COMPUTING FOURIER AND LAPLACE TRANSFORMS BY MEANS OF POWER SERIES EVALUATION

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1. NOTATIONS AND ASSUMPTIONS

Let f be a real-valued function, defined for nonnegative arguments. We shall discuss some aspects of the numerical evaluation of the Laplace transform

$$(1.1) \qquad (Lf)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt \ ,$$

and the Fourier transform

$$(1.2) \qquad (Ff)(\omega) = \int_0^\infty e^{i\omega t} f(t)\,dt \ .$$

It will turn out to be advantageous to treat (1.1) and (1.2) separately, even if (1.2) is obtained by setting $\lambda = -i\omega$ in (1.1). We shall confine our discussion to the cases λ and ω real. We observe that the two-sided Fourier transform can be cast on the form of (1.2) since

$$\int_{-\infty}^\infty e^{i\omega t} f(t)\,dt = \int_0^\infty e^{i\omega t} f(t)\,dt + \int_0^\infty e^{-i\omega t} f(-t)\,dt \ .$$

Therefore, the inverse Laplace transform may be calculated by means of evaluating integrals of the type of (1.1) and (1.2). (See e.g. [1] and [4].) In our treatment we shall assume that $f(t)$ may be calculated for an arbitrary argument t with known, finite accuracy. In order to assess the accuracy of the calculated values of (1.1) and (1.2) we must know that f

belongs to a class of functions with certain *qualitative* characteristics. For example, merely to assume that f is bounded and continuous does not suffice to derive error bounds, even if a large number of functional values $f(t)$ are calculated.

2. TRANSFORM INTEGRALS AND POWER SERIES

Let $h > 0$ be a fixed number. We find

$$\int_0^\infty e^{-\lambda t} f(t)\,dt = \sum_{n=0}^{\infty} \int_{nh}^{(n+1)h} e^{-\lambda t} f(t)\,dt = \sum_{n=0}^{\infty} e^{-\lambda hn} \int_0^h e^{-\lambda t} f(nh+t)\,dt \ .$$

Thus

$$(2.1) \quad \int_0^\infty e^{-\lambda t} f(t)\,dt = \sum_{n=0}^{\infty} x^n a_n \ , \ x = e^{-\lambda h} \ , \ a_n = \int_0^h e^{-\lambda t} f(nh+t)\,dt \ .$$

Rewriting (1.2) in the same way we obtain

$$(2.2) \quad \int_0^\infty e^{i\omega t} f(t)\,dt = \sum_{n=0}^{\infty} z^n b_n \ , \ z = e^{i\omega h} \ , \ b_n = \int_0^h e^{i\omega t} f(nh+t)\,dt \ .$$

An obvious strategy would be first to calculate a_n and b_n by means of numerical integration and then evaluate the power series, preferably using a convergence acceleration scheme; such as those discussed in [2] and [4]. We observe that the rate of convergence of the power series is influenced by the choice of h . In particular, if we take $h = \pi/\omega$ in (2.2) we get

$$(2.3) \quad \int_0^\infty e^{i\omega t} f(t)\,dt = \sum_{n=0}^{\infty} (-1)^n b_n \ , \quad b_n = \int_0^{\pi/\omega} e^{i\omega t} f(nh+t)\,dt \ .$$

This alternating series may, for a fairly large class of functions f , be evaluated using the Euler transformation which in this case is equivalent to repeated averaging of the partial sums of the alternating series in (2.3). This scheme can be shown to converge, if f is a polynomial in t, or if f admits a representation as the Stieltjes' integral

$$(2.4) \qquad f(t) = \int_0^\infty e^{-tx} d\alpha(x) ,$$

where α is of bounded variation on $[0,\infty]$ and independent of t. Computational experiments have been carried out and it turned out that fairly many functional values were required to determine a_n and b_n in (2.1) and (2.2) with high accuracy. This fact has given an incentive to try other approaches, since a_n and b_n depend on λ and ω and must hence be evaluated for each value of these parameters. Performing a change of variables in (2.1) and (2.2) we get the expressions

$$(2.5) \qquad (Lf)(\lambda) = \frac{1}{\lambda}\int_0^\infty e^{-t} f(t/\lambda)\,dt = \sum_{n=0}^{\infty} x^n a_n ,$$

$$x = e^{-h} , \quad a_n = \frac{1}{\lambda}\int_0^h e^{-t} f\left(\frac{nh+t}{\lambda}\right) dt ,$$

$$(2.6) \qquad (Ff)(\omega) = \frac{1}{\omega}\int_0^\infty e^{-it} f(t/\lambda)\,dt = \sum_{n=0}^{\infty} z^n b_n$$

$$z = e^{ih\omega} , \quad b_n = \frac{1}{\omega}\int_0^h e^{it} f\left(\frac{nh+t}{\omega}\right) dt .$$

Assume that N terms are required to evaluate the power series in (2.5) and (2.6) with desired accuracy for all λ and ω. These N terms are completely determined by the values of f in the bounded intervals

$$[0,\frac{Nh}{\lambda}] \;, \tag{2.7}$$

$$[0,\frac{Nh}{\omega}] \;. \tag{2.8}$$

Assume now that we know a function g together with its transforms Lg and Fg. Then we find

$$(Fg)(\omega) - (Ff)(\omega) = \sum_{n=0}^{\infty} z^n \frac{1}{\omega} \int_0^h e^{it} (g(\frac{nh+t}{\omega}) - f(\frac{nh+t}{\omega}))dt \;. \tag{2.9}$$

If g approximates f well on the interval (2.8) and there is a convergence acceleration scheme which delivers a good estimate of the series in (2.9) using N terms, then $(Fg)(\omega)$ is a good approximation of $(Ff)(\omega)$. A related statement holds for $(Lf)(\lambda)$. We note that if ω is large, then (2.8) defines a short interval but if ω is small, then the interval (2.8) is long. Therefore a general strategy is to approximate f well close to 0, if ω is large, while good approximations for t large are required for ω small. A similar rule holds for the calculation of $(Lf)(\lambda)$.

3. CALCULATION OF Lf AND Ff FOR LARGE PARAMETERS λ AND ω.

We illustrate the general ideas put forward above by a numerical example:

<u>EXAMPLE</u> Calculate Lf and Ff for $f(t) = \ln(1+t)$. Close to the origin we may approximate f by the first few terms of its Taylor expansion,

$$\ln(1+t) = t - t^2/2 + t^3/3 \ldots$$

Entering this expression into (1.1) we get

$$\int_0^{\infty} e^{-\lambda t}\ln(1+t)dt = \lambda^{-2} - \lambda^{-3} + 2\lambda^{-4} - 6\lambda^{-5} + 24\lambda^{-6} - 120\lambda^{-7} + \ldots, \tag{3.1}$$

which is the asymptotic expansion delivered by Watson's lemma. For $\lambda = 10$ one would truncate the expansion (3.1) after 6 terms getting the estimate 0.009152. In this particular case the expansion (3.1) could also be derived by means of integration by parts and then a simple expression is also obtained for the remainder term. Ff is defined by means of analytic continuation. Taking the real part of (Ff)(a) we find

$$\int_0^\infty \cos\omega t \,\ln(1+t)\,dt = -\omega^{-2} + 2\omega^{-4} - 24\omega^{-6} + \dots , \tag{3.2}$$

Instead of approximating f by a polynomial we try an exponential fit. Thus we put $f(t)$ in $f_n^*(t)$ where

$$f_n^*(t) = \sum_{r=1}^{n} y_{rn} e^{-(r-1)t} , \tag{3.3}$$

where y_{rn} are determined such that f_n^* interpolates f_n at the equidistant grid $t_i = (i-1)\Delta t$, $i = 1,\dots,n$. Δt is a step-width.
Thus

$$(Lf_n^*)(\lambda) = \sum_{r=1}^{n} y_{rn}/(\lambda+r-1) . \tag{3.4}$$

This formula permits easy tabulation of $L_n^*(\lambda)$ and some sample values are given below:

<u>TABLE 1</u> Estimates of $\int_0^\infty e^{-\lambda t}\,\ell n(1+t)\,dt$ based on (3.3) and (3.4) with $\Delta t = 0.1$.

n	$\lambda = 5$	$\lambda = 10$
2	0.0333850	0.00910501
3	0.0336988	0.00913712
4	0.0340097	0.00915512
5	0.0340578	0.00915598
6	0.0340753	0.00915633
7	0.0340808	0.00915630
8	0.0340829	0.00915635
9	0.0340837	0.00915632
10	0.0340841	0.00915636
11	0.0340843	0.00915636
12	0.0340843	

We find immediately

$$(Ff_n^*)(\omega) = \sum_{r=1}^{n} y_{rn}/(r-1-i\omega) ,$$

giving the approximation

$$\int_0^\infty \cos\omega t\ \ell n(1+t)\,dt \approx \sum_{r=1}^{n} (r-1)y_{rn}/(\omega^2+(r-1)^2) . \tag{3.5}$$

We give some sample values:

<u>TABLE 2</u> Estimates of $\int_0^\infty \cos\omega t\ \ell n(1+t)\,dt$ based on (3.3) and (3.4) with $\Delta t = 0.1$.

n	$\lambda = 10$	$\lambda = 20$
2	-0.00991635	-0.00249763
3	-0.00989802	-0.00248774
4	-0.00980488	-0.00248724
5	-0.00981593	-0.00248823
6	-0.00982182	-0.00248773
7	-0.00981973	-0.00248789
8	-0.00981828	-0.00248783
10	-0.00981929	-0.00248792
11	-0.00981928	-0.00248790
12	-0.00981911	

We note that the accuracy of the approximations based on (3.3) compares favourably with those based on power expansion of $\ell n(1+t)$. Another strategy to determine exponential approximations of f is described in [3]. The methods given there require that f admits a representation (2.4). Under this condition it is fairly easy to derive an error bound by applying one of the convergence schemes in [2] to (2.9).

4. REMARKS ON THE CASE OF SMALL PARAMETER VALUES

As stated at the end of Section 2, (2.9) requires that f is well approximated for t large by a function f^* , whose transform (1.1) or (1.2) is known. Put $u = 1/t$ and $g(u) = f(1/u)$. If we can construct a polynomial P which approximates g accurately for $0 \le u \le \frac{1}{T}$, where $T > 0$ is a known number, then we get

$$\int_T^\infty e^{i\omega t} f(t)\,dt \approx \int_T^\infty e^{i\omega t} p(1/t)\,dt \; . \tag{4.1}$$

To evaluate the integral at the right hand side of (4.1) we need to determine the numbers

$$c_r(\omega) = \int_T^\infty e^{i\omega t} t^{-r} dt \; , \quad r = 0,1,\ldots \; .$$

Integrating by parts we find the stable recurrence relation

$$c_{r+1}(\omega) = \frac{e^{i\omega T}}{rT^r} + \frac{i\omega}{r} c_r(\omega) \; , \quad r = 1,2,\ldots \; .$$

The starting value $c_1(\omega)$ must be calculated numerically. We find

$$c_1(\omega) = \int_T^\infty \frac{e^{i\omega t}}{t}\,dt = \int_{T\omega}^\infty \frac{e^{it}}{t}\,dt \; ,$$

and the latter integral is easily calculated using numerical integration in conjunction with convergence acceleration as described earlier. In order to evaluate (1.2) we also need to determine

$$\int_0^T e^{i\omega t} f(t)\,dt \;,$$

which can be achieved by means of standard numerical methods. The Laplace transforms are treated in a similar way.

REFERENCES

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