Lecture 3: Three fundamental theorems of Singularity theory in o-minimal structures

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Introduction

This note is devoted to the study of three fundamental theorems of Singularity theory in o-minimal structures: Morse-Sard's theorem, density of Morse functions on definable sets, and Transversality theorem. The main results come from [L1] and [L2].

Morse-Sard's theorem in o-minimal structures is proved in Section 1. There is no restriction on differentiability class and the dimensions of manifolds involved. Note that, in general case, for differentiable mappings $f : N \to M$ of class C^p, Morse-Sard's theorem requires $p > \max(\dim N - \dim M,\ 0\ )$. See [**Wh**] and [**Y**] for examples that the theorem does not hold for mappings of low smoothness.

In Morse theory it is proved that the topological shape of a space can be described via data given by Morse functions defined on the space. For Morse theory of compact smooth manifolds we refer readers to the book by Milnor [**Mi**], for Morse theory of singular spaces we refer to the book by Goresky and MacPherson [**GM**]. [**GM**] proves the density and openness of Morse functions on closed Whitney stratified subanalytic sets in the space of smooth functions endowed with Whitney topology (see also the contributions by the following authors: [**Mo**], [**Mi**], [**La**], [**Be**], [**P**], [**O**] and [**Br**]). In Section 2 we present similar results for definable sets in o-minimal structures. The proofs are based on Sard's theorem in o-minimal context. Note that the spiral $\{(x,y) \in \mathbb{R}^2 : x = e^{-\varphi^2}\cos\varphi, y = e^{-\varphi^2}\sin\varphi, \varphi \geq 0\} \cup \{(0,0)\}$ or the oscillation $\{(x,y) \in \mathbb{R}^2 : y = x\sin\dfrac{1}{x}, x > 0\} \cup \{(0,0)\}$ has no Morse functions, even though the first one is a closed Whitney stratified set (see Remark 2.4). Therefore, in some sense, our results show a tameness property of definable sets.

In Section 3 we present Thom's transversality theorem for maps and sets definable in o-minimal structures. Whitney's paper [**Wh**] provides examples of functions which are nonconstant on a connected set of critical points (see also examples in [**Y**]). Therefore, due to Morse-Sard's theorem, in the general case (see [**Le**]) and also in the $\mathcal{X}$-version of the theorem given by Shiota [**S2**], there are some restrictions on differentiability class and the dimensions of manifolds involved. For the restrictions in the general case we refer the readers to [Le, 7. Th.1] and [S2. Th.

II.5.4 (3)]. In o-minimal structures, however, the theorem holds for any C^1 definable submanifolds of any dimensions of the jet spaces. This can be seen as an example of the tamenes of ominimality. Our proof is quite elementary, using some standard arguments of Singularity theory in the o-minimal setting, and a tricky computing of rank of Jacobian matrix.

In this note we fix an o-minimal structure on $(\mathbb{R}, +, \cdot)$. "Definable" means definable in this structure. Let p be a positive integer.

Acknowledgements. This lecture is partially supported by the Grant-in-Aid for Scientific Research (No. 20540075) of Ministry of Education, Science and Culture of Japan, and HEM 21 Invitation Fellowship Programs for Research in Hyogo, and Vietnam's National Foundation for Science and Technology Development (NAFOSTED).

1. Morse-Sard's theorem

Definition 1.1 (Definable Whitney topology). Let N, M be C^p definable submanifolds of $\mathbb{R}^n, \mathbb{R}^m$, respectively. Let $\mathcal{D}^p(N, M)$ denote the set of C^p definable mappings from N to M. On this space the definable Whitney topology is defined as follows (see [**E**] or [**S1**]).
First consider the case $M = \mathbb{R}$. Let $v_1, \cdots, v_s$ be C^{p-1} definable vector fields on N, such that $v_1(x), \cdots, v_s(x)$ span the tangent space T_xN of N at each $x \in N$. For each $f \in \mathcal{D}^p(N, \mathbb{R})$, and positive continuous definable function ε on N, the ε-neighborhood of f in this topology is defined by

$$U_\varepsilon(f) = \{g \in \mathcal{D}^p(N, \mathbb{R}) : |v_{i_1} \cdots v_{i_k}(g - f)| < \varepsilon, 1 \le k \le p, 1 \le i_1, \cdots, i_k \le s\},$$

where vf is the derivative of f along v, i.e. $vf(x) = df(x)(v(x))$.
Note that this topology does not depend on the choice of $v_1, \cdots, v_s$.
The topology on $\mathcal{D}^p(N, \mathbb{R}^m) = \mathcal{D}^p(N, \mathbb{R}) \times \cdots \times \mathcal{D}^p(N, \mathbb{R})$ is the product topology.
For the general case, $\mathcal{D}^p(N, M)$ is a subspace of $\mathcal{D}^p(N, \mathbb{R}^m)$ with induced topology.
In this topology we have the following propositions which are proved in [**E**]:

Proposition 1.2. *Let T_N be a definable open neighborhood of N in $\mathbb{R}^n$. Then the restriction map*

$$\mathcal{D}^p(T_N, M) \to \mathcal{D}^p(N, M), \ f \mapsto f|_N$$

is continuous.

Proposition 1.3. *Let T_M be a C^p definable submanifolds of $\mathbb{R}^m$. Let $\pi : M \to T_M$ be a C^p definable mapping. Then the mapping*

$$\mathcal{D}^p(N, T_M) \to \mathcal{D}^p(N, M), \ f \mapsto \pi \circ f$$

is continuous.

Theorem 1.4 (Morse-Sard's theorem). *Let N and M be C^p definable manifolds, and $f : N \to M$ be a C^p definable map. For each $s \in \mathbb{N}$, let*

$$\Sigma_s(f) = \{x \in N : \operatorname{rank} df(x) < s\} \ \text{ and } \ C_s(f) = f(\Sigma_s(f)).$$

Then $C_s(f)$ is definable and $\dim C_s(f) < s$.

Proof: c.f. [W, Thm. 2.7.] It is easy to see that $C_s(f)$ is definable. To prove the second part we suppose, contrary to the assertion, that $\dim C_s(f)) \geq s$. Then, by Definable Choice, there exist a definable subset U of $C_s(f)$ of dimension $\geq s$ and a definable C^p mapping $g : U \to \Sigma_s(f)$ such that $f \circ g = id_U$. So $\operatorname{rank} df(g(y))dg(y) \geq s$, for all $y \in U$. Hence $\operatorname{rank} df(x) \geq s$, for all $x \in g(U)$. This is a contradiction. □

2. Density of Morse functions on definable sets

Definition 2.1 (Tangents to definable sets). Let X be a definable subset of $\mathbb{R}^n$. Let $\mathcal{S}$ be a definable C^p Whitney stratification of X. Note that if S is a definable submanifold of $\mathbb{R}^n$, then the tangent bundle TS and the cotangent bundle T^*S are definable submanifolds of $T\mathbb{R}^n$ and $T^*\mathbb{R}^n$, respectively.
For $S \in \mathcal{S}$, the *conormal bundle* of S in $\mathbb{R}^n$ is defined by

$$T^*_S\mathbb{R}^n = \{(\xi, x) \in (\mathbb{R}^n)^* \times S : \xi|_{T_xS} = 0\}.$$

Note that $T^*_S\mathbb{R}^n$ is a definable submanifold of $T^*\mathbb{R}^n$ of dimension n.
A *generalized tangent space* Q at $x \in S$ is any plane of the form

$$Q = \lim_{y \to x} T_yR,$$

where $R \in \mathcal{S}$ and $S \subset \overline{R}$.
The cotangent vector (ξ, x) is *degenerate* if there exists a generalized tangent space Q at x, $Q \neq T_xS$ such that $\xi|_Q = 0$.

Proposition 2.2. *The set of degenerate cotangent vectors which are conormal to S is a conical definable set of dimension $\leq n-1$.*

Proof. Let R be a stratum in $\mathcal{S}$ with $S \subset \overline{R} \setminus R$, and $\dim R = r$. Consider the mapping

$$g : R \to G_r(\mathbb{R}^n), \text{ defined by } g(x) = T_xR,$$

where $G_r(\mathbb{R}^n)$ denotes the Grassmannian of the r-dimensional vector subspaces of $\mathbb{R}^n$.
The graph g of this mapping is a definable set of dimension r. So its closure $\overline{g}$ in $\mathbb{R}^n \times G_r(\mathbb{R}^n)$ is a definable set, and hence $\dim(\overline{g} \setminus g) \leq r-1$.
Let

$$A_R = \{(\xi, x, Q) \in T^*_S\mathbb{R}^n \times G_r(\mathbb{R}^n) : (x, Q) \in \overline{g} \setminus g, \xi|_Q = 0.\}$$

Then A_R is definable. For each $(x, Q) \in \overline{g} \setminus g$ the fiber $A_R \cap (\mathbb{R}^n)^* \times (x, Q)$ has dimension $\leq n-r$. Hence, $\dim A_R \leq \dim(\overline{g} \setminus g) + (n-r) = n-1$.
Since there are a finite number of strata R in $\mathcal{S}$ such that $S \subset \overline{R} \setminus R$, the set of degenerate cotangent vectors which are conormal to S is of dimension $\leq n-1$. □

Definition 2.3 (Morse functions on stratified sets). (c.f. [**Be**], [**La**], [**P**] and [**GM**]). Let X be a subset of $\mathbb{R}^n$ and $\mathcal{S}$ be a C^p Whitney stratification of X.
A *Morse function* f on X is the restriction of a C^p function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions:

(M1) For each $S \in \mathcal{S}$, the critical points of $f|_S$ are nondegenerate, i.e. if $\dim S \geq 1$, then the Hessian of $f|_S$ at each critical point is nonsingular.
(M2) For every critical point $x \in S$ of $f|_S$, and for each generalized tangent space Q at x with $Q \neq T_xS$, $d\tilde{f}(x)|_Q \neq 0$, i.e. $d\tilde{f}(x)$ is not a degenerate cotangent vector (where $d\tilde{f}(x)$ denotes the derivative of $\tilde{f}$ at x).

REMARK 2.4. The property of being a Morse function depends strictly on the particular stratification of X. For example, every zero dimensional stratum is a critical point of f. We also note that all lines through $(0,0)$ are generalized tangent spaces of the spiral as well as of the oscillation mentioned at the beginning of this note. So there are no Morse functions on these sets.

REMARK 2.5. From the definition, one can check that if $\mathcal{S}'$ is a definable C^p Whitney stratification of X compatible with $\mathcal{S}$, and Z is a union of strata of $\mathcal{S}$, and f is a Morse function on $(X,\mathcal{S}')$, then f is Morse on $(X,\mathcal{S})$ and $(Z,\mathcal{S}|_Z)$.

Throughout this section, let X be a definable closed subset of $\mathbb{R}^n$, which is endowed with a definable C^p Whitney stratification $\mathcal{S}$.

Let T be a definable C^p manifold. Let $F : T\times\mathbb{R}^n \to \mathbb{R}$, $F(t,x) = f_t(x)$ be a definable C^p function. Define $\Phi : T\times\mathbb{R}^n \to T^*\mathbb{R}^n$ by $\Phi(t,x) = (df_t(x), x)$. Consider the set of "Morse parameters" $M(F,X) = \{t\in T : f_t|_X \text{ is a Morse function}\}$. Note that $M(F,X)$ is a definable set.

THEOREM 2.6 (Parameter version of the density for Morse functions). *If Φ is a submersion, then $M(F,X)$ is an open subset of T and* $\dim(T\setminus M(F,X)) < \dim T$.

PROOF. For each $S\in\mathcal{S}$, consider the following sets

$M_1 = M_1(S) = \{t\in T : f_t|_S \text{ has nondegenerate critical points}\}$, and

$M_2 = M_2(S) = \{t\in T : df_t(x) \text{ is a nondegenerate covector for each } x\in S\}$.

It is easy to check that M_1 and M_2 are definable sets.
Now we claim that $\dim(T\setminus M_1\cap M_2) < \dim T$. To prove the claim, let

$$D = D(S) = \{(\xi,x)\in T^*_S\mathbb{R}^n : \xi \text{ is a degenerate cotangent vector at } x\}.$$

Then D is a definable set. Let $\Phi_S : T\times S\to T^*S, \Phi_S(t,x) = ((df_t|S)(x), x)$, and $\pi : T\times\mathbb{R}^n\to T$ be the natural projection. Since Φ is submersive, Φ_S is transverse to the zero section S of T^*S. So the set $V_1 = \Phi_S^{-1}(S)$ is a definable submanifold of $T\times S$. Furthermore, $t\in M_1$ if and only if t is not a critical value of $\pi|_{V_1}$. By Morse-Sard's theorem, $\dim(T\setminus M_1) < \dim T$.
On the other hand, Φ is transverse to each stratum of any Whitney stratification of D, and by Proposition 2.2, $\dim D\le n-1$, the set $V_2 = \Phi^{-1}(D)$ is a definable set of dimension $\le \dim T - 1$. So $\dim(T\setminus M_2) = \dim\pi(V_2) \le \dim T - 1$.
Since the collection $\mathcal{S}$ is finite, the claim implies $\dim(T\setminus M(F,X)) < \dim T$.
Openness of $M(F,X)$ follows from the second part of Theorem 2.9 below. $\square$

COROLLARY 2.7. *Consider the square of distance function*

$$F : \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R},\quad F(t,x) = \|t-x\|^2.$$

Let $M = \{t\in\mathbb{R}^n : F(t,.) \text{ is a Morse function on } X\}$. Then M is definable, open and dense in $\mathbb{R}^n$.

COROLLARY 2.8. *Let $f : \mathbb{R}^n\to\mathbb{R}$ be a definable C^p function. Consider the linear deformations of f: $f+L$, where L is a linear form of $\mathbb{R}^n$. Let*

$$M = \{L\in L(\mathbb{R}^n,\mathbb{R}) :\ f+L \text{ is a Morse function on } X\}.$$

Then M is definable, open and dense in $L(\mathbb{R}^n,\mathbb{R})$.

THEOREM 2.9. *The set of definable C^p functions on $\mathbb{R}^n$ which are Morse on X and have distinct critical values is dense and open in $\mathcal{D}^p(\mathbb{R}^n)$.*

Before proving the theorem, we prepare some lemmas.

Let $d(\cdot,\cdot)$ and $\|\cdot\|$ denote the distance and the norm on $\mathbb{R}^n$ induced by the Euclidean inner product, respectively. Let $L(\mathbb{R}^n,\mathbb{R})$ denote the space of linear forms on $\mathbb{R}^n$, and $L_2(\mathbb{R}^n,\mathbb{R})$ denote the space of bilinear forms on $\mathbb{R}^n$. For $L \in L(\mathbb{R}^n,\mathbb{R})$, $B \in L_2(\mathbb{R}^n,\mathbb{R})$, and $T \in G_k(\mathbb{R}^n)$, as usual, we define

$$\begin{array}{rcl} \|L|_T\| & = & \sup\{|L(v)| : v \in T, \|v\| = 1\}, \\ \|B|_T\| & = & \sup\{|B(u,v)| : u, v \in T, \|u\| = \|v\| = 1\}, \end{array}$$

and $\det B|_T$ to be the determinant of the matrix representation of $B|_T$ with respect to an orthonormal basis of T.
By the definition we have

LEMMA 2.10. *The following mappings:*

$$\begin{array}{rcl} L(\mathbb{R}^n,\mathbb{R}) \times G_k(\mathbb{R}^n) & \to & \mathbb{R}, \quad (L,T) \mapsto \|L|_T\|, \\ L_2(\mathbb{R}^n,\mathbb{R}) \times G_k(\mathbb{R}^n) & \to & \mathbb{R}, \quad (B,T) \mapsto \|B|_T\|, \\ L_2(\mathbb{R}^n,\mathbb{R}) \times G_k(\mathbb{R}^n) & \to & \mathbb{R}, \quad (B,T) \mapsto \det B|_T, \end{array}$$

are continuous and semialgebraic.

Let $f \in \mathcal{D}^p(\mathbb{R}^n)$. To test the Morsity of $f|_S$ at $x \in S \in \mathcal{S}$, we define

$$m_{f,S}(x) = \|df(x)|_{T_xS}\| + |\det d^2 f(x)|_{T_xS}| \frac{d(x,\partial S)}{1 + d(x,\partial S)},$$

where $d^2f(x)$ is the second derivative of f at x, $\partial S = \overline{S} \setminus S$, and $d(x,\emptyset) = 1$.
Note that, by Lemma 2.10, $m_{f,S}$ is a continuous definable function on S, and $f|_S$ is Morse at x if and only if $m_{f,S}(x) > 0$. In general, $m_{f,S}$ can not be continuously extended to the closure $\overline{S}$. Instead, for a Morse function f, $m_{f,S}$ is bounded from below by the restriction of a positive continuous function on $\mathbb{R}^n$, constructed as follows.

LEMMA 2.11. *Let $f \in \mathcal{D}^p(\mathbb{R}^n)$. Then f is a Morse function on X if and only if there exists a positive continuous definable function m_f on $\mathbb{R}^n$, such that for each $S \in \mathcal{S}$, $m_f(x) \le m_{f,S}(x), \forall x \in S$.*

PROOF. Assume f is Morse on X. To construct m_f, we imitate the arguments of the proof of Lemma 6.12. in [**C**].
For each $S \in \mathcal{S}$, let

$$\mu(r) = \inf\{m_{f,S}(x) : x \in S, \|x\| \le r\}.$$

Then $\mu(r)$ is defined when $r \ge r_0$, for some $r_0 > 0$. So $\mu : [r_0, +\infty) \to \mathbb{R}$ is a positive definable nonincreasing function. To prove $\mu(r) > 0$, suppose to the contrary that $\mu(r) = 0$. Then there exists a sequence (x_k) in S, $\|x_k\| \le r$, and $m_{f,S}(x_k) \to 0$. By the boundedness, taking subsequence if necessary, we can assume that $x_k \to x \in \overline{S}$ and $T_{x_k}S \to Q$. This implies

$$\|df(x)|_Q\| + |\det d^2 f(x)|_Q| \frac{d(x,\partial S)}{1 + d(x,\partial S)} = 0.$$

If $x \in S$, then $Q = T_xS$ and hence the above equality contradicts condition M1.
If $x \in \partial S$, then Q is a generalized tangent space. Since X is closed and the strata

of $\mathcal{S}$ satisfy Whitney condition (a), the above equality contradicts condition M2.
By Monotonicity theorem, there exists $a \geq r_0$ such that μ is continuous on $[a, +\infty)$.
Let $\theta : \mathbb{R} \to [0,1]$ be a continuous nondecreasing definable function such that $\theta = 0$ on $(-\infty, a]$, $\theta = 1$ on $[a+1, +\infty)$. Define $\phi_S : \mathbb{R}^n \to \mathbb{R}$ by $\phi_S(x) = \theta(\|x\|)\mu(\|x\|) + (1-\theta(\|x\|))\mu(a+1)$. Then ϕ_S is positive, continuous, definable and by construction $\phi_S \leq m_{f,S}$ on S.
Define $m_f = \min\{\phi_S : S \in \mathcal{S}\}$. Then m_f has the desired properties.
Now let us assume conversely that m_f is a positive continuous definable function on $\mathbb{R}^n$, such that for each $S \in \mathcal{S}$, $m_f(x) \leq m_{f,S}(x), \forall x \in S$. Then $m_{f,S}(x) > 0$, and hence $f|_S$ satisfies condition M1. On the other hand, if a sequence of points $x_k \in S$ converges to $y \in \partial S$, and $T_{x_k}S$ converges to Q, then the above inequality implies $\|df(y)|_Q\| \geq m_f(y) > 0$, and hence f satisfies condition M2.
Therefore, f is Morse on X. $\square$

Roughly speaking, the following lemma says that in the (ϵ, δ)-formulation of continuity, δ can be chosen to be continuously dependent on ϵ and the variables.

Lemma 2.12. *Let $\psi : F \to \mathbb{R}$ be a continuous definable function. Suppose F is a closed subset of $\mathbb{R}^n$. Then there exists a positive continuous definable function $\delta : \mathbb{R}_+ \times F \to \mathbb{R}$ satisfying:*

$$x' \in F, \|x' - x\| < \delta(\epsilon, x) \ \Rightarrow \ |\psi(x') - \psi(x)| < \epsilon.$$

Proof. Let

$$A = \{(\epsilon, x, \delta) : \epsilon > 0, x \in F, \delta > 0 (\forall x' \in F, \|x' - x\| < \delta \Rightarrow |\psi(x') - \psi(x)| < \epsilon)\}.$$

Then A is a definable set. For each $(\epsilon, x) \in \mathbb{R}_+ \times F$, define

$$\overline{\delta}(\epsilon, x) = \min\{\sup\{\delta : (\epsilon, x, \delta) \in A\}, 1\}.$$

Since ψ is a continuous definable function, $\overline{\delta}$ is well-defined, definable and positive on $\mathbb{R}_+ \times F$. For $r > 0$, define

$$\mu(r) = \frac{1}{2} \inf\{\overline{\delta}(\epsilon, x) : \epsilon \geq \frac{1}{r}, x \in F, \|x\| \leq r\}.$$

Then $\mu : (0, +\infty) \to \mathbb{R}$ is a definable nonincreasing function. Moreover, $\mu(r) > 0$, for all $r > 0$. Indeed, by the uniform continuity of ψ on $\{x \in F : \|x\| \leq r+1\}$, there exists $\delta_0 \in (0,1)$, such that if $x, x' \in F, \|x\| \leq r+1, \|x'\| \leq r+1$, and $\|x - x'\| < \delta_0$, then $|\psi(x) - \psi(x')| < \frac{1}{r}$. So, by the definition of $\overline{\delta}$, if $\epsilon \geq \frac{1}{r}, x \in F$ and $\|x\| \leq r$, then $\overline{\delta}(\epsilon, x) \geq \overline{\delta}(\frac{1}{r}, x) \geq \delta_0 > 0$. Therefore, $\mu(r) \geq \frac{\delta_0}{2} > 0$.
Repeat the arguments of the proof of Lemma 2.11 for this μ, keep the notations there, and then define $\delta : \mathbb{R}_+ \times F \to \mathbb{R}$, by

$$\delta(\epsilon, x) = \theta(\max(\frac{1}{\epsilon}, \|x\|))\mu(\max(\frac{1}{\epsilon}, \|x\|)) + (1 - \theta(\max(\frac{1}{\epsilon}, \|x\|)))\mu(a+1).$$

It is easy to check that δ has the desired properties. $\square$

Lemma 2.13. *Let U be an open definable subset of $\mathbb{R}^n$. Let $\varepsilon : U \to \mathbb{R}$ be a positive continuous definable function. Then there exists a positive definable C^p function $\varphi : U \to \mathbb{R}$, such that*

$$|\partial^{\alpha} \varphi| < \varepsilon, \ \forall |\alpha| \leq p.$$

Proof. For $n = 1$: By finiteness of the number of the connected components of U, we can assume $U = (a, b)$. Moreover, by Cell Decomposition, ε can be smoothened at unsmooth points in an elementary way. So we reduced to the case that ε is of class C^p.

If $\lim\limits_{t\to a^+} \varepsilon(t) > 0$ and $\lim\limits_{t\to b^-} \varepsilon(t) > 0$, then take $\varphi = c$, where c is a constant, $0 < c < \min\limits_{t\in(a,b)} \varepsilon(t)$.

If $\lim\limits_{t\to b^-} \varepsilon(t) = 0$, then by Motonicity, ε' is strictly increasing on a neighborhood of b, and tends to 0 at b. Repeating the previous argument for $\varepsilon'', \cdots, \varepsilon^{(p)}$, we get $a < b' < b$, such that $\varepsilon, \varepsilon', \cdots, \varepsilon^{(p)}$ are strictly monotone on (b', b), tend to 0 at b, and $|\varepsilon(t)| < 1, \cdots, |\varepsilon^{(p)}(t)| < 1$, $\forall t > b'$. Similarly for the case $\lim\limits_{t\to a^+} \varepsilon(t) = 0$, the above conditions satisfy for ε on (a, a'), with $a < a' < b$. If we take a constant M big enough, then $\varphi = \dfrac{\varepsilon^{p+1}}{M}$ has the desired properties.

For general n: Let $\varepsilon_1(x) = \dfrac{\varepsilon(x)}{N(1 + \|x\|^p)}$, where N is a positive number. Fix $x_0 \in U$. For $t \in \mathbb{R}$, let

$$\alpha(t) = \min\{\varepsilon_1(x) : \|x - x_0\|^2 \le t^2, d(x, \partial U) \ge \frac{d(x_0, \partial U)}{1 + t^2}\}.$$

Applying the case $n = 1$, we have a positive definable C^p function $g : \mathbb{R} \to \mathbb{R}$, such that $|g^{(k)}| < \alpha, \forall k \in \{0, \cdots, p\}$. Now let $\varphi(x) = g(\|x\|^2)$. Then by the chain rule, when N is big enough, we have $|\partial^\alpha \varphi(x)| < N\varepsilon_1(x)\|x\|^{|\alpha|} < \varepsilon(x)$, $\forall x \in U, \forall |\alpha| < p$. □

Proof of Theorem 2.9. We divide the proof into two parts: density and openness.

Density: Let $f \in \mathcal{D}^p(\mathbb{R}^n)$, and $\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be a positive continuous definable function. We will find a Morse function on X in the ε- neighborhood of f.

Let $N = 1 + n2^p$. By Lemma 2.13, there exists a positive definable C^p function $\varphi : \mathbb{R}^n \to \mathbb{R}$, such that $|\partial^\alpha \varphi(x)| < \dfrac{\varepsilon(x)}{N(1 + \|x\|)}, \forall |\alpha| \le p$.

Consider the following family

$$F : I^{n+1} \times \mathbb{R}^n \to \mathbb{R},\ F(t, x) = f_t(x) = f(x) + t_0\varphi(x) + \sum_{i=1}^{n} t_i x_i \varphi(x),$$

where $I = (-1, 1)$, $t = (t_0, t_1, \cdots, t_n)$, and $x = (x_1, \cdots, x_n)$.

To apply Theorem 2.6, we check that

$$\Phi(t, x) = \left(\sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}(x) + t_0 \frac{\partial \varphi}{\partial x_j}(x) + t_j \varphi(x) + \sum_{i=1}^{n} t_i x_i \frac{\partial \varphi}{\partial x_j}(x) \right) dx_j, x \right)$$

is submersive. Indeed, since $\varphi(x) \neq 0$, the rank of the Jacobian $J\Phi(t,x) =$

$$\begin{pmatrix} O & O & O & \cdots & O & I_n \\ \dfrac{\partial \varphi}{\partial x_1}(x) & \varphi(x) + x_1\dfrac{\partial \varphi}{\partial x_1}(x) & x_2\dfrac{\partial \varphi}{\partial x_1}(x) & \cdots & x_n\dfrac{\partial \varphi}{\partial x_1}(x) & * \\ \dfrac{\partial \varphi}{\partial x_2}(x) & x_1\dfrac{\partial \varphi}{\partial x_2}(x) & \varphi(x) + x_2\dfrac{\partial \varphi}{\partial x_2}(x) & \cdots & x_n\dfrac{\partial \varphi}{\partial x_2}(x) & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \dfrac{\partial \varphi}{\partial x_n}(x) & x_1\dfrac{\partial \varphi}{\partial x_n}(x) & x_2\dfrac{\partial \varphi}{\partial x_n}(x) & \cdots & \varphi(x) + x_n\dfrac{\partial \varphi}{\partial x_n}(x) & * \end{pmatrix}$$

is $n + \operatorname{rank}(\varphi'(x), \varphi(x)e_1 + x_1\varphi'(x), \varphi(x)e_2 + x_2\varphi'(x), \cdots, \varphi(x)e_n + x_n\varphi'(x)) = 2n$, (where $e_1, \cdots, e_n$ denote the standard basis of $\mathbb{R}^n$).
So the set $\{t \in I^{n+1} : f_t \text{ is Morse on } X\}$ is dense in I^{n+1}.
On the other hand, by Leibniz's rule, it is easy to see that for each $\alpha \in \mathbb{N}^n$, $|\alpha| \le p$, we have

$$|\partial^\alpha(f_t - f)(x)| \le |t_0||\partial^\alpha\varphi(x)| + \sum_{i=1}^n |t_i||\partial^\alpha(x_i\varphi)(x)| < (1 + n2^p)\frac{\varepsilon(x)}{N} = \varepsilon(x).$$

Therefore, there exists $t \in I^{n+1}$, such that f_t is a Morse function on X in the ε-neighborhood of f in $\mathcal{D}^p(\mathbb{R}^n)$.
To get a Morse function with distinct critical values, we construct it as follows. Suppose f is a Morse function on X. For each $S \in \mathcal{S}$, the set of critical points of $f|_S$ is finite, because it is definable and discrete. So f has only finitely many critical points on X. Let $x_1, \cdots, x_q$ be the critical points of $f|_S$, of all S in $\mathcal{S}$. Let $r > 0$ be small enough so that the balls $B(x_i, r)$, $i = 1, \cdots, q$, are disjoint. For $i = 1, \cdots, q$, choose a definable C^p function $\lambda_i : \mathbb{R}^n \to [0,1]$, such that $\lambda_i = 0$ on $\mathbb{R}^n \setminus B(x_i, r)$, and $\lambda_i(x) = 1$ on $B(x_i, \frac{r}{2})$. Consider the approximations of f of the form:

$$g = f + c_1\lambda_1 + \cdots + c_q\lambda_q.$$

Then in any neighborhood U of f, we can choose $c_1, \cdots, c_q$ so that $g \in U$, g is a Morse function on X with the set of critical points being $\{x_1, \cdots, x_q\}$, and $g(x_i) \neq g(x_j)$, when $i \neq j$. This completes the proof of the density part.
Openness: Let $f \in \mathcal{D}^p(\mathbb{R}^n)$ be a Morse function on X with distinct critical values. We will find a neighborhood of f that contains only Morse functions on X with distinct critical values.
By Lemma 2.11, there is a positive continuous definable function $m_f : \mathbb{R}^n \to \mathbb{R}$, such that for each $S \in \mathcal{S}$

$$m_f(x) \le \|df(x)|_{T_xS}\| + |\det d^2f(x)|_{T_xS}|\frac{d(x, \partial S)}{1 + d(x, \partial S)}, \forall x \in S.$$

Define $\varepsilon_X = \frac{1}{3}m_f$.
By Lemma 2.10 and 2.12, there exists a positive continuous semialgebraic function $\delta : \mathbb{R}_+ \times L_2(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$, satisfying the following condition for all $k \in \{0, \cdots, n\}$ and all $T \in G_k(\mathbb{R}^n)$:

$$B' \in L_2(\mathbb{R}^n, \mathbb{R}), \|B - B'\| < \delta(\epsilon, B) \ \Rightarrow\ |\det B|_T - \det B'|_T| < \epsilon.$$

Let $\varepsilon = \min\{\varepsilon_X, \delta(\varepsilon_X, d^2f)\}$. Then ε is a positive continuous definable function on $\mathbb{R}^n$. By the construction of ε, if $g \in \mathcal{D}^p(\mathbb{R}^n)$, $\|dg - df\| < \varepsilon$ and $\|d^2g - d^2f\| < \varepsilon$, then for each $x \in S \in \mathcal{S}$,

$$m_{g,S}(x) = \|dg(x)|_{T_xS}\| + |\det d^2g(x)|_{T_xS}| \frac{d(x,\partial S)}{1+d(x,\partial S)} > m_{f,S}(x) - 2\varepsilon_X(x) \geq \varepsilon(x).$$

So, by Lemma 2.11, g is a Morse function on X.
Moreover, since f has only a finite number of critical points and takes distinct values at them, we can reduce ε so that if $g \in \mathcal{D}^p(\mathbb{R}^n)$, $|g-f| < \varepsilon$, $\|dg - df\| < \varepsilon$, and $\|d^2g - d^2f\| < \varepsilon$, then g is Morse on X, the set of critical points of $g|_X$ is close to that of f, and g still has distinct critical values. We have constructed a neighborhood of f in $\mathcal{D}^p(\mathbb{R}^n)$ containing only Morse functions on X with distinct critical values. □

To apply Morse theory to definable sets, one needs the following corollary:

COROLLARY 2.14. *There exists a definable C^p Morse function on X which is proper and has distinct critical values.*

PROOF. By Corollary 2.7, there exists a definable C^p Morse function f on X which is proper. An approximation of f which has distinct critical values is constructed in the proof of Theorem 2.7. □

Using the same arguments as in [**P**, Th.2] one obtains:

COROLLARY 2.15. *If $f \in \mathcal{D}^p(\mathbb{R}^n)$ $(p \geq 3)$ is a Morse function on X which is proper and has distinct critical values, then f is stable in the sense that there exists an open neighborhood U of f in $\mathcal{D}^p(\mathbb{R}^n)$ such that for each $g \in U$, one can find homeomorphisms $h : X \to X$ and $\lambda : \mathbb{R} \to \mathbb{R}$, such that $g \circ h = \lambda \circ f$.*

REMARK 2.16. For the density of Morse functions to be true, X is not required to be closed. However, for the openess to be true, X must be closed and the stratification must satisfy Whitney's condition (a) (i.e. for every pair strata (Γ, Γ') of the stratification with $\Gamma \subset \overline{\Gamma}'$, given a sequence of points (x_k) in Γ' converging to a point y of Γ such that $T_{x_k}\Gamma'$ converging to a vector subspace T of $\mathbb{R}^m$, then $T_y\Gamma \subset T$). See an example in [**P**] (see also [**T**]).

3. Transversality theorem

DEFINITION 3.1 (Definable jet bundles). Let N, M be C^p definable submanifolds of $\mathbb{R}^n, \mathbb{R}^m$, respectively. Let $0 < r \leq p$. Let $J^r(N,M)$ denote the space of all r-jets of maps from N to M (see for example [H] for the definition). We define the *definable r-jet space* by

$$J^r_{\mathcal{D}}(N,M) = \{j^r f \in J^r(N,M) : f \in \mathcal{D}^r(N,M)\}.$$

To see that this space is a definable set, we can construct it as follows (c.f. [**S2**]).
Let $P^r(\mathbb{R}^n)$ denote the set of all polynomials in n variable of degree $\leq r$ which have their constant term equal zero. Let

$$R = \#\{\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n : 1 \leq |\alpha| = \alpha_1 + \cdots + \alpha_n \leq r\}.$$

We identify $P^r(\mathbb{R}^n)$ with $\mathbb{R}^R$, by $\displaystyle\sum_{1 \leq |\alpha| \leq r} a_\alpha X^\alpha \leftrightarrow (\alpha! a_\alpha)_{1 \leq |\alpha| \leq r}$

For the case $N = U$ and $M = V$ being open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, it is easy to see that

$$J^r_{\mathcal{D}}(U,V) = J^r(U,V) \equiv U \times V \times \prod_{i=1}^{m} P^r(\mathbb{R}^n) \equiv U \times V \times \mathbb{R}^{mR}.$$

For $f \in \mathcal{D}^r(U,V)$, we denote and identify the r-jet of f at x by

$$j^r f(x) = (x, f(x), \sum_{1\le|\alpha|\le r} \frac{\partial^{\alpha} f(x)}{\alpha!}(X-x)^{\alpha}) \equiv (x, f(x), \partial^{\alpha} f(x))_{1\le|\alpha|\le r},$$

where $\partial^{\alpha} f = \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, when $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{N}^n$.
For general N and M, take C^r definable tubular neighborhoods (T_N, π_N, ρ_N) and (T_M, π_M, ρ_M) of N and M in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Note that such neighborhoods exist by [E, Th.1.9]. Then we have

$$\begin{array}{rcl} J^r_{\mathcal{D}}(N,M) & = & \{j^r(\pi_M \circ f \circ \pi_N)(x) : f \in \mathcal{D}^r(T_N, T_M), x \in N\} \\ & = & \{j^r(\pi_M \circ T^r_x f \circ \pi_N)(x) : f \in J^r(T_N, T_M), x \in N\}\ , \end{array}$$

where $T^r_x f(y) = f(x) + \displaystyle\sum_{1\le|\alpha|\le r} \frac{\partial^{\alpha} f(x)}{\alpha!}(y-x)^{\alpha}$.
Indeed, for each $f \in \mathcal{D}^r(N,M)$, there exists $\tilde{f} = f \circ \pi_N \in \mathcal{D}^r(T_N, T_M)$ such that $\tilde{f}|_N = f$, so we have the first equality. By Leibniz's rule, $\partial^{\alpha}(\pi_M \circ f \circ \pi_N)(x)$ is a polynomial of $\partial^{\beta}\pi_N(x), \partial^{\delta} f(\pi_N(x))$ and $\partial^{\gamma}\pi_M(f(\pi_N(x)))$, with $|\beta|, |\delta|, |\gamma| \le |\alpha|$, so we have the second equality.
Hence, $J^r_{\mathcal{D}}(N,M)$ is a definable submanifold of $J^r(\mathbb{R}^n, \mathbb{R}^m)$.

Theorem 3.2 (Transversality theorem). *Let N and M be definable C^p manifolds. Let $\mathcal{A}$ be a finite collection of definable C^1 submanifolds of $J^r_{\mathcal{D}}(N,M)$ $(0 < r < p)$. Then the set*

$$\tau_r(\mathcal{A}) = \{f \in \mathcal{D}^p(N,M) : j^r f \textit{ is transverse to each member of } \mathcal{A}\}$$

is a dense subset of $\mathcal{D}^p(N,M)$.
Moreover, if $\mathcal{A}$ is a stratification of a closed subset and satisfies Whitney's condition (a), then $\tau_r(\mathcal{A})$ is an open subset of $\mathcal{D}^p(N,M)$.

To prove the theorem, we use the following lemmas.

Lemma 3.3. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^p function, and $\alpha \in \mathbb{N}^n$. For $\gamma \in \mathbb{N}^n$, denote $\gamma \le \alpha$ iff $\alpha - \gamma \in \mathbb{N}^n$, and $\gamma < \alpha$ iff $\gamma \le \alpha, \gamma \ne \alpha$. Then there exist real numbers a_{γ} $(\gamma < \alpha)$ such that for all $\beta \in \mathbb{N}^n$ with $|\beta| \le |\alpha|$, we have:*

$$\begin{array}{rcl} \partial^{\beta}(x^{\alpha}\varphi) & = & \displaystyle\sum_{\gamma<\alpha} a_{\gamma} x^{\alpha-\gamma}\partial^{\beta}(x^{\gamma}\varphi) \quad \textit{if } \beta \ne \alpha\ , \\ \partial^{\alpha}(x^{\alpha}\varphi) & = & \alpha!\varphi + \displaystyle\sum_{\gamma<\alpha} a_{\gamma} x^{\alpha-\gamma}\partial^{\alpha}(x^{\gamma}\varphi). \end{array}$$

Proof. First, note that

$$\partial^{\delta}(x^{\alpha}) = \begin{cases} \dfrac{\alpha!}{(\alpha-\delta)!} x^{\alpha-\delta} & \text{if } \delta \le \alpha, \\ 0 & \text{if } \delta \not\le \alpha. \end{cases}$$

Let a_γ ($\gamma < \alpha$) be the solution of the following system of linear equations of triangular form:

$$\sum_{\delta \leq \gamma < \alpha} \frac{(\gamma-\delta)!}{\gamma!} a_\gamma = \frac{\alpha!}{(\alpha-\delta)!} \quad (\delta \in \mathbb{N}^n, \delta < \alpha).$$

We shall check that the a_γ ($\gamma < \alpha$) satisfy the equations of the lemma.
For each $\beta \in \mathbb{N}^n, |\beta| \leq |\alpha|$, by Leibniz's rule and the note, the lefthand side is

$$A_\beta = \partial^\beta(x^\alpha\varphi) = \sum_{\delta\leq\beta} C^\delta_\beta \partial^\beta(x^\alpha)\partial^{\beta-\delta}\varphi = \sum_{\delta\leq\beta,\delta\leq\alpha}\left(C^\delta_\beta\frac{\alpha!}{(\alpha-\delta)!}x^{\alpha-\delta}\right)\partial^{\beta-\delta}\varphi.$$

On the other hand, by Leibniz's rule and the note, the righthand side is

$$\begin{aligned}
B_\beta &= \sum_{\gamma<\alpha} a_\gamma x^{\alpha-\gamma}\partial^\beta(x^\gamma\varphi) \\
&= \sum_{\gamma<\alpha} a_\gamma x^{\alpha-\gamma}\left(\sum_{\delta\leq\beta} C^\delta_\beta\partial^\delta(x^\gamma)\partial^{\beta-\delta}\varphi\right) \\
&= \sum_{\delta\leq\beta}\left(\sum_{\gamma<\alpha} a_\gamma x^{\alpha-\gamma} C^\delta_\beta \partial^\delta(x^\gamma)\right)\partial^{\beta-\delta}\varphi \\
&= \sum_{\delta\leq\beta}\left(\sum_{\delta\leq\gamma<\alpha} a_\gamma x^{\alpha-\gamma} C^\delta_\beta \partial^\delta(x^\gamma)\right)\partial^{\beta-\delta}\varphi \\
&= \sum_{\delta\leq\beta}\left(\sum_{\delta\leq\gamma<\alpha} C^\delta_\beta \frac{(\gamma-\delta)!}{\gamma!} a_\gamma x^{\alpha-\delta}\right)\partial^{\beta-\delta}\varphi.
\end{aligned}$$

By the definition of the a_γ 's, we have

$$B_\beta = \sum_{\delta\leq\beta,\delta<\alpha}\left(C^\delta_\beta\frac{\alpha!}{(\alpha-\delta)!}x^{\alpha-\delta}\right)\partial^{\beta-\delta}\varphi.$$

If $|\beta| \leq |\alpha|$ and $\beta \neq \alpha$, then $\{\delta \in \mathbb{N}^n : \delta \leq \beta, \delta \leq \alpha\} = \{\delta \in \mathbb{N}^n : \delta \leq \beta, \delta < \alpha\}$, and thus $A_\beta = B_\beta$
If $\beta = \alpha$, then $\{\delta \in \mathbb{N}^n : \delta \leq \beta, \delta \leq \alpha\} \setminus \{\delta \in \mathbb{N}^n : \delta \leq \beta, \delta < \alpha\} = \{\alpha\}$, and thus $A_\alpha - B_\alpha = C^\alpha_\alpha \dfrac{\alpha!}{(\alpha-\alpha)!}x^{\alpha-\alpha}\partial^{\alpha-\alpha}\varphi = \alpha!\varphi$.
This completes the proof of the lemma. □

Lemma 3.4. *Let N, J and T be definable C^p manifolds, and $\Phi : T \times N \to J$ be a definable C^p map. Let $\mathcal{A}$ be a finite collection of definable submanifolds of J. If Φ is submersive, then the set*

$$\tau(\Phi,\mathcal{A}) = \{t \in T : \Phi(t,\cdot) \ \text{ is transverse to } \mathcal{A}\}$$

is a definable set and $\dim(T \setminus \tau(\Phi,\mathcal{A})) < \dim T$.

Proof. Using the same arguments as in the proof of Theorem 2.6. □

Proof of Theorem 3.2. (c.f [S2, Th. II.5.4 (3)])
We reduce the proof to the case $N = U$ being an open subset of $\mathbb{R}^n$, and $M = \mathbb{R}^m$, by the following arguments.
Let (T_M, π_M, ρ_M) be a C^p definable tubular neighborhood of M in $\mathbb{R}^m$. Let $\pi_{M*} : J^r_{\mathcal{D}}(N, T_M) \to J^r_{\mathcal{D}}(N, M), j^r f \mapsto j^r(\pi_M \circ f)$. Let $f \in \mathcal{D}^p(N, M)$. If g is

an approximation of f in $\mathcal{D}^p(N,\mathbb{R}^m)$ and $j^r g$ is transverse to $\pi_{M*}^{-1}(A)$, for each $A \in \mathcal{A}$, then, by Proposition 1.3, $\pi_M \circ g$ is an approximation of f in $\mathcal{D}^p(N,M)$ and $j^r(\pi_M \circ g)$ is transverse to each member of $\mathcal{A}$. Therefore, we can reduce to the case $M = \mathbb{R}^m$.

On the other hand, let (T_N, π_N, ρ_N) be a C^p definable tubular neighborhood of N in $\mathbb{R}^n$. For each $A \in \mathcal{A}$ being a definable submanifold of $J^r_{\mathcal{D}}(N,\mathbb{R}^m)$, define

$$\tilde{A} = \{(x, y, a_\alpha)_{1\le|\alpha|\le r} \in J^r(T_N,\mathbb{R}^m) : (\pi_N(x), y, a_\alpha)_{1\le|\alpha|\le r} \in A\}.$$

Then $\tilde{A}$ is a definable submanifold of $J^r(T_N,\mathbb{R}^m)$. Let $f \in \mathcal{D}^p(N,\mathbb{R}^m)$. If g is an approximation of f in $\mathcal{D}^p(T_N,\mathbb{R}^m)$ and $j^r g$ is transverse to $\tilde{A}$, for each $A \in \mathcal{A}$, then, by Proposition 1.2, $g|_N$ is an approximation of f in $\mathcal{D}^p(N,\mathbb{R}^m)$ and $j^r(g|_N)$ is transverse to each member of $\mathcal{A}$ in $J^r_{\mathcal{D}}(N,\mathbb{R}^m)$. Therefore, we can reduce to the case N being an open subset of $\mathbb{R}^n$.

Now, let $f : U \to \mathbb{R}^m, f = (f_1, \cdots, f_m)$ be a C^p definable map. Let $\varepsilon : U \to \mathbb{R}$ be a positive continuous definable function. We shall construct a function in $\tau_r(\mathcal{A})$ which is in ε-neighborhood of f.

Case $m = 1$: Let $P = \#\{\alpha \in \mathbb{N}^n : |\alpha| \le p\}$, and $C = P^2(p!)^{2n}$. Let $\varphi : U \to \mathbb{R}$ be a positive C^p definable function such that

$$|\partial^\alpha \varphi(x)| < \frac{\varepsilon(x)}{C(1+\|x\|^p)}, \forall |\alpha| \le p.$$

Note that such a function exists by Lemma 2.13.

Consider the following family of definable functions

$$F(t,x) = f_t(x) = f(x) + \sum_{|\alpha|\le r} t_\alpha x^\alpha \varphi(x),$$

where $I = (-1,1), t = (t_\alpha)_{|\alpha|\le r} \in I^{R_0}$, $R_0 = \#\{\alpha \in \mathbb{N}^n : |\alpha| \le r\}$, and $x \in U$.

First, we prove that f_t is in ε-neighborhood of f. Indeed, for each $\alpha \in \mathbb{N}^n$ with $|\alpha| \le p$,

$$|\partial^\alpha (f_t - f)(x)| \le \sum_{|\beta|\le r} |t_\beta| \left|\partial^\alpha (x^\beta \varphi)(x)\right| \le \sum_{|\beta|\le r} \left|\partial^\alpha (x^\beta \varphi)(x)\right|.$$

From the proof of Lemma 3.3 and by the definition of φ, we have

$$\begin{aligned} \left|\partial^\alpha (x^\beta \varphi)\right| = \left|\sum_{\delta\le\alpha} C^\delta_\alpha \partial^\delta (x^\beta) \partial^{\alpha-\delta}\varphi\right| &= \left|\sum_{\delta\le\beta,\delta\le\alpha} C^\delta_\alpha \frac{\beta!}{(\beta-\delta)!} x^{\beta-\delta}\partial^{\alpha-\delta}\varphi\right| \\ &< \sum_{|\delta|\le p} (p!)^n (p!)^n (1+\|x\|^p)\frac{\varepsilon}{C(1+\|x\|^p)}. \end{aligned}$$

Therefore, by the selection of C, we get

$$|\partial^\alpha (f_t - f)(x)| < \varepsilon(x)\ , \forall x \in U.$$

Using Lemma 3.4, we shall prove that there are many $t \in I^{R_0}$, such that $f_t \in \tau_r(\mathcal{A})$. To this end, we need to check that

$$\Phi : I^{R_0} \times U \to J^r(U,\mathbb{R}) \equiv U \times \mathbb{R}^{R_0}, \quad \text{defined by} \quad \Phi(t,x) = (x, (\partial^\alpha f_t(x))_{|\alpha|\le r}),$$

is a submersion.
Calculating, we can write the Jacobian matrix $J\Phi(x,t)$ in block form:

$$\begin{pmatrix} O & \cdots & O & \cdots & O & \cdots & O & \cdots & I_n \\ \varphi(x) & \cdots & x^\gamma\varphi(x) & \cdots & x^\beta\varphi(x) & \cdots & x^\alpha\varphi(x) & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \partial^\gamma\varphi(x) & \cdots & \partial^\gamma x^\gamma\varphi(x) & \cdots & \partial^\gamma x^\beta\varphi(x) & \cdots & \partial^\gamma x^\alpha\varphi(x) & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \partial^\beta\varphi(x) & \cdots & \partial^\beta x^\gamma\varphi(x) & \cdots & \partial^\beta x^\beta\varphi(x) & \cdots & \partial^\beta x^\alpha\varphi(x) & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \partial^\alpha\varphi(x) & \cdots & \partial^\alpha x^\gamma\varphi(x) & \cdots & \partial^\alpha x^\beta\varphi(x) & \cdots & \partial^\alpha x^\alpha\varphi(x) & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{pmatrix}$$

where $1 \le |\gamma| < |\beta| = |\alpha|$.
Applying Lemma 3.3, we can reduce the matrix to the form

$$\begin{pmatrix} O & \cdots & O & \cdots & O & \cdots & O & \cdots & I_n \\ \varphi(x) & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \partial^\gamma\varphi(x) & \cdots & \gamma!\varphi(x) & \cdots & 0 & \cdots & 0 & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \partial^\beta\varphi(x) & \cdots & * & \cdots & \beta!\varphi(x) & \cdots & 0 & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \partial^\alpha\varphi(x) & \cdots & * & \cdots & 0 & \cdots & \alpha!\varphi(x) & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{pmatrix}.$$

Therefore, $\operatorname{rank} J\Phi(t,x) = R_0 + n, \forall (t,x) \in I^{R_0} \times U$, i.e. Φ is submersive.
For the general case: Use the same arguments as in the case $m = 1$ for the family

$$F(t,x) = \left(f_1(x) + \sum_{|\alpha|\le r} t_{1,\alpha} x^\alpha \varphi(x), \cdots, f_m(x) + \sum_{|\alpha|\le r} t_{m,\alpha} x^\alpha \varphi(x) \right),$$

where $t = (t_{i,\alpha})_{1\le i\le m, |\alpha|\le r} \in I^{mR_0}$.

Since the mapping $j^r : \mathcal{D}^p(N,M) \to \mathcal{D}^{p-r}(N, J^r_{\mathcal{D}}(N,M))$ is continuous and $\tau_r(\mathcal{A}) = (j^r)^{-1}\{F \in \mathcal{D}^{p-r}(N, J^r_{\mathcal{D}}(N,M)) : F$ is transverse to each member of $\mathcal{A}\}$, the proof of the second part of the theorem reduces to the case $\mathcal{A}$ being a stratification of a definable closed subset of M and satisfying Whitney's condition (a). Let K be the subset of $J^1_{\mathcal{D}}(N,M)$ defined by

$\alpha \in K$ iff there exist $x \in N, S \in \mathcal{A}, y \in S$ such that $\alpha_{x,y}$ is not transverse to T_yS,

where $\alpha_{x,y} : T_xN \to T_yM$ denotes the restriction of α to $J^1_{\mathcal{D}}(N,M)_{(x,y)}$.
Then by the proof of [F, Prop. 3.6], K is closed in $J^1_{\mathcal{D}}(N,M)$. Since the mapping $j : \mathcal{D}^p(N,M) \to J^1_{\mathcal{D}}(N,M), f \mapsto j^1 f$, is continuous, $j^{-1}(K)$ is closed.
Therefore, $\tau_1(\mathcal{A}) = \mathcal{D}^p(N,M) \setminus j^{-1}(K)$ is open. □

REMARK 3.5. We do not know if the theorem hold for $p = \infty$ or $p = \omega$.

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