REGULARITY AT THE BOUNDARY AND REMOVABLE SINGULARITIES FOR SOLUTIONS OF QUASILINEAR PARABOLIC EQUATIONS

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1. INTRODUCTION

The purpose of this note is to describe recent results concerning removable singularities and behavior of weak solutions of quasilinear parabolic equations of the second order at the boundary of an arbitrary domain. Specifically, we investigate the local behaviour of weak solutions of equations of the form

$$u_t = \operatorname{div} A(x,t,u,u_x) + B(x,t,u,u_x) \tag{1}$$

where A and B are, respectively, vector and scalar valued Borel functions defined on $\Omega \times R^1 \times R^n$, where Ω is arbitrary open subset of $R^{n+1}(x,t)$. The functions A and B are required to satisfy the following structure conditions:

$$\begin{gathered} |A(x,t,u,w)| \le a_0|w|^{p-1} + (a_1|u|)^{p-1} + a_2^{p-1} \\ |B(x,t,u,w)| \le b_0|w|^p + b_1|w|^{p-1} + (b_2|u|)^{p-1} + b_3^{p-1} \\ A(x,t,u,w) \cdot w \ge |w|^p - (c_1|u|)^p - c_2^p \,. \end{gathered} \tag{2}$$

Here, $p > 1$, $a_0 > 0$, $b_0 > 0$ and the remaining coefficients are non-negative functions of (x,t) that are required to belong to specified Lebesgue classes. For the purposes of this exposition, we will simply require a_1^p, a_2^p, b_1^p, b_2^{p-1}, b_3^{p-1}, c_1^p and c_2^p to lie in $L^q(\Omega)$ where $\frac{n}{pq} + \frac{1}{q} < 1$.

Essentially all research concerned with the behavior of weak solutions of (1) has been restricted to the case $p = 2$. For example, interior regularity, i.e., Hölder continuity of weak solutions of (1) has been established by several authors, c.f., [KO], [LSU], [AS], [T]. Landis, [L], announced a criterion for continuity of solutions of the heat equation at the boundary of an arbitrary open set, although a complete development of his results has apparently never appeared. Recently, Evans and Gariepy, [EG], established a characterization of regular boundary points for the heat equation which is in the same spirit as the Wiener criterion for Laplace's equation. Other results concerning boundary regularity of linear parabolic equations include [E], [L1], [L2], [PI], [EK].

Closely associated with the problem of regularity at the boundary of weak solutions of (1) is the question of determining conditions under which a compact set $K \subset \Omega$ is removable for solutions of (1). Some results in this direction were obtained in [A] for linear equations and in [EP] for equations of the form (1) with $p = 2$. For a general development of removability results for a wide class of higher order linear equations, the reader is referred to [HP].

2. CONTINUITY AT THE BOUNDARY

If $U \subset \Omega$ is an open set, a bounded function $u \in W^{1,p}_{loc}(U)$ is said to be a weak solution of (1) in U if

$$\iint -u\phi_t + A(x,t,u,u_x) \cdot \phi_x - B(x,t,u,u_x)\phi = 0$$

for all $\phi \in C_0^\infty(U)$.

The fundamental solution of the heat operator $H = \frac{\partial}{\partial t} - \Delta$ is given by

$$G(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t) , & t > 0 \\ 0 , & t \le 0 . \end{cases}$$

For any set $E \subset R^{n+1}$, the *classical parabolic capacity* is defined by

$$C(E) = \sup\{\mu(R^{n+1})\}$$

where the supremum is taken over all non-negative Borel measures μ supported in E whose potential, $G * \mu$, is everywhere bounded above by 1 .

For the purpose of describing boundary regularity results, let $z_0 = (x_0,t_0) \in \partial\Omega$. For $\alpha > 0$, let

$$R_\alpha(r) = B(x_0,r) \times (t_0 - \tfrac{3}{4}\alpha r^2, t_0 + \tfrac{1}{4}\alpha r^2)$$

where $B(x_0,r)$ is the ball in R^n with center x_0 and radius r . We associate with $R_\alpha(r)$ a subcylinder $R^*_\alpha(r) = B(x_0,\frac{r}{2}) \times (t_0 - \frac{2}{3}\alpha r^2, t_0 - \frac{1}{3}\alpha r^2)$. If $z_0 \in \partial\Omega$, $k \in R^1$ and $u \in W^{1,2}(\Omega)$, we say that

$$u(z_0) \le k \quad \text{weakly}$$

if for every $\ell > k$, there is an $r > 0$ such that $\eta(u-\ell)^+ \in W_0^{1,2}(\Omega)$ whenever $\eta \in C_0^\infty[U(z_0,r)]$ where $U(z_0,r)$ denotes the ball in R^{n+1} with center z_0 and radius r . A similar definition is given for $u(z_0) \ge k$ weakly and consequently it is clear what is meant by $u(z_0) = k$ weakly. The following result comes from [Z1] and [GZ2].

<u>THEOREM</u> *Let* Ω *be an open subset of* R^{n+1} *with* $z_0 \in \partial\Omega$. *Let* $u \in W^{1,2}(\Omega)$ *be a bounded weak solution of* (1) *with structure* (2) *and* $p = 2$ *such that* $u(z_0) = k$ *weakly. If for some* $\alpha > 0$,

$$(3) \qquad \int_0 \frac{C[R_\alpha^*(r)-\Omega]}{C[R_\alpha^*(r)]} \frac{dr}{r} = \infty ,$$

then

$$\lim_{\substack{z \to z_0 \\ z \in \Omega}} u(z) = k .$$

In the case that $\Omega = D \times (0,T)$ where D is an open subset of R^n , it is not difficult to show that if $z_0 = (x_0, t_0) \in \partial\Omega$, then (3) holds if and only if

$$(4) \qquad \int_0 \frac{\gamma[B(x_0,r)-D]}{\gamma[B(x_0,r)]} \frac{dr}{r} = \infty$$

where γ is classical Newtonian capacity in R^n . But (4) is precisely the Wiener criterion for continuity of solutions of Laplace's equation at $x_0 \in \partial D$. Hence, we have the following.

<u>COROLLARY</u> *If* $x_0 \in \partial D$ *is a regular point for Laplace's equation, then* $z_0 = (x_0, t_0) \in \partial[D\times(0,T)]$ *is a regular point for bounded weak solutions of* (1) *with structure* (2) *and* $p = 2$.

The author has recently proved that weak bounded solutions of (1), (2) with $p > 1$ are continuous in Ω , [Z2]. However, the question of extending the above theorem to cover the case of all $p > 1$ remains open.

3. REMOVABLE SINGULARITIES

Suppose $D \subset R^n$ is an open set and $K \subset D$ a compact set of zero Newtonian capacity. It is well known that a bounded harmonic function defined in $D - K$ can be defined on K in such a way that the resulting function is harmonic throughout D. Likewise a single point is removable for any harmonic function which is $o(r^{2-n})$ in a neighbourhood of the point. These two results represent extreme cases concerning the size of the singular set. Carleson [C] provided an interpolation between them by relating the Hausdorff dimension of the singular set to the Lebesgue class of the harmonic function. Serrin [S] extended Carleson's results to solutions of the elliptic counterpart of (1) and (2) by using the notion of s-capacity in place of Hausdorff measure. Below we state analogous results for solutions of parabolic equations of the form (1).

One of the difficulties encountered in the parabolic situation is the selection of the appropriate capacity which is used to balance the size of the singular set against the Lebesgue class of the solution. Unlike the corresponding elliptic case, no obvious definition is suggested by the analysis of the equation. The capacity we employ is defined for each compact set K as

$$\Gamma_s(K) = \inf\left\{\iint |\nabla v(x,t)|^s dxdt + \int \left\|\frac{\partial v}{\partial t}\right\|_{-1,s'}^{s'} dt\right\}$$

where the infimum is taken over all $v \in C_0^\infty(R^{n+1})$ such that $v \geq 1$ on K. Here, $s > 1$, $\frac{1}{s} + \frac{1}{s'} = 1$ and $\|\frac{\partial v}{\partial t}\|_{-1,s'}$ denotes the norm of $\frac{\partial v}{\partial t}(\cdot,t)$ when taken as an element of $W^{1,s'}(R^n)$. It can be shown that, for each compact set $K \subset R^{n+1}$, $\Gamma_2(K) = C(K)$ where C is classical parabolic capacity defined in §2 above, c.f., [PM]. Also, it is shown in

[GZ2] that Γ_s is strictly weaker than the capacity employed in [A] or [EP] in the sense that there exist compact sets K whose capacity, as defined in [A] or [EP], is positive but $\Gamma_s(K) = 0$. If $\Gamma_2(K) = C(K) = 0$, it can be shown [GZ1] that there is a sequence of smooth functions $\{\alpha_i\}$ with the following properties :

$$0 \le \alpha_i \le 1$$

$$\alpha_i = 0 \quad \text{on a neighbourhood of} \quad K$$

$$\|\nabla\alpha_i\|_2 \to 0 \quad \text{as} \quad i \to \infty$$

$$\|H^{\vee}\alpha_i\|_1 \to 0 \quad \text{and} \quad \alpha_i \to 0 \quad \text{a.e.} \quad \text{as} \quad i \to \infty ,$$

where $H^{\vee} = -\frac{\partial}{\partial t} - \Delta$ is the adjoint to the heat operator. This information is critical in establishing the following result, [GZ1].

<u>THEOREM</u> *Let* K *be a compact subset of an open set* $\Omega \subset R^{n+1}$. *Let* $u \in L^{\infty}_{loc}(\Omega) \cap W^{1,2}_{loc}(\Omega - K)$ *be a weak solution in* $\Omega - K$ *of* (1) *and* (2) *with* $p = 2$. *If* $\Gamma_2(K) = 0$, *then* $u \in W^{1,2}_{loc}(\Omega)$ *and* u *is a weak solution of* (1) *in* Ω.

Clearly this result is optimal for the class of equations under consideration, for if $\Gamma_2(K) > 0$, then the capacity equilibrium potential of K is a bounded function on R^{n+1} that satisfies the heat equation on $R^{n+1} - K$ but not on R^{n+1}.

In order to consider weak solutions of (1), (2) for all $p > 1$, we assume that the constant b_0 that appears in (2) is zero. A typical interpolatory result analogous to that of Serrin's cited above takes the following form, [Z2].

THEOREM *Let* $K \subset \Omega$ *be a compact set with* $\Gamma_s(K) = 0$. *If* $u \in L^q_{loc}(\Omega) \cap W^{1,p}_{loc}(\Omega - K)$ *with* $p \le s$ *is a weak solution of* (1), (2) *in* $\Omega - K$, *then* u *is a weak solution in* Ω *provided* $(p-1)(\frac{s}{s-p}) < q$.

A more detailed analysis can be provided by considering solutions that lie in the spaces $L^{p,q}_{loc}(\Omega)$ with norm

$$\left\{\iint \left(\iint |u|^p dx\right)^{q/p} dt\right\}^{1/q} .$$

The definition of the Γ capacity must then be modified accordingly.

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