Lecture 1: O-minimal structures

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Introduction

In Real Algebraic and Analytic Geometry the following traditional classes of sets and their geometries are considered:
(1) The class of semialgebraic sets (Whitney-Łojasiewicz, in the 50's) (see [**BCR**]).
(2) The class of semianalytic sets (Łojasiewicz, in the 60's) (see [**L**]).
(3) The class of subanalytic sets (Gabrielov-Hironaka-Hardt and Krakovian school, in the 70's) (see [**BM**], [**Hi**], [**LZ**]).
These classes of sets have many nice properties. Semialgebraic and subanalytic sets form the so-called *Tarski-type systems*, that is the corresponding class is closed under boolean operators and under proper projections. In particular, these classes have the finiteness property: each set in these classes has locally only finite number of connected components and each of the components also belongs to the corresponding class.

In some problems, we have to treat functions like x^α or $\exp(-1/x)$, where $x > 0$ and α is an irrational number, which are not subanalytic at 0. Naturally, it requires an extension of classes mentioned above. According to van den Dries [**D1**], the finiteness is the most remarkable in the sense that if a Tarski-type system has this property, it will preserve many nice properties of semi and subanalytic sets. Van den Dries, Knight, Pillay and Steinhorn gave the name *o-minimal structures* for such systems and developed the general theory ([**D1**],[**KPS**], [**PS**]). Note that Shiota had a similar program ([**S1**],[**S2**]). Khovanskii's results on Fewnomials [**Kh**], and a notable theorem of Wilkie on model completeness [**W1**] confirm the o-minimality of the real exponential field.

The theory of o-minimal structures is a wide-ranging generalization of semialgebraic and subanalytic geometry. Moreover, one can view the subject as a realization of Grothendieck's idea of *topologie modérée*, or tame topology, in "Esquisse d'un Programme" (1984). In recent years, o-minimality of many remarkable structures have been proved (see the examples in 1.3–1.9) and many interesting results have been established in the theory of o-minimal structures on the real field (see Lectures 2,3).

This note is a part of [**L1**]. Mainly we follow the proofs of [**D2**] and [**C**] with some changes. The definition and some examples of o-minimal structures are given in Section 1. In Section 2 we give some important properties of o-minimal structures. In the last section we sketch the idea of constructing an analytic-geometric category corresponding to an o-minimal structure.

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1. Definition and examples of o-minimal structures

MOTIVATION 1.1. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a class of real-valued functions on spaces $\mathbb{R}^n,\ n \in \mathbb{N}$. Similar to semialgebraic or subanalytic sets, it is natural to construct a class of subsets of $\mathbb{R}^n,\ n \in \mathbb{N}$, as follows.
First consider basic sets of the forms

$$\{x \in \mathbb{R}^n : f(x) > 0\}, \quad \text{where } f \in \mathcal{F}_n,\ n \in \mathbb{N}.$$

Then starting from these basic sets we create new sets by taking finite unions, finite intersections, complements, Cartesian products, and linear projections (or proper projections) onto smaller dimensional Euclidean spaces. Repeating these operators with the new sets that arise, we get a class of subsets of $\mathbb{R}^n, n \in \mathbb{N}$, which is closed under usual topological operators (e.g. taking closure, interior, boundary, ...).
We are interested in the case that the new sets are not so complicated and pathological as Cantor sets, Borel sets, nonmeasurable sets..., since it promises a "tame topology" for the class of sets that we constructed. The corresponding category of spaces and maps between them may yield a rich algebraic-analytic-topological structure.

DEFINITION 1.2. A *structure* on the real field $(\mathbb{R}, +, \cdot)$ is a sequence $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ such that the following conditions are satisfied for all $n \in \mathbb{N}$:

(D1) $\mathcal{D}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$.
(D2) If $A \in \mathcal{D}_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A \in \mathcal{D}_{n+1}$.
(D3) If $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.
(D4) $\mathcal{D}_n$ contains $\{x \in \mathbb{R}^n : P(x) = 0\}$ for every polynomial $P \in \mathbb{R}[X_1, \cdots, X_n]$.

Structure $\mathcal{D}$ is called to be *o-minimal* if

(D5) Each set in $\mathcal{D}_1$ is a finite union of intervals and points.

A set belonging to $\mathcal{D}$ is called *definable* (in that structure). *Definable maps* in structure $\mathcal{D}$ are maps whose graphs are definable sets in $\mathcal{D}$.

EXAMPLE 1.3. Given a collection of real-valued functions $\mathcal{F}$, the smallest structure on $(\mathbb{R}, +, \cdot)$ containing the graphs of all $f \in \mathcal{F}$ is denoted by $(\mathbb{R}, +, \cdot, \mathcal{F})$.

EXAMPLE 1.4. Let $\mathbb{R}_{\text{alg}}$ be the smallest structure on $(\mathbb{R}, \cdot, +)$. By Tarski-Seidenberg's Theorem a subset $X \subset \mathbb{R}^n$ is definable in $\mathbb{R}_{\text{alg}}$ if and only if X is semialgebraic. Obviously, $\mathbb{R}_{\text{alg}}$ is o-minimal.

EXAMPLE 1.5. Let $\mathbb{R}_{\text{an}} = (\mathbb{R}, +, \cdot, \mathcal{A})$, where $\mathcal{A}$ is the class of all restricted analytic functions on $[-1, 1]^n$ $(n \in \mathbb{N})$. Definable sets in $\mathbb{R}_{\text{an}}$ are finitely subanalytic sets (see [**D2**]), i.e $X \subset \mathbb{R}^n$ is definable in $\mathbb{R}_{\text{an}}$ if and only if X is subanalytic in the projective space $\mathbf{P}^n(\mathbb{R})$, where we identify $\mathbb{R}^n$ with an open set of $\mathbf{P}^n(\mathbb{R})$

via $(x_1, \cdots, x_n) \mapsto (1 : x_1 : \cdots : x_n)$. By Grabrielov's Theorem on the complement and a Łojasiewicz result on connected components of semianalytic sets (see [**BM**],[**L**],[**LZ**]) $\mathbb{R}_{\mathrm{an}}$ is o-minimal.

EXAMPLE 1.6. Let $\mathbb{R}_{\exp} = (\mathbb{R}, +, \cdot, \exp)$. Wilkie [**W1**] proved that $\mathbb{R}_{\exp}$ is model complete, as a direct consequence of this theorem each definable sets in $\mathbb{R}_{\exp}$ is the image of the zero set of a function in $\mathbb{R}[x_1, \cdots, x_N, \exp(x_1), \cdots, \exp(x_N)]$, for some $N \in \mathbb{N}$ under a natural projection (see [**L2**]). Then by a Khovanskii result on fewnomials [**Kh**], $\mathbb{R}_{\exp}$ is an o-minimal structure. An analytic proof of Wilkie's theorem is given in [**LR**]. Note that $x^\alpha, \exp(-1/x)$ ($x > 0$ and α is irrational) are definable functions in $\mathbb{R}_{\exp}$ but not subanalytic at 0.

EXAMPLE 1.7. Let $\mathbb{R}_{\mathrm{an},\exp} = (\mathbb{R}, +, \cdot, \mathcal{A}, \exp)$, where $\mathcal{A}$ as in 3.2. Extending Wilkie's method, van den Dries and Miller [**DM1**] proved that $\mathbb{R}_{\mathrm{an},\exp}$ is also o-minimal. (see also [**LR**] for an analytic proof)

EXAMPLE 1.8. Let $f_1, \cdots, f_k : \mathbb{R}^n \to \mathbb{R}$ be a *Pfaffian chain*, i.e. $f_1, \cdots, f_k$ are smooth functions and there exist polynomials $P_{ij} \in \mathbb{R}[X_1, \cdots, X_n, Y_1, \cdots, Y_i]$ such that for all $x \in \mathbb{R}^n$

$$\frac{\partial f_i}{\partial x_j}(x) = P_{ij}(x, f_1(x), \cdots, f_i(x)) \qquad (i = 1, \cdots, k; j = 1, \cdots, n).$$

Let $\mathcal{P} = \mathcal{P}(f_1, \cdots, f_k)$ be the class of all functions of the form $f(x) = Q(x, f_1(x), \cdots, f_k(x))$, where $Q \in \mathbb{R}[X_1, \cdots, X_n, Y_1, \cdots, Y_k]$, (they are called *Pfaffian functions*). Then, by Khovanskii's Theory [**Kh**] and by a result of Wilkie [**W2**], $(\mathbb{R}, +, \cdot, \mathcal{P})$ is o-minimal.

EXAMPLE 1.9. Examples of Pfaffian functions:
a) The polynomials are Pfaffian functions.
b) Let $f_1(x) = e^x, f_n(x) = e^{f_{n-1}(x)}$ ($n \in \mathbb{N}$). Then $(f_1, \ldots, f_k)$ is a Pfaffian chain, since $f_n' = f_{n-1}' f_n = f_1 \cdots f_n$.
c) Let $f(x) = (x^2 + 1)^{-1}$, $g(x) = \arctan x$. Then (f, g) is a Pfaffian chain, since $f' = -2xf$, $g' = f$.

2. Some properties of o-minimal structures

Throughout this section $\mathcal{D}$ denotes an o-minimal structure on $(\mathbb{R}, +, \cdot)$. "Definable" means definable in $\mathcal{D}$.

DEFINITION 2.1. A *first-order formula* (of the language of $\mathcal{D}$) is constructed according to the following rules:

- If $P \in \mathbb{R}[X_1, \cdots, X_n]$, then $P \star 0$, where $\star \in \{=, >, <\}$, is a formula.
- If A is a definable set, then $x \in A$ is a formula.
- If ϕ and ψ are formulas, then their conjunction $\phi \wedge \psi$, their disjunction $\phi \vee \psi$, and the negation $\neg\phi$ are formulas.
- If $\phi(x, y)$ is a formula and A is a definable set, then $\exists x \in A, \phi(x, y)$ and $\forall x \in A, \phi(x, y)$ are formulas.

We use the relations between logical notations and boolean algebras: Let x, y be variables ranging over nonempty sets X, Y, and let $\phi(x, y)$ and $\psi(x, y)$ be first-order formulas on $(x, y) \in X \times Y$ defining sets

$\Phi = \{(x, y) \in X \times Y : \phi(x, y)\}$, and $\Psi = \{(x, y) \in X \times Y : \psi(x, y)\}$.

Then

$\phi(x, y) \vee \psi(x, y)$ defines $\Phi \cup \Psi$,

$\phi(x,y) \wedge \psi(x,y)$ defines $\Phi \cap \Psi$,
$\neg\phi(x,y)$ defines $X \times Y \setminus \Phi$,
$\exists x\phi(x,y)$ defines $\pi_Y(\Phi)$, where $\pi_Y : X \times Y \to Y$ is the natural projection,
$\forall x\phi(x,y)$ defines $Y \setminus \pi_Y(X \times Y \setminus \Phi)$.

From these relations and the definition of structures, we have:

Proposition 2.2. *Let $\phi(x_1, \cdots, x_n)$ be a first-order formula of the language of $\mathcal{D}$. Then the set $\{(x_1, \cdots, x_n) \in \mathbb{R}^n : \phi(x_1, \cdots, x_n)\}$ is definable.*

Theorem 2.3 (Elementary properties). (i) *The closure, the interior, and the boundary of a definable set are definable.*
(ii) *Images and inverse images of definable sets under definable maps are definable.*
(ii) *Compositions of definable maps are definable.*

Proof. To prove these properties we use the definition and Proposition 2.2. If A is definable subset of $\mathbb{R}^n$, then its closure is

$$\overline{A} = \{x \in \mathbb{R}^n : \forall \epsilon \in \mathbb{R}, \epsilon > 0 \Rightarrow \exists y \in \mathbb{R}^n, (y \in A) \wedge (\sum_{i=1}^{n} (x_i - y_i)^2 < \epsilon^2)\},$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots y_n)$. By Proposition 2.2, $\overline{A}$ is definable. The interior and the boundary of A can be expressed by $\mathrm{int}\,(A) = \mathbb{R}^n \setminus \overline{\mathbb{R}^n \setminus A}$ and $\mathrm{bd}\,(A) = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$, so they are definable.
Let $f : X \to Y$ be a definable function and $A \subset X, B \subset Y$ be definable subsets. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the natural projections. Denote the graph of f by f. Then $f(A) = \pi_Y(f \cap A \times Y)$ and $f^{-1}(B) = \pi_X(f \cap X \times B)$. So they are definable.
Let $f : X \to Y, g : Y \to Z$ be definable maps. Then $g \circ f = \pi(f \times Z \cap X \times g)$, where $\pi : X \times Y \times Z \to X \times Z$ defined by $\pi(x,y,z) = (x,z)$. So $g \circ f$ is definable. □

Exercise 2.4. Let $f : A \to \mathbb{R}$ be a definable function and $p \in \mathbb{N}$. Prove that the set $C^p(f) = \{x \in A : f \text{ is of class } C^p \text{ at } x\}$ is definable, and the partial derivatives $\partial f/\partial x_i$ are definable functions on $C^p(f)$.

Exercise 2.5. Let $f : A \to \mathbb{R}$ be a definable function. Suppose that f is bounded from below. Let $g : A \to \mathbb{R}^m$ be a definable mapping. Prove that the function $\varphi : g(A) \to \mathbb{R}$, defined by $\varphi(y) = \inf_{x \in g^{-1}(y)} f(x)$, is a definable function.

Note that these properties hold for any structure not necessary o-minimal. Important results in the subject of o-minimality are the Monotonicity theorem, the Cell decomposition theorem and their consequences.

Theorem 2.6 (Monotonicity theorem). *Let $f : (a,b) \to \mathbb{R}$ be a definable function and $p \in \mathbb{N}$. Then there are points $a = a_0 < \cdots < a_k = b$ such that $f|_{(a_i, a_{i+1})}$ is C^p, and either constant or strictly monotone, for $i = 0, \cdots, k-1$.*

Proof. First we establish four claims.

Claim 1: Let $D(f)$ denote the set of discontinuity of f. Then $D(f)$ is a finite set.

By Proposition 2.2 $D(f)$ is definable and hence it is finite or contains an interval.

Contrary to the claim, suppose $D(f)$ contains an interval I. Then $f(I')$ contains an interval for all subinterval I' of I. By induction, we can construct a sequence of intervals $[\alpha_n, \beta_n] \subset I$ such that $\alpha_n < \alpha_{n+1} < \beta_{n+1} < \beta_n$, $\beta_n - \alpha_n < 1/n$, and $f([\alpha_n, \beta_n])$ is contained in an interval of length smaller than $1/n$. Clearly f is continuous at the point $x_0 \in \cap_{n\in\mathbb{N}}[\alpha_n, \beta_n] \subset D(f)$, a contradiction.

Claim 2: f has left derivative $f'_-(x)$ and right derivative $f'_+(x)$ in $\mathbb{R} \cup \{-\infty, +\infty\}$ for every $x \in (a, b)$.

Suppose for instancce that at $x \in (a, b)$,

$$\liminf_{h\to 0^+} \frac{f(x+h) - f(x)}{h} < \limsup_{h\to 0^+} \frac{f(x+h) - f(x)}{h}$$

Then there exists $k \in \mathbb{R}$, such that for each $\varepsilon > 0$, there are $h_1, h_2 \in (0, \varepsilon)$,

$$f(x + h_1) - f(x) < kh_1 \text{ and } f(x + h_2) - f(x) > kh_2 \qquad (*)$$

On the other hand, $(a, b) \setminus \{x\}$ is the union of three definable subsets

$\{y \in (a, b) \setminus \{x\} : f(y) > f(x) + k(y - x)\}$,
$\{y \in (a, b) \setminus \{x\} : f(y) < f(x) + k(y - x)\}$,
$\{y \in (a, b) \setminus \{x\} : f(y) = f(x) + k(y - x)\}$.

So there is $\varepsilon > 0$ such that $f(x + h) - f(x) > kh, \forall h \in (0, \varepsilon)$ or $f(x + h) - f(x) < kh, \forall h \in (0, \varepsilon)$ or $f(x + h) - f(x) = kh$, $\forall h \in (0, \varepsilon)$. This contradicts $(*)$. Similarly for the existence of $f'_-(x)$.

Claim 3: If f is continuous and $f'_+ > 0$ (resp. $f'_- > 0$) on an interval I, then f is strictly increasing on I.

Otherwise, there are $c < d$ both in I such that $f(c) > f(d)$. Since $f'_+(c) > 0$, there exists $e \in (c, d)$, $f(e) > f(c)$. By continuity, $f|_{[c,d]}$ attains its maximum at a point $x_0 \in (c, d)$ and hence $f'_+(x_0) \le 0$, contradiction. Similarly for $f'_- > 0$.

Claim 4: If f is continuous on I, then for all but finitely many points in I, we have $f'_+ = f'_-$. Hence f is differentiable outside a finite subset of I.

First note that the set on which $f'_-(x)$ or $f'_+(x) \in \{-\infty, +\infty\}$ is finite. Otherwise, by definability, there would be a subinterval of I on which $f'_- = \infty$ or $f'_+ = \infty$. Suppose for instance that $f'_+ = +\infty$ on a subinterval J. Take $a < b$, both in J and set $g(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}x$, for $x \in J$. We have $g'_+ = +\infty$ on J. By Claim 2, g is strictly increasing on J, which contradicts $g(b) = g(a)$.

So f'_- and f'_+ take values in $\mathbb{R}$ outside a finite subset of I. By Claim 1, f, f'_-, f'_+ are continuous outside a finite subset of I. Suppose that there is $x_0 \in I$, such that f, f'_-, f'_+ are continuous at x_0, but $f'_-(x_0) < f'_+(x_0)$. By continuity, there are a subinterval J, $x_0 \in J \subset I$, and $k \in \mathbb{R}$ such that $f'_- < k < f'_+$ on J. Claim 2 implies that $x \mapsto f(x) - kx$ is at the same time strictly increasing and strictly decreasing on J, which is impossible. Similarly for the other cases.

Now we prove the theorem. By induction on p it suffices to prove the desired result for $p = 1$. By the claims there exists a finite subset D' of (a, b) such that on each component C of $(a,b)\setminus D'$, $f|_C$ is of class C'. Using Claim 1 for $(f|_C)'$ and definability of the sets $\{(f|_C)'<0\},\{(f|_C)'=0\},\{(f|_C)'<0\}$ for each component C, we get a finite subset D'' of (a,b) such that on each component of $(a,b)\setminus D''$ f is C^1 and either constant or strictly monotone. $\square$

EXERCISE 2.7. Let $f:(a,b)\to\mathbb{R}$ be definable and $c\in(a,b)$. Prove that $\lim_{x\to a_+ f(x)}$, $\lim_{x\to b_-} f(x), \lim_{x\to c_+} f(x), \lim_{x\to c_-} f(x)$ exist in $\mathbb{R}\cup\{-\infty,+\infty\}$.

Note: From Monotonicity, the germs at $+\infty$ of definable functions on $\mathbb{R}$ forms a *Hardy field* , i.e a set of germs at $+\infty$ of real-valued functions on neighborhoods of $+\infty$ that is closed under differentiation and that forms a field with usual addition and multiplication of germs.

DEFINITION 2.8. Let $p\in\mathbb{N}$. *C^p cells* in $\mathbb{R}^n$ are connected definable submanifolds of $\mathbb{R}^n$ which are defined by induction on n as follows:

• The C^p cells in $\mathbb{R}$ are points or open intervals.
• If $C\subset\mathbb{R}^n$ is a C^p cell and $f,g:C\to\mathbb{R}$ are definable functions of class C^p such that $f<g$, then the sets:
$\Gamma(f)=\{(x,t):t=f(x)\}$ (the graph), $(f,g)=\{(x,t):f(x)<t<g(x)\}$, $C\times\mathbb{R}$, $(-\infty,f)=\{(x,t):t<f(x)\}$, and $(f,+\infty)=\{(x,t):f(x)<t\}$ (the bands) are C^p cells in $\mathbb{R}^{n+1}$.

EXERCISE 2.9. Prove that for each nonempty cell C, there is a definable homeomorphism $h:C\to\mathbb{R}^d$ for some $d\in\mathbb{N}$.

A *C^p cell decomposition* of $\mathbb{R}^n$ is defined by induction on n:

• A C^p cell decomposition of $\mathbb{R}$ is a finite collection of intervals and points $\{(-\infty,a_1),\cdots,(a_k,+\infty),\{a_1\},\cdots,\{a_k\}\}$, where $a_1<\cdots<a_k$, $k\in\mathbb{N}$.
• A C^p cell decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into cells C, such that the collection of all the projections $\pi(C)$ is a C^p cell decomposition of $\mathbb{R}^n$, where $\pi:\mathbb{R}^{n+1}\to\mathbb{R}^n$ is the projection on the first n coordinates.

We say that a decomposition *compatible with* a class $\mathcal{A}$ of subsets of $\mathbb{R}^n$, if each $A\in\mathcal{A}$ is a union of some cells of the decomposition.

THEOREM 2.10 (Uniform finiteness - Cell decomposition - Piecewise smoothness). *(U_n) Let A be a definable subset of $\mathbb{R}^n$, such that for every $x\in\mathbb{R}^{n-1}$, the set $A_x=\{y\in\mathbb{R}:(x,y)\in A\}$ is finite. Then there exists $l\in\mathbb{N}$ such that $\#A_x\le l$, for every $x\in\mathbb{R}^{n-1}$.*

(I_n) For $A_1,\cdots,A_k\in\mathcal{D}_n$, there exists a C^p cell decomposition of $\mathbb{R}^n$ compatible with $\{A_1,\cdots,A_k\}$.

(II_n) For each definable function $f:A\to\mathbb{R}$, $A\subset\mathbb{R}^n$, there exists a C^p cell decomposition of $\mathbb{R}^n$ compatible with A such that for each cell $C\subset A$ of the decomposition the restriction $f|_C$ is of class C^p.

PROOF. We only give the proof of the theorem for $p=0$. Then using Theorem 2.11, one can get the theorem for $p>0$ (Exercise).
The proof of the theorem for $p=0$ is by induction on n. For $n=1$, (U_1) is trivial, (I_1) follows from the o-minimality, (II_1) is a consequence of Monotonicity theorem. From now on, assume $(U_m),(I_m),(II_m)$ hold for all m such that $0<m<n$.

2.10.1 Proof of (U_n)**.** We can assume that, for every $x \in \mathbb{R}^{n-1}$, A_x is contained in $(-1,1)$ (we can replace A with its image by $(x,y) \mapsto (x, y/\sqrt{1+y^2})$).
For $x \in \mathbb{R}^{n-1}$ with $A_x \neq \emptyset$, define $f_i(x)$ by

$$A_x = \{f_1(x), \cdots, f_{\#(A_x)}(x)\},\ f_1(x) < \cdots < f_{\#(A_x)}(x).$$

Note that, for each $i \in \{1, 2, \cdots\}$, the function f_i is definable (on its domain).
Call $a \in \mathbb{R}^{n-1}$ *good* if $f_1, \cdots, f_{\#(A_a)}$ are defined and continuous on an open box $B \subset \mathbb{R}^{n-1}$ containing a, and $(B \times \mathbb{R}) \cap A = \Gamma(f_1|_B) \cup \cdots \cup \Gamma(f_{\#(A_a)}|_B)$.
Call $a \in \mathbb{R}^{n-1}$ *bad* if it is not good.

Claim 1: The set of good points is definable.
Let $a \in \mathbb{R}^{n-1}$. Let $b \in [-1,1]$. We say that (a,b) is *normal* if there exists an open box $C = B \times (c,d)$ containing (a,b) such that $A \cap C$ is either empty or the graph of a continuous definable function $B \to (c,d)$.
If a is good, then clearly (a,b) is normal for every $b \in [-1,1]$.
Now assume a is bad. Let f_l be the first function f_i such that a is in the closure of the domain of f_i and there is no open box containing a on which f_i is defined and continuous. Set $\beta(a) = \liminf_{x\to a} f_l(x)$. Then $(a, \beta(a))$ is not normal (Otherwise, suppose $(a, \beta(a))$ is normal. There is an open box $B \times (c,d)$ containing $(a, \beta(a))$ whose intersection with A is the graph of a continuous function $g : B \to (c,d)$. We can assume that $f_l(x) > c$ for all $x \in B$ such that $f_l(x)$ is defined. If $l > 1$ and $\beta(a) = f_{l-1}(a)$, we would deduce $g = f_{l-1}|B$ since B is connected. We would have $f_l(x) \geq d$ for all $x \in B$ such that $f_l(x)$ is defined, which contradicts $\beta(a) < d$. Hence, we can assume $l = 1$ or $f_{l-1} < c$ on B. It follows that $g = f_l|B$, which contradicts the definition of l.)
We have shown that $a \in \mathbb{R}^{n-1}$ is good if and only if for all $b \in [-1,1]$, (a,b) is normal. From this we deduce the claim.

Claim 2: The set of good points is dense.
Otherwise, there is an open box $B \subset \mathbb{R}^{n-1}$ contained in the set of bad points. Consider the definable function $\beta : B \to [-1,1]$ defined as above. By (II_{n-1}), we can assume that β is continuous. For $x \in B$, we define $\beta_-(x)$ (resp. $\beta_+(x)$) to be the maximum (resp. minimum) of the $y \in A_a$ such that $y < \beta(x)$ (resp. $y > \beta(x)$), if such y exists. Using (II_{n-1}) and shrinking B, we can assume that β_- (resp. β_+) either nowhere defined on B or is continuous on B. Then the set of $(x,y) \in A \cap (B \times \mathbb{R})$ such that $y \neq \beta(x)$ is open and closed in $A \cap (B \times \mathbb{R})$. Shrinking B, we can assume that the graph of $\beta|B$ is either disjoint from A or contained in A. The first case contradicts the definition of β. In the second case, $(x, \beta(x))$ would be normal for every $x \in B$, which contradicts what was proved in Claim 1.

Now we prove (U_n). By (I_{n-1}), there is a cell decomposition of $\mathbb{R}^{n-1}$ compatible with the set of good points. Let C be a cell of dimension $n-1$. Since good points are dense, every $x \in C$ is good. Take $a \in C$. The set of $x \in C$ such that $\#(A_x) = \#(A_a)$ is definable, open and closed in C. By connectedness of C, it is equal to C. If D is a cell of smaller dimension, we can use a definable homeomorphism $D \to \mathbb{R}^d$ and the assumption that (U_d) holds to prove that $\#(A_x)$ is uniformly bounded for $x \in D$. Since there are finitely many cells, the proof is completed.

2.10.2 Proof of (I_n)**.** Let A be the set of $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that y belongs to the frontier of one of $A_{1,x}, \cdots, A_{k,x}$. Clearly A is definable and satisfies the assumptions of (U_n). Hence $\#(A_x)$ has a maximum l for $x \in \mathbb{R}^{n-1}$, and A is the union of the graphs of functions $f_1, \cdots, f_l$ defined at the beginning of the proof of (U_n). For $x \in \mathbb{R}^{n-1}$, define the type $\tau(x)$ being the following data:
• $\#(A_x)$,
• $\{(i,j) : 1 \leq i \leq \#A_x, 1 \leq j \leq k, f_i(x) \in A_{j,x}\}$,
• $\{(i,j) : 0 \leq i \leq \#A_x, 1 \leq j \leq k, (f_i(x), f_{i+1}(x)) \subset A_{j,x}\}$ $(f_0 = -\infty, f_{A_x+1} = +\infty)$.
Then $T = \{\tau(x) : x \in \mathbb{R}^{n-1}\}$ is finite, and for all $t \in T$ the set $\{x \in \mathbb{R}^{n-1} : \tau(x) = t\}$ is definable. By (I_{n-1}) there is a cell decomposition $\mathcal{P}_1$ of $\mathbb{R}^{n-1}$ such that two points in the same cell have the same type. Moreover, using (II_{n-1}), we can assume the cell decomposition is such that for each cell C and $i = 1, \cdots, l$, either f_i is defined nowhere on C or $f_i|_C$ is continuous. Then $\mathcal{P} = \mathcal{P}_1 \cup \{\Gamma(f_i|C), (f_i|C, f_{i+1}|C), C \in \mathcal{P}_1, i = 0, \cdots, l\}$ is a cell decomposition compatible with $A_1, \cdots, A_k$.

2.10.3 Proof of (II_n)**.** First consider the case that A is an open box $B \times (a,b)$.

Claim 1: Suppose that $f(x,\cdot)$ is continuous and monotone on (a,b) for every $x \in B$, and $f(\cdot,y)$ is continuous on B for every $y \in (a,b)$. Then f is continuous on $B \times (a,b)$.

Indeed, take $(x_0,y_0) \in B \times (a,b)$ and I an interval containing $f(x_0,y_0)$. By continuity of $f(x_0,\cdot)$, we find $y_1 < y_0 < y_2$ such that $f(x_0,y_i) \in I$ for $i = 1,2$. By continuity of $f(\cdot,y_i)$, we can find an open box $B' \ni x_0$ in B such that $f(B' \times \{y_i\}) \subset I$ for $i = 1,2$. It follows from the monotonicity of $f(x,\cdot)$ that $f(B' \times (y_1,y_2))$ is contained in I. This proves the continuity of f.

Claim 2: There is an open box $A' \subset A$ such that $f|A'$ is continuous.

Take an open box $B \times (a,b)$ contained in A. For every $x \in B$, let

$$\lambda(x) = \inf\{y \in (a,b) : f(x,\cdot) \text{ is continuous and monotone on } (a,y)\}$$

The function λ is well-defined and definable. The Motonicity implies $\lambda(x) > a$ for all $x \in B$. Applying (II_{n-1}) to λ and replacing B with a smaller open box, we can assume that λ is continuous, and there is $c > a$ such that $\lambda > c$. Replacing b with c, we can assume that for every $x \in B$, $f(x,\cdot)$ is continuous and monotone on (a,b). Now consider the set C of points $(x,y) \in B \times (a,b)$ such that $f(\cdot,y)$ is continuous at x. The set C is definable. It follows from (II_{n-1}) that for every $y \in (a,b)$ the set x such that $f(\cdot,y)$ is continuous at x is dense in B. Hence, C is dense in A. Applying (I_n), we deduce that C contains an open box of A. Replacing A with this box, we can assume that for every $y \in (a,b)$, $f(\cdot,y)$ is continuous. So the claim is followed by Claim 1.

Now we prove (II_n). Let D be the set of discotinuity of f. Then D is definable. By (I_n), there is a cell decomposition $\mathcal{P}_1$ of $\mathbb{R}^n$ compatible with $\{A, D\}$. Let $C \in \mathcal{P}_1$, $C \subset A$. We dishtinguish 2 cases.
Case 1: C is an open cell. the claims show that $C \cap D = \emptyset$, i.e. $f|C$ is continuous.

Case 2: C is not an open cell. Then there is a definable homeomorphism $p : C \to \mathbb{R}^d$. Applying (II_d) to $f \circ p^{-1}$, then applying (I_n), we obtain a finite partition of C into cells $C'_1, \cdots C'_{k(C)}$ such that $f|_{C'_i}$ is continuous for $i = 1, \cdots, k(C)$.
Let $\mathcal{P}$ be the cell decomposition of $\mathbb{R}^n$ consisting of:
• $C \in \mathcal{P}_1$ such that $C \cap A = \emptyset$,
• $C \in \mathcal{P}_1$ such that $C \subset A$ and C open,
• $C'_1, \cdots C'_{k(C)}$, where $C \in \mathcal{P}_1, C \subset A$ and C is not open.
Then $\mathcal{P}$ satisfies the demands of (II_n). □

THEOREM 2.11 (C^p smoothness). *Let $f : U \to \mathbb{R}$ be a definable function, with U open subset of $\mathbb{R}^n$. For $p \in \mathbb{N}$, let $C^p(f) = \{x \in U : f$ is of class C^p at $x\}$. Then $C^p(f)$ is a definable dense subset of U. In particular $U \setminus C^p(f)$ has empty interior.*

PROOF. Using Proposition 2.2, one can check that $C^p(f)$ is definable. We prove the density by induction on p. The case $p = 0$ is proved in Claim 2 of 2.10.3. By induction, it is sufficient to prove that for each $i \in \{1, \cdots, n\}$, the complement of the set where the partial derivative $\partial f/\partial x_i$ exists has empty interior. Otherwise, there would exist an open box where $\partial f/\partial x_i$ does not exist. Considering the restriction of f to an interval of a line parallel to the x_i axis contained in this box, we obtain a contradiction with Motonicity theorem. □

Note: For all presently known o-minimal structures on the real field Theorem 2.10 still holds true if we replace "C^p" by "analytic", i.e. $p = \omega$.

EXERCISE 2.12. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 definable function. Let $C = \{x \in \mathbb{R}^n : Df(x) = 0\}$. Prove that $f(C)$ is finite.

THEOREM 2.13 (Theorem on components). *Every definable set has only finitely many connected components and each component is also definable.*

PROOF. The proof follows from Theorem 2.10 (I_n). □

THEOREM 2.14 (Definable choice). *Let $A \subset \mathbb{R}^m \times \mathbb{R}^n$ be a definable set and let $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ be the projection on the first m coordinates. Then there exists a definable map $\rho : \pi(A) \to \mathbb{R}^m \times \mathbb{R}^n$ such that $\pi(\rho(x)) = x$, for all $x \in \pi(A)$.*

PROOF. It is sufficient to consider the case $n = 1$. Take a cell decomposition of $\mathbb{R}^{m+1}$ compatible with A. Then $\pi(A)$ is the union of the images by π of cells contained in A. Hence, we can assume that A is a cell, and consequently $\pi(A)$ is a cell. If A is the graph of $f : \pi(A) \to \mathbb{R}$, take $\rho = f$. If A is a band (f, g), then if f, g are bounded, take $\rho = \frac{1}{2}(f + g)$, if f is bounded, $g = +\infty$, take $\rho = f + 1$, and if $f = -\infty$, g is bounded, take $\rho = g - 1$. □

THEOREM 2.15 (Curve selection). *Let A be a definable subset of $\mathbb{R}^n$, and $a \in \overline{A} \setminus A$. Let $p \in \mathbb{N}$. Then there exists a C^p definable curve $\gamma : (0, 1) \to A \setminus \{a\}$ such that $\lim_{t \to 0^+} \gamma(t) = a$.*

PROOF. Let $X = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : x \in A, 0 < \|x - a\| < t\}$. Let $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the natural projection. Since $a \in \overline{A} \setminus A$, we have $\pi(X) = \{t \in \mathbb{R} : t > 0\}$. Applying Definable choice and Monotonicity theorem, we find $\varepsilon > 0$ and a C^p definable map $\delta : (0, \varepsilon) \to A \setminus \{a\}$ such that $\|\delta(t) - a\| < \varepsilon$. Take $\gamma : (0, 1) \to \mathbb{R}^n$ defined by $\gamma(t) = \delta(t\varepsilon)$. □

Note: For sets definable in $\mathbb{R}_{\text{alg}}$ or $\mathbb{R}_{\text{an}}$, by Puiseux lemma, γ can be chosen to be analytic on $(-1,1)$. For sets in structure $\mathbb{R}_{\text{exp}}$ or $\mathbb{R}_{\text{an,exp}}$, the theorem holds true for analytic curve γ; but, in general, it cannot be analytically extended to 0 (e.g. $S = \{(x,y) : x > 0, y = \exp(-1/x)\}$).

The Curve selection replaces the use of sequences in many situations.

Exercise 2.16. Prove that a definable function $f : A \to \mathbb{R}$ is continuous if and only if for every continuous definable $\gamma : [0,1) \to A$, $\lim\limits_{t\to 0_+} f(\gamma(t)) = f(\gamma(0))$.

Exercise 2.17. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 definable function. Prove that if a is a regular or isolated singular point of f, then there exists $\varepsilon_0 > 0$ such that for every sphere S_ε centered at a with radius $\varepsilon < \varepsilon_0$, S_ε is transverse to the hypersurface $Z = \{x \in \mathbb{R}^n : f(x) = f(a)\}$, i.e. $\operatorname{grad} f(x)$ and $x - a$ are linearly independent for all $x \in Z \cap S_\varepsilon$.

Definition 2.18. Let $A \subset \mathbb{R}^n$ be a definable set. The dimension of A is defined by

$$\dim A = \sup\{\ \dim C : C \text{ is a } C^1\text{-submanifold contained in } A\}.$$

Theorem 2.19. (i) *If $A \subset B$ are definable sets, then $\dim A \le \dim B$.*
(ii) *If $A_1, \cdots, A_p$ are definable subsets of $\mathbb{R}^n$, then $\dim \cup_{i=1}^p A_i = \max\limits_{1\le i\le p} \dim A_i$.*
(iii) *Let $f : A \to \mathbb{R}^m$ is definable. If $\dim f^{-1}(y) \le k$, for all $y \in f(A)$, then*

$$\dim f(A) \le \dim A \le \dim f(A) + k.$$

(iv) *If A is a definable set, then $\dim(\overline{A}\setminus A) < \dim A$. In particular, $\dim \overline{A} = \dim A$.*

Exercise 2.20. Construct a surjective, continuous function $f : [0,1] \to [0,1]^2$. (Hint. Peano curves).

Exercise 2.21. Find an example of $A \subset \mathbb{R}^n$ such that $\dim(\overline{A} \setminus A) = \dim A$. (Hint. e.g. the oscillation $A = \{(x,y) \in \mathbb{R}^2 : x > 0, y = \sin\frac{1}{x}\}$).

Proof. (i) and (ii) are obvious.
(iii) Let $X = \Gamma(f) \subset \mathbb{R}^n \times \mathbb{R}^m$, and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ denote the natural projection. Then $\dim A = \dim X$, $\dim f(A) = \dim \pi(X)$, and $\dim f^{-1}(y) = \dim(\pi_2^{-1}(y) \cap X)$, for all $y \in f(A) = \pi(X)$. Therefore, it is sufficient to prove (iii) for $A := X$ and $f := \pi|_X$.
Let $C \subset X$ be a C^1-submanifold of dimension $\dim X$. Then the definable set $C_0 = \{x \in C : \operatorname{rank}_x \pi|_C \text{ is maximal }\}$ is open in C. By the rank theorem, each fiber $\pi|_{C_0}^{-1}(y)$, $y \in \pi(C_0)$, is a submanifold of dimension $\dim C_0 - \operatorname{rank} \pi|_{C_0}$. Therefore,

$$\dim X = \dim C_0 \le \dim \pi(X) + k.$$

On the other hand, by Cell decomposition, we can represent $X = \cup_{i=1}^p C_i$, where C_i is a C^1-cell, $\pi(C_i)$ is a C^1-cell, and $\pi|_{C_i}$ has constant rank, $i \in \{1, \cdots, p\}$. Then

$$\dim X = \max_{1\le i\le p} \dim C_i \ge \max_{1\le i\le p} \operatorname{rank} \pi|_{C_i} = \max_{1\le i\le p} \dim \pi(C_i) = \dim \pi(X).$$

(iv) We will prove that there exist $\varepsilon > 0$, and a definable subset U of $\overline{A} \setminus A$ with $\dim U = \dim(\overline{A} \setminus A)$, and a definable injective map $\gamma : U \times (0,\varepsilon) \to A$. Then, by (iii), we have $\dim(\overline{A} \setminus A) = \dim U = \dim \gamma(U \times (0,\varepsilon)) - 1 \le \dim A - 1$, and hence

$\dim(\overline{A} \setminus A) < \dim A$.
Using a locally definable homeomorphism, we reduce the proof of the existence of γ to the following lemma.

Lemma 2.22. *(Wing Lemma). Let $V \subset \mathbb{R}^k$ be a nonempty open definable set, and $A \subset \mathbb{R}^k \times \mathbb{R}^l$ be a definable set. Suppose $V \subset \overline{A} \setminus A$. Then there exist a nonempty open subset U of V, $\varepsilon > 0$, and a definable map $\gamma : U \times (0, \varepsilon) \to A$, such that $\gamma(y,t) = (y, \rho(y,t))$ and $\|\rho(y,t)\| = t$, for all $y \in U, t \in (0, \varepsilon)$.*

Proof Let $X = \{(y,x,t) : y \in V, x \in A, 0 < t < 1, \|x-y\| < t, \pi(x) = y\}$, where $\pi : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$ is the natural projection. Note that X is definable.
Let $\pi_2(y,x,t) = (y,t)$, and $\pi_2(X)_y = \{t : (t,y) \in \pi_2(X)\}$. Define

$$\epsilon(y) = \inf \pi_2(X)_y \ , \quad (\inf \emptyset := 1).$$

Then $\epsilon : V \to \mathbb{R}$ is definable, and if $\epsilon(y) > 0$ then $(0, \epsilon(y)) \cap \pi_2(X)_y = \emptyset$.

Claim: $\dim\{y \in V : \epsilon(y) > 0\} < \dim V = k$.
Suppose to the contrary that the dimension is k. Then, by Cell decomposition, there is an open ball $B \subset V$ and $c > 0$ such that $\epsilon > c$ on B. This implies $B \not\subset \overline{A} \setminus A$, a contradiction.
Now let $V_0 = \{y \in V : \epsilon(y) = 0\}$. Then $\dim V_0 = k$, and, by the definition, $V_0 \times (0,1) \subset \pi_2(X)$. By Definable choice and Cell decomposition, there exists an open set $V' \subset V_0$, $\delta > 0$, and a continuous definable map: $V' \times (0, \delta) \to A$, $(y,t) \mapsto (y, \theta(y,t))$. Let $\tau(y) = \sup_{0<s<\delta} \|\theta(y,s)\|$. Then for $y \in V'$, $t < \tau(y)$, there exists $x \in A$, such that $\pi(x) = y$ and $\|x - y\| = t$. Again by Definable choice and Cell decomposition it is easy to prove the existence of the ε, U, γ satisfying the demands of the lemma. $\square$

3. Globalization

The notions of definable sets and their properties can be globalized in a natural way to arbitrary analytic manifolds. Here we sketch the idea due to van den Dries and Miller [**DM2**] (c.f. [**S2**]).

Definition 3.1. We say that an *analytic-geometric category* $\mathcal{C}$ is given if each manifold M (= real analytic, Hausdorff manifold with a countable basis for its topology) is equipped with a collection $\mathcal{C}(M)$ of subsets of M such that the following conditions are satisfied for all manifolds M and N:

(AG1) $\mathcal{C}(M)$ is a boolean algebra of subsets of M, with $M \in \mathcal{C}(M)$.
(AG2) If $A \in \mathcal{C}(M)$, then $A \times \mathbb{R} \in \mathcal{C}(M \times \mathbb{R})$.
(AG3) If $f : M \to N$ is a proper analytic map and $A \in \mathcal{C}(M)$, then $f(A) \in \mathcal{C}(N)$.
(AG4) If $A \in \mathcal{C}(M)$ and $(U_i)_{i\in I}$ is an open covering of M, then $A \in \mathcal{C}(M)$ if and only if $A \cap U_i \in \mathcal{C}(U_i)$, for all $i \in I$.
(AG5) Every bounded set in $\mathcal{C}(\mathbb{R})$ has finite boundary.

It is proved in [**DM2**] that $\mathcal{C}$ is a category with its objects being pairs (A, M), where M is a manifold and $A \in \mathcal{C}(M)$, and its morphisms $(A, M) \to (B, N)$ being maps $f : A \to B$ whose graphs belong to $\mathcal{C}(M \times N)$.

Definition 3.2. Let $\mathcal{D}$ be an o-minimal structure on $\mathbb{R}_{\text{an}}$. One can construct an analytic-geometry category $\mathcal{C}$ by defining the collection $\mathcal{C}(M)$ in a manifold M to be those set $A \subset M$ such that for each $x \in M$, there exist an open neighborhood

U of x, an open set $V \subset \mathbb{R}^n$, and an analytic homeomorphism $h : U \to V$ such that $h(A \cap U) \in \mathcal{D}_n$.

From the definition it follows that the category $\mathcal{C}_{\rm an}$ of subanalytic sets and continuous subanlytic maps is the smallest analytic-geometric category corresponding to the structure $\mathbb{R}_{\rm an}$.

3.3 Properties of analytic-geometry categories. Results in Section 2 can be translated to the setting of analytic-geometric categories. Perhaps the only thing one has to change is to replace "finite" by "locally finite".

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