understood to apply to expressions like 'almost everywhere' and 'almost all' with respect to subsets of $\mathbb{R}^n$. The *centred Hardy-Littlewood maximal function* of $f \in L^1([0,1]^2)$ is given by

$$M(f)(x) = \sup_{r>0} \frac{\int_{C_r} |f(x+t)|\, dt}{|C_r|}, \quad x \in [0,1]^2. \tag{2.1}$$

In the formula above, the function f is put equal to zero outside the square $[0,1]^2$ and $C_r = [-r,r] \times [-r,r]$ for $r > 0$. The maximal function $M(f)$ is equivalent to the maximal function obtained by averaging over centred disks [14, Exercise 2.1.3], but for the purposes of the present note it is convenient to emphasise the product structure of the unit square. According to Lebesgue's differentiation theorem [14, Corollary 2.1.16], if $f \in L^1([0,1]^2)$, we have

$$\lim_{r\to 0+} \frac{\int_{C_r} f(x+t)\,dt}{|C_r|} = f(x) \tag{2.2}$$

for almost all $x \in [0,1]^2$, so that $|f| \le M(f)$ almost everywhere and the set L_f of *Lebesgue points* $x \in [0,1]^2$ of f where

$$\lim_{r\to 0+} \frac{\int_{C_r} |f(x+t)-f(x)|\,dt}{|C_r|} = 0$$

has full measure in $[0,1]^2$.

Let $\phi :]-1,1[\to [0,\infty[$ be a continuous function with compact support and $\int_{-1}^1 \phi(t)\,dt = 1$. For the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\varphi(x,y) = \phi(x)\phi(y)$, for $x, y \in]-1,1[$ and zero outside $]-1,1[^2$, we set $\varphi_\epsilon(x) = \epsilon^{-2}\varphi(x/\epsilon)$, $x \in \mathbb{R}^2$, $\epsilon > 0$. Then a variant of Lebesgue's differentiation theorem for an integrable function f shows that $\varphi_\epsilon * f \to f$ in $L^p([0,1]^2)$ for $1 \le p < \infty$ and almost everywhere as $\epsilon \to 0+$ [14, Corollary 2.1.17].

We are interested in the class of bounded linear operators $T_k : L^2([0,1]) \to L^2([0,1])$ with a distinguished kernel $k : [0,1]^2 \to \mathbb{C}$ for which $|k|$ also defines a bounded linear operator $T_{|k|} : L^2([0,1]) \to L^2([0,1])$ (absolute integral operators) and the intersection $L_k \cap \mathrm{diag}$ of the Lebesgue set L_k of k with the *diagonal* $\mathrm{diag} = \{(x,x) : x \in [0,1]\}$ has full linear measure. Because constant functions belong to $L^2([0,1])$, the kernel k necessarily belongs to $L^1([0,1]^2))$, so we first look at a subspace of $L^1([0,1]^2)$ consisting of functions f for which $L_f \cap \mathrm{diag}$ has full linear measure.

3. The Banach function space of traceable functions

Let $(\Sigma, \mathcal{B}, \mu)$ be a σ-finite measure space. The space of all μ-equivalence classes of scalar functions measurable with respect to $\mathcal{B}$ is denoted by $L^0(\mu)$. It is equipped with the topology of convergence in μ-measure over sets of finite measure and vector operations pointwise μ-almost everywhere. Any Banach space X that is a subspace of $L^0(\mu)$ with the properties that

(i) X is an order ideal of $L^0(\mu)$, that is, if $g \in X$, $f \in L^0(\mu)$ and $|f| \le |g|$ μ-a.e., then $f \in X$, and
(ii) if $f, g \in X$ and $|f| \le |g|$ μ-a.e., then $\|f\|_X \le \|g\|_X$,

is called a *Banach function space* (based on $(\Sigma, \mathcal{B}, \mu)$) [22, §2.6]. The set of $f \in X$ with $f \ge 0$ μ-a.e. is writen as X_+.

The map $J : [0,1] \to [0,1]^2$ defined by $J(x) = (x,x)$, $x \in [0,1]$, maps $[0,1]$ homeomorphically onto diag. For $f \in L^1([0,1]^2)$, the extended real number $\rho(f) \in [0,\infty]$ is defined by $\rho(f) = \|f\|_1 + \int_0^1 M(f)\circ J(x)\,dx$.

Proposition 3.1. *The space $L^1(\rho) = \{f \in L^1([0,1]^2) : \rho(f) < \infty\}$ with norm ρ is a Banach function space continuously embedded in $L^1([0,1]^2)$.*

Proof. Properties (i) and (ii) follow from the observation that $M(f) \le M(g)$ everywhere if $|f| \le |g|$ almost everywhere on $[0,1]^2$. According to [22, Proposition 2.6.2], it is enough to prove that $L^1(\rho)$ has the Riesz-Fischer property. Suppose that $f_j \geq 0$ almost everywhere for $j = 1, 2, \dots$ and $\sum_{j=1}^{\infty} \rho(f_j) < \infty$. Then monotone convergence ensures that $\sum_{j=1}^{\infty} f_j$ converges almost everywhere in $[0,1]^2$ and in $L^1([0,1]^2)$ to a nonnegative integrable function f and $M(f) \leq \sum_{j=1}^{\infty} M(f_j)$ everywhere on $[0,1]^2$ and so $\rho(f) \leq \sum_{j=1}^{\infty} \rho(f_j)$. The inequality $\|f\|_1 \leq \rho(f)$ ensures that the inclusion of $L^1(\rho)$ in $L^1([0,1]^2)$ is continuous. □

Suppose that $f \in L^1(\rho)$. By [14, Corollary 2.1.12], there exists $C > 0$ independent of f such that $\sup_{\epsilon>0} |(\varphi_\epsilon * f)(x)| \leq CM(f)(x)$ for every $x \in [0,1]^2$, so if we let $\tilde{f} = \limsup_{\epsilon \to 0+} (\varphi_\epsilon * f)$ on $[0,1]^2$, then $\tilde{f} = f$ almost everywhere on $[0,1]^2$ by [14, Corollary 2.1.17], $\tilde{f} \circ J \leq CM(f) \circ J$ and

$$\int_0^1 |\tilde{f}(x,x)|\, dx \leq C \int_0^1 M(f) \circ J(x)\, dx < \infty,$$

so in this sense, elements of $L^1(\rho)$ possess an integrable trace on diag $\subset [0,1]^2$. However, the mapping $f \longmapsto \int_0^1 \tilde{f}(x,x)\, dx$, $f \in L^1(\rho)$, may only be *sublinear*, so next we examine a subspace for which the lim sup can be replaced by a genuine limit almost everywhere on diag.

If u and v are two real valued functions defined on $[0,1]$, the tensor product $u \otimes v : [0,1]^2 \to \mathbb{R}$ of u and v is defined by $(u \otimes v)(x,y) = u(x)v(y)$, $x \in [0,1]$. A similar notation is used for the equivalence classes of functions so that $[u \otimes v] \circ J := [u.v]$. Then $L^\infty([0,1]) \otimes L^\infty([0,1])$ denotes the linear space of all finite linear combinations of elements $u \otimes v$ with $u, v \in L^\infty([0,1])$. Each element f of the finite tensor product $L^\infty([0,1]) \otimes L^\infty([0,1])$ is essentially bounded on $[0,1]^2$ and $M(f) \leq \|f\|_\infty$, so $f \in L^1(\rho)$ and $f \circ J \in L^\infty([0,1])$. Let $L^\infty([0,1]) \widehat{\otimes}_\rho L^\infty([0,1])$ denote the norm closure of the subspace $L^\infty([0,1]) \otimes L^\infty([0,1])$ in the Banach function space $L^1(\rho)$.

Proposition 3.2. *Let $f \in L^1([0,1]^2)$. Then $f \in L^\infty([0,1]) \widehat{\otimes}_\rho L^\infty([0,1])$ if and only if $\varphi_\epsilon * f \to f$ in $L^1(\rho)$ as $\epsilon \to 0+$. If $f \in L^\infty([0,1]) \widehat{\otimes}_\rho L^\infty([0,1])$ then $(\varphi_\epsilon * f) \circ J$ converges a.e. on $[0,1]$ and in $L^1([0,1])$ as $\epsilon \to 0+$.*

Proof. By an application of the Cauchy-Schwarz inequality and the L^2-bound for the Hardy-Littlewood maximal operator [14, Theorem 2.1.6], there exists $C > 0$ such that if $u, v \in L^2([0,1])$, then

$$\begin{aligned} \int_0^1 M(u \otimes v)(x,x)\, dx &\leq \int_0^1 M(u)(x) M(v)(x)\, dx \\ &\leq C \|u\|_2 \|v\|_2. \end{aligned} \tag{3.1}$$

Here $M(u)$ and $M(v)$ are the one dimensional maximal functions of u and v defined as in formula (2.1).

Suppose first that $f = u \otimes v$ for $u, v \in L^\infty([0,1])$. Then

$$\varphi_\epsilon * f = (\phi_\epsilon * u) \otimes (\phi_\epsilon * v)$$

because $\varphi = \phi \otimes \phi$ and so

$$\begin{aligned}\int_0^1 M(\varphi_\epsilon * f - f)(x,x)\,dx &\le C\int_0^1 M(\phi_\epsilon * u - u)(x)M(v)(x)\,dx\\ &\quad + \int_0^1 M(\phi_\epsilon * v - v)(x)|u|(x)\,dx\\ &\le C'(\|\phi_\epsilon * u - u\|_2\|v\|_2 + \|\phi_\epsilon * v - v\|_2\|u\|_2) \to 0\end{aligned}$$

as $\epsilon \to 0+$. Consequently, $\varphi_\epsilon * f \to f$ in $L^1(\rho)$ as $\epsilon \to 0+$ when f is a linear combination of products of functions belonging to $L^\infty([0,1])$. There exists $C > 0$ such that $\|\varphi_\epsilon * f\|_1 \le C\|f\|_1$ for every $f \in L^1([0,1]^2)$ and

$$\int_0^1 M(\varphi_\epsilon * f)(x,x)\,dx \le C\int_0^1 M(f)(x,x)\,dx, \quad \epsilon > 0. \tag{3.2}$$

To check the inequality (3.2), suppose that $\psi = \pi^{-1}\chi_{D_1}$ for the unit disk D_1 centred at zero in $\mathbb{R}^2$ and let $\tilde{\varphi}$ be the least decreasing radial majorant of φ. Because φ is continuous with compact support, $\tilde{\varphi}$ is integrable on $\mathbb{R}^2$. Then $\tilde{\varphi}_\epsilon * \psi_\delta$ is a radial function for which

$$\begin{aligned}2\pi\int_0^\infty r(\tilde{\varphi}_\epsilon * \psi_\delta)(re_1)\,dr &= \|\tilde{\varphi}_\epsilon * \psi_\delta\|_{L^1(\mathbb{R}^2)}\\ &= \|\tilde{\varphi}_\epsilon\|_{L^1(\mathbb{R}^2)}\|\psi_\delta\|_{L^1(\mathbb{R}^2)}\\ &= \|\tilde{\varphi}\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

As in the proof of [14, Theorem 2.1.10], there exists $C' > 0$ such that $\sup_{\epsilon,\delta>0}\tilde{\varphi}_\epsilon * \psi_\delta * |f| \le C'M(f)$. Because the maximal function (2.1) is equivalent to the maximal function for centred disks, there exists $C \ge 1$ such that

$$M(\varphi_\epsilon * f) \le M(\tilde{\varphi}_\epsilon * f) \le CM(f)$$

from which the inequality (3.2) follows.

Consequently, the linear map $f \longmapsto \varphi_\epsilon * f$, $f \in L^1(\rho)$, is continuous on $L^1(\rho)$ for each $\epsilon > 0$ so that if $f \in L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$ then $\varphi_\epsilon * f \to f$ in $L^1(\rho)$ as $\epsilon \to 0+$. Because $\varphi_\epsilon * f \in C([0,1]^2)$ and $C([0,1])\otimes C[0,1])$ is dense in $C([0,1]^2)$ in the uniform norm, it follows that $\varphi_\epsilon * f \in L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$ for each $\epsilon > 0$, and the limit of $\varphi_\epsilon * f$ in $L^1(\rho)$ as $\epsilon \to 0+$ also belongs to $L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$.

Let $T_* f = \sup_{\epsilon>0}|\varphi_\epsilon * f| \circ J$ for $f \in L^1(\rho)$. Then $T_* : L^1(\rho) \to L^1([0,1])$ is uniformly continuous. An argument similar to the proof of [14, Theorem 2.1.14] shows that $(\varphi_\epsilon * f) \circ J$ converges almost everywhere and in $L^1([0,1])$ as $\epsilon \to 0+$ for each $f \in L^1(\rho)$. □

Let $f \in L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$ and set $\tilde{f} = \lim_{\epsilon\to 0+}\varphi_\epsilon * f$ wherever the limit exists in $[0,1]^2$ and zero elsewhere. Writing $f^\#$ for the corresponding function with ϕ replaced by $\chi_{[-\frac{1}{2},\frac{1}{2}]}$, it follows from equation (2.2) that $f^\# = \tilde{f}$ almost everywhere on $[0,1]^2$ and $f^\# \circ J = \tilde{f} \circ J$ almost everywhere on $[0,1]$, because the last equality certainly holds when f belongs to the dense subspace $L^\infty([0,1]) \otimes L^\infty([0,1])$. In particular, $\tilde{f} \circ J \in L^1([0,1])$ and the integral $\int_B \tilde{f} \circ J(x)\,dx$, $B \in \mathcal{B}([0,1])$, does not depend on the choice of the function φ.

Example 3.3. For a continuous function f on $[0,1]^2$ equal to zero on $\mathbb{R}^2 \setminus [0,1]^2$, the continuous functions $\varphi_\epsilon * f$ converge uniformly to f on compact subsets of $]0,1[^2$ [14, Theorem 1.2.19 (2)], so that

$$f \in L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$$

and $\tilde{f} = f$. Hence, $C([0,1]^2)$ and $C([0,1]) \otimes C([0,1])$ are dense in $L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$.

Functions belonging to $W^{1,1}(\mathbb{R}^2)$ or the space $L^{\alpha,p}(\mathbb{R}^2)$ of Bessel potentials on $\mathbb{R}^2$ also admit a trace on $\mathrm{diag}(\mathbb{R}^2)$ if $p, \alpha p > 1$, see [1, Section 6.2].

There exists a continuous periodic function $\phi : \mathbb{R} \to \mathbb{C}$ with period one such that

$$\sum_{n\in\mathbb{Z}} |\hat{\phi}(n)|^p = \infty$$

for all $p < 2$ [Carlemann, 1918]. If $k(x,y) = \phi(x-y)$, then k is a continuous kernel, $M(k) \circ J \le \|\phi\|_\infty$ and $k(x,x) = \phi(0)$ for all $x \in [0,1]$ and so $\int_0^1 k(x,x)\,dx = \phi(0)$, although the Hilbert-Schmidt operator T_k is not a trace class operator. Because $k \le \|\phi\|_\infty$ and a constant function is the kernel of a finite rank operator, the trace class operators do not form a *lattice ideal* in the Banach lattice of Hilbert-Schmidt operators, despite being an operator ideal in $\mathcal{L}(L^2([0,1]))$.

Example 3.4. The kernel $\chi_{\{y<x\}}$ of the Volterra integral operator in Example 1.1 belongs to the Banach space $L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$ and the same holds true for the function $\chi_{\{y\le x\}}$ which differs from $\chi_{\{y<x\}}$ on diag, a set of measure zero in $[0,1]^2$.

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a nonsingular linear transformation, then there exists $c_T > 0$ such that $\rho(f\circ T) \le c_T\rho(f)$ if both f and $f\circ T$ are supported by $[0,1]^2$ because the collection $\{TC_r : r > 0\}$ is itself a regular family of sets whose associated maximal function is equivalent to the one defined for cubes by formula (2.1). Furthermore, if $g \in L^\infty([0,1])\otimes L^\infty([0,1])$, then $(\varphi \circ T)_\epsilon * g \to g$ in $L^1(\rho)$ as $\epsilon \to 0+$, hence $\varphi_\epsilon * (g \circ T^{-1})$ converges to $g \circ T^{-1}$ in $L^1(\rho)$ as $\epsilon \to 0+$ as well. Taking g to be the characteristic functions of squares and T to be rotation through $\pi/4$ gives $\chi_{\{y<x\}} \in L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$.

Proposition 3.5. *Every element of $L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$ has a representative function $f : [0,1]^2 \to \mathbb{R}$ for which there exist numbers $c_j \in \mathbb{R}$ and Borel subsets A_j, B_j of $[0,1]$, $j = 1,2,\dots$, such that*

$$\sum_{j=1}^\infty |c_j|(|A_j|.|B_j| + |A_j \cap B_j|) < \infty$$

and $f(x) = \sum_{j=1}^\infty c_j\chi_{A_j\times B_j}(x)$ for every $x \in [0,1]^2$ such that the sum $\sum_{j=1}^\infty |c_j|\chi_{A_j\times B_j}(x)$ is finite. In particular, $f \circ J = \sum_{j=1}^\infty c_j\chi_{A_j\cap B_j} = \tilde{f}\circ J$ almost everywhere.

Proof. Let $\mu = \lambda \otimes \lambda + \lambda \circ J^{-1}$. If $[f_0] \in L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$, then let $f = f_0$ on $[0,1]^2 \setminus \mathrm{diag}$ and set $f \circ J = \lim_{\epsilon\to 0+}(\varphi_\epsilon * f_0) \circ J$ wherever the limit exists and zero otherwise. By [14, Corollary 2.1.12], there exists $C > 0$ such that $\sup_{\epsilon>0} |(\varphi_\epsilon * f_0)(x)| \le CM(f_0)(x)$ for every $x \in [0,1]^2$.

Because $\rho([f_0]) < \infty$, f is μ-integrable. The statement now follows from [18, Proposition 2.13]. A similar statement is proved in [15, Lemma 3.2]. □

The *projective tensor product* $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ is the set of all sums

$$k = \sum_{j=1}^{\infty} \phi_j \otimes \psi_j \text{ a.e., with } \sum_{j=1}^{\infty} \|\phi_j\|_2 \|\psi_j\|_2 < \infty. \tag{3.3}$$

The norm of $k \in L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ is given by

$$\|k\|_\pi = \inf \left\{ \sum_{j=1}^{\infty} \|\phi_j\|_2 \|\psi_j\|_2 \right\}$$

where the infimum is taken over all sums for which the representation (3.3) holds. The Banach space $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ is actually the completion of the algebraic tensor product $L^2([0,1]) \otimes L^2([0,1])$ with respect to the projective tensor product norm [28, Section 6.1]. The estimate (3.1) establishes the following result.

Proposition 3.6. *The projective tensor product $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ embeds onto a proper dense subspace of $L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$.*

There is a one-to-one correspondence between the space of trace class operators acting on $L^2([0,1])$ and $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$, so that the trace class operator T_k has an integral kernel $k \in L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ given, for example, by formula (1.2). If the integral kernel k defined by equation (3.3) has the property that

$$k(x,y) = \sum_{j=1}^{\infty} \phi_j(x)\psi_j(y)$$

for all $x, y \in \Sigma$ such that the sum $\sum_{j=1}^{\infty} |\phi_j(x)\psi_j(y)|$ is finite, then k is the integral kernel of a trace class operator T_k and the equality

$$\operatorname{trace}(T_k) = \sum_{j=1}^{\infty} \int_0^1 \phi_j(x)\psi_j(x)\,dx = \int_0^1 k(x,x)\,dx$$

holds. The representation of Proposition 3.5 for elements of $L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$ may be viewed as a substitute for the representation (3.3) of an element of the projective tensor product $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$.

Another way to view the trace $\operatorname{trace}(T_k)$ of a trace class operator $T_k : L^2([0,1]) \to L^2([0,1])$ with an integral kernel k is as a type of *bilinear integral* with respect to the $L^2([0,1])$-valued vector measure $m : B \longmapsto \chi_B$, $B \in \mathcal{B}([0,1])$. For example, if $k = \sum_{j=1}^{n} \chi_{B_j} \otimes f_j$ for Borel subsets B_j of $[0,1]$ and $f_j \in L^2([0,1])$, $j = 1, \dots, n$ and $\Phi_k : [0,1] \to L^2([0,1])$ is the $L^2([0,1])$-valued simple function defined by $\Phi_k(x) = \sum_{j=1}^{n} \chi_{B_j}(x).f_j$, $x \in [0,1]$, then

$$\int_B \langle \Phi_k, dm \rangle = \sum_{j=1}^{n} \int_{B \cap B_j} f_j(x)\,dx = \int_B k(x,x)\,dx$$

and $\int_B \Phi_k \otimes dm = (\chi_B \otimes 1).k \in L^2([0,1]) \otimes L^2([0,1])$ for $B \in \mathcal{B}([0,1])$. The bilinear integrals $\int_B \langle \Phi_k, dm \rangle$ and $\int_B \Phi_k \otimes dm$ also makes sense for

$k \in L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ where T_k is a trace class operator and

$$\text{trace}(T_k) = \int_0^1 \langle \Phi_k, dm \rangle \tag{3.4}$$

independently of the integral kernel k representing the operator T_k [17].

On the other hand, if $k \in L^1([0,1]^2)$, then by Fubini's Theorem, the function $\Phi_k(x) = f(x,\cdot)$ has values in $L^1([0,1])$ for almost all $x \in [0,1]$ and $\int_0^1 \Phi_k \otimes dm = k$ is an element of $L^1([0,1])\widehat{\otimes}_\pi L^1([0,1]) \equiv L^1([0,1]^2)$ [28, 6.5]. Furthermore, if $\int_0^1 \Phi_k \otimes dm$ belongs to the subspace $L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$ of $L^1([0,1]^2)$, then $\int_B \langle \Phi_k, dm \rangle$ is defined for each Borel set B contained in $[0,1]$ by appealing to Proposition 3.5. As Example 1.1 shows, now k need not be the integral kernel of a trace class operator acting on $L^2([0,1])$.

As a matter of notation, if $T : L^2([0,1]) \to L^2([0,1])$ has an integral kernel k belonging to $L^\infty([0,1])\widehat{\otimes}_\rho L^\infty([0,1])$, then the integral $\int_0^1 \langle \Phi_k, dm \rangle$ is independent of any integral kernel k representing T, so it makes sense to write $\int_0^1 \langle T, dm \rangle$ for $\int_0^1 \langle \Phi_k, dm \rangle$.

4. Traceable operators on Banach function spaces

It is clear that the ideas of the preceding section are concerned mainly with the order properties of the Banach function space $L^2([0,1])$, although the smoothing operators $k \longmapsto \varphi_\epsilon * k$, $k \in L^1([0,1]^2)$, $\epsilon > 0$, depend on the group structure of $\mathbb{R}^2$. For a σ-finite measure space $(\Sigma, \mathcal{B}, \mu)$, the same result is achieved by taking the maximal function with respect to a suitable *filtration* $\langle \mathcal{E}_n \rangle_{n \in \mathbb{N}}$ for which $\mathcal{B} = \bigvee_n \mathcal{E}_n$. The filtration determined by dyadic partitions of $\mathbb{R}$ localised to $[0,1]$ gives the results of Section 3.

Let X be a complex Banach function space based on the σ-finite measure space $(\Sigma, \mathcal{B}, \mu)$, as defined at the beginning of Section 3. A continuous linear operator $T : X \to X$ is called *positive* if $T : X_+ \to X_+$. The collection of all positive continuous linear operators on X is written as $\mathcal{L}_+(X)$. If the real and imaginary parts of a continuous linear operator $T : X \to X$ can be written as the difference of two positive operators, it is said to be *regular*. The *modulus* $|T|$ of a regular operator T is defined by

$$|T|f = \sup_{|g| \le f} |Tg|, \quad f \in X_+.$$

The collection of all regular operators is written as $\mathcal{L}_r(X)$ and it is given the norm $T \longmapsto \||T|\|$, $T \in \mathcal{L}_r(X)$ under which it becomes a Banach lattice [22, Proposition 1.3.6]. In the case that $X = L^2(\mu)$, the same notation for the hermitian positive operator $(TT^*)^{\frac{1}{2}}$ is never used in the present work in order to avoid possible confusion.

A continuous linear operator $T : X \to X$ has an integral kernel k if $k : \Sigma \times \Sigma \to \mathbb{C}$ is a Borel measurable function such that $T = T_k$ for the operator given by

$$(T_k f)(x) = \int_\Sigma k(x,y) f(y)\, d\mu(y), \quad \mu\text{-almost all } x \in \Sigma, \tag{4.1}$$

in the sense that, for each $f \in X$, we have $\int_\Sigma |k(x,y)f(y)|\, d\mu(y) < \infty$ for μ-almost all $x \in \Sigma$ and the map $x \longmapsto \int_\Sigma k(x,y)f(y)\, d\mu(y)$ is an element of X. If $T_k \ge 0$, then $k \ge 0$ $(\mu \otimes \mu)$-a.e. on $\Sigma \times \Sigma$ [22, Theorem 3.3.5].

A continuous linear operator T is an *absolute integral operator* if it has an integral kernel k for which $T_{|k|}$ is a bounded linear operator on $L^2(\mu)$. Then $|T_k| = T_{|k|}$ [22, Theorem 3.3.5]. Then k is $(\mu \otimes \mu)$-integrable on any product set $A \times B$ with finite measure. The collection of all absolute integral operators is a lattice ideal in $\mathcal{L}_r(X)$ [22, Theorem 3.3.6].

Suppose that $T \in \mathcal{L}(X)$ has an integral kernel $k = \sum_{j=1}^n f_j \otimes \chi_{A_j}$ that is an X-valued simple function with $\mu(A_j) < \infty$. Then it is natural to view

$$\int_\Sigma \langle T, dm\rangle := \sum_{j=1}^n \int_{A_j} f_j \, d\mu = \int_\Sigma k(x,x)\, d\mu(x)$$

as a bilinear integral. Our aim is to extend the integral to a wider class of absolute integral operators acting on the Banach function space X.

Suppose that for each $n = 1, 2 \dots$, the collection $\mathcal{P}_n$ of sets belonging to the σ-algebra $\mathcal{B}$ is a countable partition of Σ into sets with finite measure such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ for each $n = 1, 2, \dots$, that is, every element of $\mathcal{P}_n$ is the union of elements of $\mathcal{P}_{n+1}$. Then the σ-algebra $\mathcal{E}_n$ generated by the partition $\mathcal{P}_n$ of Σ is the collection of all unions of elements of $\mathcal{P}_n$, so that $\mathcal{E}_n \subset \mathcal{E}_{n+1}$ for $n = 1, 2, \dots$. Suppose that $\mathcal{B} = \bigvee_n \mathcal{E}_n$, the smallest σ-algebra containing all $\mathcal{E}_n$, $n = 1, 2, \dots$. It follows that $\mathcal{B}$ is countably generated. The filtration $\langle \mathcal{E}_n\rangle_{n\in\mathbb{N}}$ is denoted by $\mathcal{E}$.

Suppose that $k \geq 0$ is a Borel measurable function defined on $\Sigma \times \Sigma$ that is integrable on every set $U \times V$ for $U, V \in \mathcal{P}_1$. For each $x \in \Sigma$, the set $U_n(x)$ is the unique element of the partition $\mathcal{P}_n$ containing x. For each $n = 1, 2, \dots$, the conditional expectation $k_n = \mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n)$ can be represented for μ-almost all $x, y \in \Sigma$ as

$$\begin{aligned}\mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n)(x,y) &= \frac{1}{\mu(U_n(x))\mu(U_n(y))}\int_{U_n(x)}\int_{U_n(y)} k(s,t)\, d\mu(s) d\mu(t)\\ &= \sum_{U,V\in\mathcal{P}_n} \frac{\int_{U\times V} k \, d(\mu\otimes\mu)}{\mu(U)\mu(V)} \chi_{U\times V}(x,y).\end{aligned}$$

The point here is that the formula above defines a *distinguished* element of the conditional expectation $\mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n)$ that possesses a *trace* on the diagonal of $\Sigma \times \Sigma$, that is, $B \longmapsto \mathbb{E}(\chi_B|\mathcal{E}_n \otimes \mathcal{E}_n)(x,y)$, $B \in \mathcal{B}$, $x, y \in \Sigma$, is a *regular conditional measure* [3, Definition 10.4.1].

Let $\mathcal{N}$ be the set of all $x \in \Sigma$ for which there exists $n = 1, 2, \dots$ such that $\mu(U_n(x)) = 0$. Then $\mu(U_m(x)) = 0$ for all $m > n$ because $\mathcal{P}_m$ is a refinement of $\mathcal{P}_n$ if $m > n$. Moreover $\mathcal{N}$ is μ-null because

$$\mathcal{N} \subset \bigcup_{n=1}^\infty \bigcup \{U \in \mathcal{P}_n : \mu(U) = 0\ \}.$$

If $0 \leq k_1 \leq k_2$ $(\mu \otimes \mu)$-a.e., then

$$\mathbb{E}(k_1|\mathcal{E}_n \otimes \mathcal{E}_n)(x,y) \leq \mathbb{E}(k_2|\mathcal{E}_n \otimes \mathcal{E}_n)(x,y), \quad n = 1, 2, \dots,$$

for all $(x, y) \in \mathcal{N}^c \times \mathcal{N}^c$. In particular,

$$\mathbb{E}(k_1|\mathcal{E}_n \otimes \mathcal{E}_n)(x,x) \leq \mathbb{E}(k_2|\mathcal{E}_n \otimes \mathcal{E}_n)(x,x), \quad n = 1, 2, \dots,$$

for all $x \in \mathcal{N}^c$ and the representation

$$\mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n)(x,x) = \sum_{U \in \mathcal{P}_n} \frac{\int_{U\times U} k\, d(\mu\otimes\mu)}{\mu(U)^2}\chi_U(x).$$

on the diagonal is valid μ-almost everywhere. Although $\mathrm{diag}(\Sigma \times \Sigma) = \{(x,x) : x \in \Sigma\ \}$ may be a set of $(\mu\otimes\mu)$-measure zero, the application of the conditional expectation operators $k \longmapsto \mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n)$, $n = 1, 2, \dots$, has the effect of regularising k. By an appeal to the Martingale Convergence Theorem [3, Theorem 10.2.3], k_n converges $(\mu\otimes\mu)$-a.e. to k as $n \to \infty$.

For any Borel measurable function $f : \Sigma \times \Sigma \to \mathbb{C}$ that is integrable on every set $U \times V$ for $U, V \in \mathcal{P}_1$, let

$$M_{\mathcal{E}}(f)(x,y) = \sup_{n\in\mathbb{N}} \mathbb{E}(|f||\mathcal{E}_n \otimes \mathcal{E}_n)(x,y), \quad x, y \in \Sigma, \tag{4.2}$$

be the maximal function associated with the martingale $\langle \mathbb{E}(|f||\mathcal{E}_n \otimes \mathcal{E}_n)\rangle_{n\in\mathbb{N}}$. The maximal function associated with dyadic partitions $\mathcal{P}_n$ of $[0, 1)$ into intervals $[(k-1)/2^n, k/2^n)$, $k = 1, 2, \dots, 2^n$ of length 2^{-n} is equivalent to the maximal function considered in Section 3 [14, Exercise 2.1.12]

Let $\mathfrak{C}_1(\mathcal{E}, X)$ denote the collection of absolute integral operators $T_k : X \to X$ whose integral kernels k have the property that $\mathbb{E}(|f||\mathcal{E}_1 \otimes \mathcal{E}_1)$ takes finite values and

$$\int_\Sigma M_{\mathcal{E}}(k)(x,x)\, d\mu(x) < \infty.$$

Where convenient, if k is the integral kernel of T, the maximal function $M_{\mathcal{E}}(k)$ is also written as $M_{\mathcal{E}}(T)$. The map $J : \Sigma \to \Sigma \times \Sigma$ defined by $J(x) = (x,x)$ for $x \in \Sigma$ maps Σ bijectively onto $\mathrm{diag}(\Sigma \times \Sigma)$.

Theorem 4.1. *The space $\mathfrak{C}_1(\mathcal{E}, X)$ is a lattice ideal in $\mathcal{L}_r(X)$, that is, if $S, T \in \mathcal{L}_r(X)$, $|S| \le |T|$ and $T \in \mathfrak{C}_1(\mathcal{E}, X)$, then $S \in \mathfrak{C}_1(\mathcal{E}, X)$. Moreover, $\mathfrak{C}_1(\mathcal{E}, X)$ is a Dedekind complete Banach lattice with the norm $\|\cdot\|_{\mathfrak{C}_1(\mathcal{E},X)}$ defined by*

$$\|T\|_{\mathfrak{C}_1(\mathcal{E},X)} = |||T||| + \int_\Sigma M_{\mathcal{E}}(T) \circ J\, d\mu, \qquad T \in \mathfrak{C}_1(\mathcal{E}, X). \tag{4.3}$$

Proof. If $S, T \in \mathcal{L}_r(X)$, $|S| \le |T|$ and $T \in \mathfrak{C}_1(\mathcal{E}, X)$, then S is an absolute integral operator by [22, Theorem 3.3.6]. If k_1 is the integral kernel of S and k_2 is the integral kernel of T, then by [22, Theorem 3.3.5], the inequality $|k_1| \le |k_2|$ holds $(\mu\otimes\mu)$-a.e. . Then $|M_{\mathcal{E}}(k_1)(x,x)| \le |M_{\mathcal{E}}(k_2)(x,x)|$ for all $x \in \Sigma$, so that

$$\int_\Sigma M_{\mathcal{E}}(k_1) \circ J\, d\mu \le \int_\Sigma M_{\mathcal{E}}(k_2) \circ J\, d\mu < \infty.$$

Hence $S \in \mathfrak{C}_1(\mathcal{E}, X)$ and $\|S\|_{\mathfrak{C}_1(\mathcal{E},X)} \le \|T\|_{\mathfrak{C}_1(\mathcal{E},X)}$.

To show that $\mathfrak{C}_1(\mathcal{E}, X)$ is complete in its norm, suppose that

$$\sum_{j=1}^\infty \left(|||T_j||| + \int_\Sigma M_{\mathcal{E}}(T_j) \circ J\, d\mu \right) < \infty$$

for $T_j \in \mathfrak{C}_1(\mathcal{E}, X)$. Then $T = \sum_{j=1}^\infty T_j$ in the space of regular operators on X. The inequality $|T| \le \sum_{j=1}^\infty |T_j|$ ensures that T is an absolute integral operator with kernel k by [22, Theorem 3.3.6] and $|k| \le \sum_{j=1}^{\infty} |k_j|$ $(\mu \otimes \mu)$-a.e..

Suppose first that X is a real Banach function space. Each positive part T_j^+ of T_j, $j = 1, 2, \dots$ has an integral kernel k_j^+. By monotone convergence, there exists a set of full μ-measure on which

$$\mathbb{E}(k^+|\mathcal{E}_n \otimes \mathcal{E}_n)(x,x) \le \sum_{j=1}^{\infty} \mathbb{E}(k_j^+|\mathcal{E}_n \otimes \mathcal{E}_n)(x,x)$$

for each $n = 1, 2, \dots$. Taking the supremum and applying the monotone convergence theorem pointwise and under the sum shows that

$$M_{\mathcal{E}}(k^+)(x,x) \le \sum_{j=1}^{\infty} M_{\mathcal{E}}(k_j^+)(x,x)$$

for μ-almost all $x \in \Sigma$ and $\int_\Sigma M_{\mathcal{E}}(k^+) \circ J \, d\mu < \infty$. Applying the same argument to T^- and then the real and imaginary parts of T ensures that $T \in \mathfrak{C}_1(\mathcal{E}, X)$ and

$$|||T||| + \int_\Sigma M_{\mathcal{E}}(T) \circ J \, d\mu \le \sum_{j=1}^{\infty} \left(|||T_j||| + \int_\Sigma M_{\mathcal{E}}(T_j) \circ J \, d\mu \right).$$

Dedekind completeness is inherited from $\mathcal{L}_r(X)$ [22, Theorem 1.3.2] and $L^1(\mu)$ [22, Example v) p. 9]. □

As in Section 3, we may define $\tilde{k} = \limsup_{n\to\infty} \mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n)$ for the integral kernel k of an operator $T \in \mathfrak{C}_1(\mathcal{E}, X)$ so that $\int_\Sigma \tilde{k} \circ J \, d\mu \le \|T\|_{\mathfrak{C}_1(\mathcal{E},X)}$. The same function $\tilde{k} : \Sigma \times \Sigma \to \overline{\mathbb{R}}$ is obtained for any integral kernel k associated with the operator T. The integral $\int_\Sigma \tilde{k} \circ J \, d\mu$ is denoted as $\int_\Sigma \langle T, dm \rangle$, which is the notation used in [15].

Proposition 4.2 ([5, Theorem 4.2])**.** *Let $T : L^2(\mu) \to L^2(\mu)$ be an absolute integral operator whose integral kernel is square integrable on any set of finite $(\mu \otimes \mu)$-measure. If $(Tu, u) \ge 0$ for all $u \in L^2(\mu)$, then T is trace class if and only if $T \in \mathfrak{C}_1(\mathcal{E}, L^2(\mu))$, and in this case* $\mathrm{trace}(T) = \int_\Sigma \langle T, dm \rangle$.

The statement above is a slight generalisation of Brislawn's result [5, Theorem 4.2] by localisation on sets of finite measure and the introduction of the filtration $\mathcal{E}$ independent of any topology on Σ. Extensions of Brislawn's results to nuclear operators between Banach spaces appear in the papers [7, 8, 9, 10, 11, 12].

As the case of the Volterra integral operator considered in Example 1.1 shows, the assumption that the operator is Hilbert space positive in Proposition 4.2 cannot be omitted.

We next see when the limsup can be replaced by a genuine limit, as is the case for trace class operators on $L^2(\mu)$. In order that an integrable X-valued simple function defines the integral kernel of a finite rank operator $T : X \to X$, we assume that $\chi_A \in X$ if $\mu(A) < \infty$ and every function $f \in X$ is integrable on any set of finite measure. What is actually required is that both X and the Köthe dual

$$X^\times = \{f \in L^0(\mu) : \int_\Sigma fg \, d\mu < \infty \text{ for all } g \in X\}$$

of X be order dense in $L^0(\mu)$, see [22, Theorem 3.3.7].

The closure in $\mathfrak{C}_1(\mathcal{E}, X)$ of the collection of all finite rank operators $T : X \to X$ with integral kernels of the form $k = \sum_{j=1}^n f_j \otimes \chi_{B_j}$, for $f_j \in X$, $B_j \in \mathcal{B}$, with $\mu(B_j) < \infty$, $j = 1, \dots, n$ and $n = 1, 2, \dots$, is denoted by $X \widehat{\otimes}_{\mathcal{E}} X^\times$.

The following statement is the martingale analogue of Proposition 3.2, proved along the same lines.

Proposition 4.3. *Suppose that k is the integral kernel of the operator operator $T \in \mathfrak{C}_1(\mathcal{E}, X)$ and T_n has the integral kernel $\mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n)$ for $n = 1, 2, \dots$. Then $T \in X \widehat{\otimes}_{\mathcal{E}} X^\times$ if and only if $T_n \to T$ in $\mathfrak{C}_1(\mathcal{E}, X)$ as $n \to \infty$. If $T \in X \widehat{\otimes}_{\mathcal{E}} X^\times$, then $\mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n) \circ J$ converges a.e. on Σ and in $L^1(\mu)$ as $n \to \infty$.*

For $T \in X \widehat{\otimes}_{\mathcal{E}} X^\times$, we have $\int_\Sigma \langle T, dm \rangle = \lim_{n\to\infty} \int_\Sigma \mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n) \circ J \, d\mu$.

4.1. **Dependence on the filtration $\mathcal{E}$.** The filtration $\mathcal{E} = \langle \mathcal{E}_n \rangle_{n\in\mathbb{N}}$ is assumed above to be constructed from an increasing sequence $\langle \mathcal{P}_n \rangle_{n\in\mathbb{N}}$ of countable partitions of Σ into measurable sets. Such a filtration is constructed in [5] on any second countable space. The essential property of the filtration $\mathcal{E}$ is that there exists a natural regular conditional measure $B \longmapsto \mathbb{E}(\chi_B|\mathcal{E}_n)$, $B \in \mathcal{B}$, for each $n = 1, 2, \dots$. Of course, the assumption that the choice of partitions is possible could be avoided simply by choosing a family of regular conditional measures associated with some filtration.

The existence of the filtration $\mathcal{E}$ constructed from partitions imposes conditions on the measure space $(\Sigma, \mathcal{B})$. As noted above, $\mathcal{B}$ must be countably generated. Moreover, such a filtration $\mathcal{E}$ is associated with a natural topology $\tau_{\mathcal{E}}$ on $\Sigma \setminus \mathcal{N}$ for which $\mathcal{U}_{\mathcal{E}}(x) = \{U_n(x)\}_n$ is a neighbourhood base for $\tau_{\mathcal{E}}$ for $x \notin \mathcal{N}$ and $\Sigma \setminus \mathcal{N}$ supports μ. Let $\mathcal{U}_{\mathcal{E}'}(x)$ be a neighbourhood base at $x \in \Sigma \setminus \mathcal{N}$ for another partition filtration $\mathcal{E}'$, enlarging $\mathcal{N}$ if necessary.

Suppose that there exist constants $c_1, c_2 > 0$ such that for every $x \in \Sigma \setminus \mathcal{N}$, the following two conditions hold:

(i) for every $U \in \mathcal{U}_{\mathcal{E}}(x)$, there exists $V \in \mathcal{U}_{\mathcal{E}'}(x)$ such that $U \subset V$ and $\mu(V) \le c_1 \mu(U)$,
(ii) for every $V \in \mathcal{U}_{\mathcal{E}'}(x)$, there exists $U \in \mathcal{U}_{\mathcal{E}}(x)$ such that $V \subset U$ and $\mu(U) \le c_2 \mu(V)$.

Then for each $f \in L^1(\mu \otimes \mu)$, the inequalities

$$c_2^2 M_{\mathcal{E}'}(f) \le M_{\mathcal{E}}(f) \le c_1^2 M_{\mathcal{E}'}(f)$$

hold on $(\Sigma \setminus \mathcal{N})^2$ for the maximal functions defined by equation (4.2) with respect to either filtration $\mathcal{E}$ and $\mathcal{E}'$. Consequently, for any other filtration $\mathcal{E}'$ such that $\mathcal{E}$ and $\mathcal{E}'$ satisfy (i) and (ii), the maximal functions $M_{\mathcal{E}}(f)$ and $M_{\mathcal{E}'}(f)$ are equivalent and so $\mathfrak{C}_1(\mathcal{E}, X) = \mathfrak{C}_1(\mathcal{E}', X)$.

Now suppose that $\tau_{\mathcal{E}}$ is a Hausdorff topology, that is, if $x, y \in \Sigma \setminus \mathcal{N}$ and $x \ne y$, then there exists $n = 1, 2, \dots$ such that $U_n(x) \cap U_n(y) = \emptyset$ and every decreasing sequence of sets from $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ has nonempty intersection. Then the filtration $\mathcal{E}$ is closely associated with a metric topology, because according to [29, Lemma 9, p. 98], the Hausdorff topological space $(\Sigma \setminus \mathcal{N}, \tau_{\mathcal{E}})$ is a Lusin space for which $\mathcal{B}$ is the associated Borel σ-algebra on $\Sigma \setminus \mathcal{N}$ [29, Theorem 5, p. 101], so there exists a metric $d_{\mathcal{E}}$ on $\Sigma \setminus \mathcal{N}$ whose topology is stronger than $\tau_{\mathcal{E}}$ and $(\Sigma \setminus \mathcal{N}, d_{\mathcal{E}})$ is complete and separable. Then $\mathcal{B}$ is also the Borel σ-algebra for the metric $d_{\mathcal{E}}$ [29, Corollary 2, p. 101].

Let us say that the filtration $\mathcal{E}$ is *μ-compatible* if conditions (i) and (ii) are satisfied with respect to the neighbourhood bases $\mathcal{U}_{\mathcal{E}}(x)$ and the collection $\mathcal{V}(x)$ of open balls $B_r(x)$, $r > 0$, for the metric $d_{\mathcal{E}}$ centred at $x \in \Sigma \setminus \mathcal{N}$ replacing $\mathcal{U}_{\mathcal{E}'}(x)$ above. If $\mathcal{E}$ is μ-compatible, then the maximal function $M_{\mathcal{E}}(f)$ and the metric maximal function $M_{d_{\mathcal{E}}}(f)$ defined by

$$M_{d_{\mathcal{E}}}(f)(x,y) = \sup_{r>0} \frac{\int_{B_r(x)\times B_r(y)} f\, d(\mu\otimes\mu)}{\mu(B_r(x))\mu(B_r(y))}, \quad x, y \in \Sigma \setminus \mathcal{N},$$

are equivalent. Condition (ii) and the Martingale Convergence Theorem [3, Theorem 10.2.3] ensures the validity of the Lebesgue differentiation theorem with respect to the metric $d_{\mathcal{E}}$ and the measure μ, even without the assumption that μ is a doubling measure with respect to $d_{\mathcal{E}}$. Indeed, it is clear that for μ-compatible filtrations $\mathcal{E}$, many of the martingale results in harmonic analysis [21] translate for the metric space $(\Sigma \setminus \mathcal{N}, d_{\mathcal{E}})$, where the filtration $\mathcal{E}$ plays the role of the filtration of dyadic cubes in Euclidean space. See also [31, 32] for results in harmonic analysis on filtered measure spaces.

4.2. **Connection with other generalised traces.** An axiomatic treatement of traces on operator ideals is given in [23, 24] with recent updates in [25, 26, 27]. The starting point is the Calkin theorem [25, Theorem 2.2] which asserts that the collection of all operator ideals on a separable Hilbert space $\mathcal{H}$ is in one-to-one correspondence with symmetric sequence ideals. The correspondence is obtained from the singular values of operators in the ideal. A *trace* on an operator ideal $\mathfrak{U}(\mathcal{H})$ then corresponds to a unitarily invariant linear functional on $\mathfrak{U}(\mathcal{H})$ or, equivalently, a symmetric linear functional on the corresponding sequence ideal [25, Theorem 6.2]. A particular example that has assumed importance recently because of noncommutative geometry is the *Dixmier trace* defined on the Marcinkiewicz operator ideal. The Dixmier trace is an example of a singular trace because it vanishes on all finite rank operators, see [6, 20] for example.

By contrast, in this note, the emphasis with the Hardy-Littlewood maximal function approach to traces is on the *Banach lattice* of all absolute integral operators T on a Banach function space X, so that $T \geq 0$ implies $\int_\Sigma \langle T, dm\rangle \geq 0$—just what is needed in the proof of the Cwikel-Lieb-Rosenbljum inequality for dominated semigroups in [15]. A result of D. Lewis [19] shows that on an infinite dimensional Hilbert space, the collection of all Hilbert-Schmidt operators is the only Banach operator ideal isomorphic to a Banach lattice, despite the observation that a symmetric sequence ideal is itself a Riesz space. For a choice of Banach limit $\omega \in (\ell^\infty)'$, the map

$$T \longmapsto \int_\Sigma \omega(\{\mathbb{E}(k|\mathcal{E}_n \otimes \mathcal{E}_n) \circ J\}_{n=1}^\infty)\, d\mu, \quad T \in \mathfrak{C}_1(\mathcal{E}, X),$$

is continuous and linear on $\mathfrak{C}_1(\mathcal{E}, X)$, so there may be many possible choices of a continuous trace on the whole Banach lattice $\mathfrak{C}_1(\mathcal{E}, X)$ depending on ω.

The *Selberg trace formula* also relates regularised traces (geometric information) to asymptotic estimates for eigenvalues (spectral information) of a Laplacian, see [2] for a survey of this deep subject.

References

[1] D. Adams and L. Hedberg, Function spaces and potential theory, Grundlehren der Mathematischen Wissenschaften, **314**, Springer-Verlag, Berlin, 1996.

[2] J. Arthur, An introduction to the trace formula, *Harmonic analysis, the trace formula, and Shimura varieties*, Clay Math. Proc. **4**, Amer. Math. Soc., Providence, RI, 2005, 1–263

[3] V. Bogachev, *Measure Theory*, Springer-Verlag, Berlin, 2007.

[4] C. Brislawn, Kernels of trace class operators, *Proc. Amer. Math. Soc.* **104** (1988), 1181–1190.

[5] ———, Traceable integral kernels on countably generated measure spaces, *Pacific J. Math.* **150** (1991), 229–240.

[6] A. Carey and F. Sukochev, Dixmier traces and some applications in non-commutative geometry, *Russian Math. Surveys* **61** (2006), 1039–1099.

[7] M. Castro, V. Menegatto and A. Peron, Integral operators generated by Mercer-like kernels on topological spaces, *Colloq. Math.* **126** (2012), 125–138.

[8] ———, Traceability of positive integral operators in the absence of a metric, *Banach J. Math. Anal.* **6** (2012), 98–112.

[9] J. Delgado, Trace formulas for nuclear operators in spaces of Bochner integrable functions, *Monatsh. Math.* **172** (2013), 259–275.

[10] ———, The trace of nuclear operators on $L^p(\mu)$ for σ-finite Borel measures on second countable spaces, *Integral Equations Operator Theory* **68** (2010), 61–74.

[11] ———, A trace formula for nuclear operators on L^p, *Pseudo-differential operators: complex analysis and partial differential equations*, Oper. Theory Adv. Appl. **205**, 181–193.

[12] J. Ferreira, V. Menegatto and C. Oliveira, On the nuclearity of integral operators, *Positivity* **13** (2009), 519–541.

[13] I. Gohberg, S. Goldberg and N. Krupnik, *Traces and determinants of linear operators*, Operator Theory: Advances and Applications **116**, Birkhäuser Verlag, Basel, 2000.

[14] L. Grafakos, *Classical Fourier analysis*, Graduate Texts in Mathematics **249**, 2nd Ed., Springer, New York, 2008.

[15] B. Jefferies, The CLR inequality for dominated semigroups, *Math. Phys. Anal. Geom.* **17** (2014), 115–137, DOI 10.1007/ s11040-014-9145-6.

[16] ———, Traceability of positive integral operators, *submitted*.

[17] B. Jefferies and S. Okada, Bilinear integration in tensor products, *Rocky Mountain J. Math.* **28** (1998), 517–45.

[18] I. Kluvánek, *Integration structures*, Australian Nat. Univ., Canberra, Proc. Centre for Mathematical Analysis **18**, 1988.

[19] D. Lewis, An isomorphic characterization of the Schmidt class, *Compos. Math.* **30** (1975), 293–297.

[20] S. Lord and F. Sukochev and D. Zanin, *Singular traces*, De Gruyter Studies in Mathematics, **46**, De Gruyter, Berlin, 2013.

[21] R.L. Long, *Martingale spaces and inequalities*, Peking University Press, Beijing; Friedr. Vieweg & Sohn, Braunschweig, 1993.

[22] P. Meyer-Neiberg, *Banach lattices*, Springer-Verlag, Berlin, 1991.

[23] A. Pietsch, *Eigenvalues and s-Numbers*, Geest & Portig, Leipzig, and Cambridge Univ. Press, 1987.

[24] ———, Traces and shift invariant functionals, *Math. Nachr.* **145** (1990), 7–43.

[25] ———, Traces on operator ideals and related linear forms on sequence ideals (part I), *Indag. Math. (N.S.)* **25** (2014), 341–365.

[26] ———, Traces on operator ideals and related linear forms on sequence ideals (part II), *Integral Equations Operator Theory* **79** (2014), 255–299.

[27] ———, Traces on operator ideals and related linear forms on sequence ideals (part III), *J. Math. Anal. Appl.* **421** (2015), 971–981.

[28] H. Schaefer, *Topological Vector Spaces*, Graduate Texts in Mathematics **3**, Springer-verlag, Berlin/Heidel-berg/New York, 1980.

[29] L. Schwartz, *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*, Tata Institute Publications, Oxford University Press, Bombay, 1973.
[30] B. Simon, *Trace ideals and their applications.* 2nd Ed., Mathematical Surveys and Monographs **120** American Mathematical Society, Providence, RI, 2005.
[31] H. Tanaka and Y. Terasawa, A characterization for the boundedness of positive operators in a filtered measure space, *Arch. Math. (Basel)*, **101**, 2013, 559–568.
[32] ———, Positive operators and maximal operators in a filtered measure space, *J. Funct. Anal.*, **264** 2013, 920–946.

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