STEFAN PROBLEMS

John van der Hoek

§0. Introduction

In 1899 STEFAN [1] posed the following problem : A heat conducting material occupies the space $-\infty < x < \infty$. Initially ($t = 0$), the liquid phase occupies $-\infty < x < 0$ at temperature $T_1 > 0$ and the solid phase occupies $0 < x < \infty$ at temperature $T_2 < 0$. It is required to determine temperatures $u^1(x,t)$ and $u^2(x,t)$ at position x and time t of the liquid and solid phases, respectively, and the position $x = s(t)$ of the free or moving boundary between the phases as a function of time (t). Stefan showed that the following thermal balance operates between the two phases :

$$(1) \qquad -\lambda\rho \frac{ds}{dt} = \left(\kappa_1 \frac{\partial u^1}{\partial x} - \kappa_2 \frac{\partial u^2}{\partial x}\right)\bigg|_{x=s(t)}$$

where λ = latent heat, ρ = density of the material in its original phase state, κ_1 and κ_2 are coefficients of conductivity corresponding to the liquid and solid phases, respectively. Condition (1) will be referred to as the Stefan free boundary condition. This relationship holds, for example, at the interface between water and ice in the process of melting ice (see RUBINSTEIN [2]). The full formulation of Stefan's problem would also include in addition to (1),

$$(2) \qquad c_1 \frac{\partial u^1}{\partial t} = \frac{\partial}{\partial x}\left(\kappa_1 \frac{\partial u^1}{\partial x}\right), \; s(t) < x < \infty, \; t > 0$$

$$(3) \qquad c_2 \frac{\partial u^2}{\partial t} = \frac{\partial}{\partial x}\left(\kappa_2 \frac{\partial u^2}{\partial x}\right), \; -\infty < x < s(t), \; t > 0$$

(4) $u^1(x,0) = T_1 > 0, \quad -\infty < x < 0$

(5) $u^2(x,0) = T_2 < 0, \quad 0 < x < \infty$

(6) $u^1(s(t),t) = u^2(s(t),t) = 0, \quad t > 0$

where c_1 and c_2 are specific heats of the respective media which will in general depend on the spacial variable x. Stefan problems are free boundary problems where the free boundary is characterised (in part) by the Stefan free boundary condition. Such free boundary problems are also referred to as moving boundary problems since the position of the free boundary is a function of time. For other types of moving boundary problems, the reader is referred to WILSON et al. [3], WILSON et al. [4], RUBINSTEIN [2] and FURZELAND [5]. The Stefan free boundary condition can also be expressed in space dimension $n \geqslant 2$. (see FRIEDMAN [6]).

§1. One Phase Stefan Problem

When the temperature in the solid phase is maintained at the melting temperature, the two phase problem of §0 becomes a one phase Stefan problem. Under some simplifying assumptions and under scaling of independent variables, the problem in one space dimension becomes one of finding functions $u = u(x,t)$ and $s = s(t)$ satisfying,

(7) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < s(t),\ 0 < t < T$$

(8) $u(x,0) = h(x), \quad 0 < x < b$

where, $b = s(0)$, $h(b) = 0$, $h(x) > 0$ if $x \neq b$

(9) $u(0,t) = f(t), \quad 0 < t < T,\ f > 0$

(10) $u(s(t),t) = 0, \quad 0 < t < T$

(11) $$-\frac{\partial u}{\partial x}(s(t)-,t) = \frac{ds}{dt}(t), \quad 0 < t < T.$$

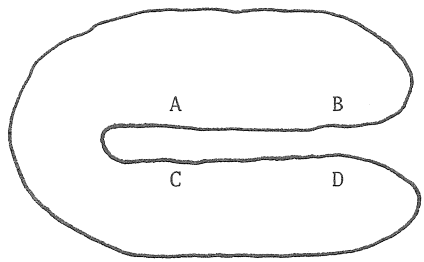
A recent survey of results concerning problem (7)-(11) is given in FRIEDMAN [7] for one and more space dimensional problems. The one phase Stefan problem yields several formulations, for example the weak formulation due to FRIEDMAN [6] and the formulation as a variational inequality due to DUVAUT [8]. For the one dimensional one phase Stefan problem, many results have been obtained on existence and regularity of the solution u and on the nature and regularity of the free boundary $x = s(t)$. For example, FRIEDMAN [9] has shown that if $f(t)$ in (9) is real analytic in t for $0 < t < T$ then so is $s(t)$ as a function of t. For space dimension $n \geqslant 2$ very little is known. The most recent result is due to CAFFARELLI and FRIEDMAN [10] who prove that the temperature in the one phase n-dimensional Stefan problem is continuous.

§3. Two and Multiphase Stefan Problems

A recent survey of results for the general multiphase Stefan problem is CANNON [11]. In 1968, FRIEDMAN [6], [12] developed a theory of weak solutions of these Stefan problems. As for one phase Stefan problems, many results are known in the case of one dimensional space variable but little known for space dimension $n \geqslant 2$. For the one dimensional problem we refer the reader to CANNON, HENRY and KOTLOW [13] for existence of classical solutions and to SCHAEFFER [14] and FRIEDMAN [9] for regularity of the free boundary. No results are known for the regularity of temperatures or for the free boundary for space dimension $n \geqslant 2$. In 1975, DUVAUT [15] gave a formulation of the two phase Stefan problem as an evolution variational inequality. It has been used by some authors (see for example KIKUCHI and ISHIKAWA [16]) to provide numerical results.

§4. Some Related Comments

One of the major differences between Stefan problems in one space dimension and for space dimension $n \geqslant 2$ is the fact that the former allows analysis via integral equation methods (see CANNON et al. [13]) whereas the latter does not. As FRIEDMAN [6] points out the complications in space dimension $n \geqslant 2$ are of a physical nature. "Thus, even if the data are very smooth the solution need not be smooth in general. For example, when a body of ice having the shape



keeps growing, the interfaces AB and CD may eventually coincide. Then, in the next movement the whole joint boundary will disappear. Thus the free boundary varies in a discontinuous manner."

The Stefan problems have been studied with various types of boundary conditions. The most usual formulations have used temperature and heat flux specification on the boundary of interest. FASANO and PRIMICERIO [17] have studied in one dimension, boundary conditions which involve a non linear relationship between the heat flux and the temperature on the boundary. CANNON and VAN DER HOEK [18]-[20] have studied the one and two phase Stefan problem in one space dimension in the case where the boundary condition (for example (9)) is replaced by a specification of energy or heat content in one of the phases. In the one phase problem this amounted to specifying the quantity

$$\int_0^{s(t)} u(x,t)\,dx$$

as a function of t for $0 < t < T$.

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Department of Mathematics,
The University of Adelaide,
Adelaide, S.A. 5001
AUSTRALIA