MINIMAL SURFACES WITH FREE BOUNDARIES

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We will in this lecture give a survey of some recent results for minimal surfaces with free boundaries. To this end, we consider boundary configurations $\langle\Gamma, S\rangle$ in $\mathbb{R}^3$ consisting of *a fixed part* Γ and a *free part* S . The fixed part Γ is the union of Jordan arcs $\Gamma_1, \ldots, \Gamma_m$, and the free part consists of surfaces $S_1, \ldots, S_n$ in $\mathbb{R}^3$ with or without boundary. Each of the curves Γ_j is either a closed curve, or else an arc with end points on S . In the following, the fixed part Γ may be void, whereas the free part S is always assumed to be non-empty.

A mapping $X : \bar{\Omega} \to \mathbb{R}^3$ of some Riemann surface Ω with boundary $\partial\Omega$ into $\mathbb{R}^3$ is called a *solution of the free boundary problem for the configuration* $\langle\Gamma, S\rangle$ if the following properties are satisfied:

(i) $X \in C^0(\bar{\Omega}, \mathbb{R}^3) \cap C^2(\Omega, \mathbb{R}^3)$;

(ii) X is a harmonic mapping ;

(iii) X maps Ω conformally onto $X(\Omega)$, except for isolated branch points in Ω ;

(iv) $X(\partial\Omega) \subset \Gamma \cup S$;

(v) the surface $\mathcal{M} = X(\bar{\Omega})$ intersects S at $\Sigma \cap \operatorname{int} S$ perpendicularly. Here Σ denotes the *free trace* $X(\partial\Omega)$ of the minimal surface $\mathcal{M}$ on the free boundary S.

Obviously, property (v) does not make sense since we assumed X to be only continuous at the boundary $\partial\Omega$. Therefore, (v) has to be understood in a weak sense. However, it follows from well known regularity theorems that X is smooth up to the boundary provided that Γ and S are smooth; cf. [16], §315 and §512. General regularity theorems for solutions of free boundary problems that are neither area minimizing nor *a priori* assumed to be continuous up to the boundary have recently been proved by Grüter-Hildebrandt-Nitsche [6] and by Dziuk [4]; the case $\partial S \neq \phi$ has been treated by Hildebrandt and Nitsche [9], [10] ;see also [6]. Thus we can assume that X is sufficiently regular at $\partial\Omega$.

In the following we shall mostly be concerned with disk-type minimal surfaces solving the free boundary problem for some configuration $\langle\Gamma,S\rangle$.

First we consider the *number of solutions* for a free boundary problem. It is well known that for boundaries consisting of a single smooth Jordan curve it is still undecided whether they can bound infinitely many minimal surfaces. Even more difficult to decide seems to be the question of whether or not a closed curve can bound infinitely many minimal disks. It is, however, trivial to find supporting surfaces S which bound continua of minimal disks. For instance, the sphere, the cylinder, or the torus furnish simple examples. Yet, in these cases, all minimal surfaces belonging to the same family are congruent to each other. On the other hand, H.A. Schwarz [18] has already described boundary configurations $\langle\Gamma,S\rangle$ that bound denumerably many noncongruent minimal surfaces.

Let, for example, S be a circular cylinder surface and Γ_1, Γ_2 be two straight arcs which are perpendicular to each other as well as to the cylinder axis and pass through the axis at different heights. This configuration bounds denumerably many left and right winding helicoids which meet the cylinder at a right angle. Only two of these helicoids are area minimizing; the others are but stationary.

However, worse can occur. As Gulliver and Hildebrandt [8] have shown, there exist real analytic boundary configurations $\langle\Gamma,S\rangle$ that bound 1-parameter families of noncongruent (and even nonisometric) and area minimizing minimal surfaces of fixed topological type. We quote only one of the examples of [8]:

There exists a real-analytic, embedded surface S *of the type of the torus, and a homology class* $[\gamma_0]$ *in* $H_1(S;\mathbb{Z})$, *so that* γ_0 *bounds a family of stationary minimal surfaces of the type of the disk, which have smallest area among all oriented surfaces in* G *with boundary on* S *and homologous in* S *to* γ_0. (cf. figure 1). Here G denotes the solid body with $\partial G = S$.

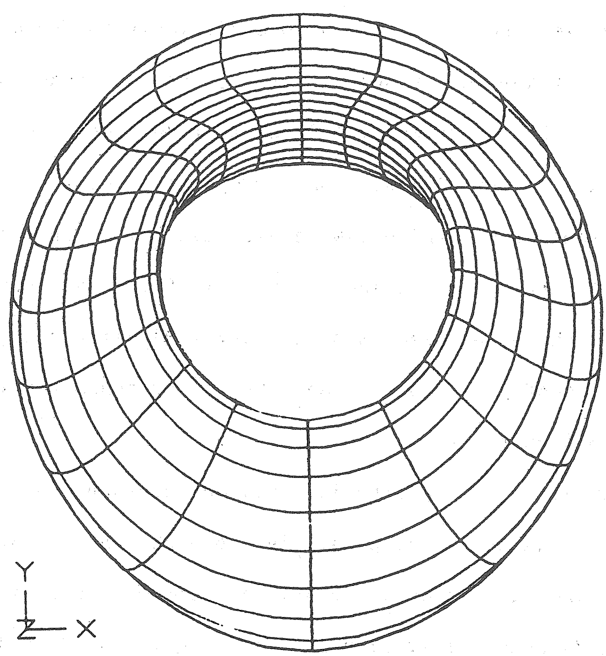

Fig. 1

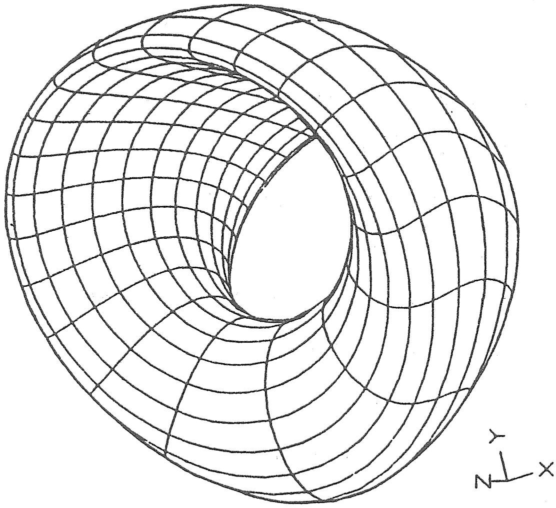

Fig. 2.

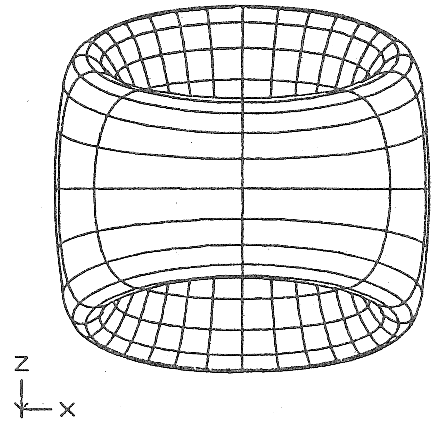

Fig. 3.

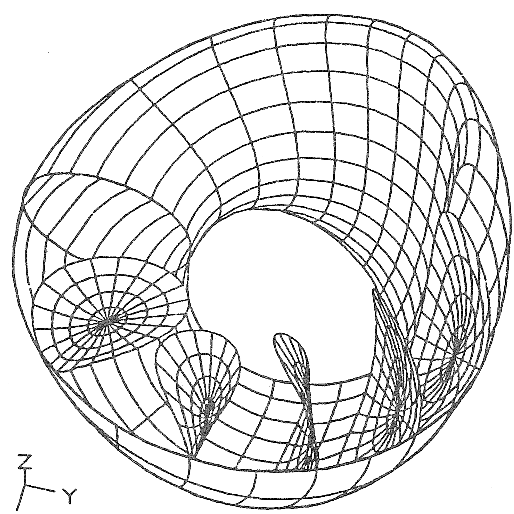

Fig. 4.

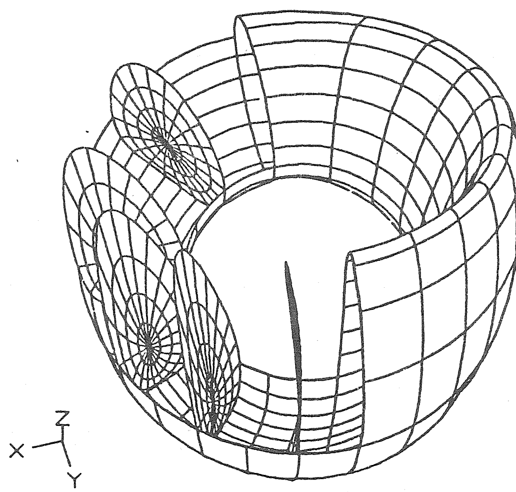

Fig. 5.

Figures 1 - 5 have been taken from [8]

On the other hand, Tomi [21] proved the following remarkable result:

If a compact, analytic, and H-convex body G *in* $\mathbb{R}^3$ *has the properties that there is a closed Jordan curve in* G *which cannot be contracted in* G, *and secondly, that the free boundary problem for* G *admits infinitely many minimizing solutions of disk-type contained in* G, *then* G *must be homeomorphic to a solid torus, and the set of all solutions form an analytic family of embedded minimal disks.*

As Tomi pointed out, the assumption of analyticity of ∂G is crucial, since one can produce smooth H-convex bodies of arbitrary genus which possess continua of solutions.

Let now $X : \bar{B} \to \mathbb{R}^3$ be a disk-type minimal surface that intersects a smooth surface S orthogonally. If r, θ are polar cooordinates about the center of B, we shall write $X = X(r,\theta)$, and the conformality relations take the form

(1) $$r^2|X_r|^2 = |X_\theta|^2 \ , \quad X_r \cdot X_\theta = 0.$$

On the other hand, we infer from the equation

$$\Delta X = 0 \quad \text{on} \quad B$$

the relation

$$\int_0^{2\pi} X_r(1,\theta)\, d\theta = 0.$$

The boundary condition (v) implies that $X_r(1,\theta)$ is a normal vector to S at $X(1,\theta)$. If we assume that S is orientable, we can pick a unit normal vector field N on S such that

$$X_r = N(X)|X_r| = N(X)|X_\theta|$$

holds on ∂B, whence we obtain

(2) $$\int_\Sigma N(X)ds = 0$$

where Σ denotes the free trace $X : \partial B \to \mathbb{R}^3$, and ds is the arc element of Σ.

The relation (2), which yields a necessary but, in general, not sufficient condition for a closed curve Σ on S to be the solution of a free boundary problem for <S> ,was first derived by B.Smyth [19]. He used it to establish the following theorem:

If S *is the boundary of a tetrahedron in* $\mathbb{R}^3$, *then there exist exactly three disk-type minimal surfaces which solve the free boundary problem for* <S>. *Each of these solutions is non-planar and embedded ; in fact, each is a graph over some planar domain.*

The proof rests on the following well known observation that can easily be proved:

If X^* *is the adjoint surface of a minimal surface* $X : B \to \mathbb{R}^3$ *that maps some subarc* C *of* ∂B *into a straight line* L , *then* X^* *maps* C *into a plane* E *which intersects* L *perpendicularly, and the adjoint surface* X^* *meets* E *along the trace* $X^* : C \to \mathbb{R}^3$ *at a right angle. Moreover, the converse of this statement is also true.*

Therefore, the adjoint surface X^* of a solution $X : B \to \mathbb{R}^3$ of the free boundary problem for a polyhedron S is bounded by a polygon Γ , the edges of which intersect the planes of the corresponding faces of S orthogonally. If S has only four faces F_1, F_2, F_3, F_4, the polygon Γ is, up to similarity, uniquely determined by S and by the circuit in which $X(\partial B)$ runs through the faces F_κ. Moreover one may reverse the procedure. There are three quadrilaterals corresponding in the described way to the three essentially different circuits of the faces of F , and each of these quadrilaterals bounds a unique minimal graph. From these graphs, one can retrieve three solutions of the free boundary problem for S by passing to their adjoint surfaces.

We note that, for a particular tetrahedron, this procedure had already been carried out by H.A.Schwarz; cf. [18]. Unfortunately this simple method does not work for polyhedra with five or more faces.

Nevertheless there have recently been proved existence theorems for disk-type minimal surfaces stationary within a smooth ,closed, and convex surface S. Struwe [20] established the existence of at least one disk-type solution but had to allow for possible selfintersections of the minimal surface, whereas Grüter and Jost [7] established the existence of embedded disk-type solutions. Jost has lately proved that there always exist at least three solutions provided that the curvature of S satisfies a suitable pinching condition.

What are the stationary minimal surfaces within a sphere S? Clearly the disks passing through the centre of S and with boundary on S are solutions. J.C.C.Nitsche [17] recently proved that these are the only disk-type minimal surfaces stationary within S, by employing the fact that for each minimal surface $X(w)$, $w = u + iv$, the expression

$$(L-N) - 2iM$$

is a holomorphic function of w (here L,M, and N denote the coefficients of the second fundamental form of $X(w)$). On the other hand, there certainly exist solutions of higher topological type for S. For instance, if we choose $a = (x_0^2 + \cosh^2 x_0)^{-1/2}$, where x_0 is the positive root of the equation

$$\tanh x = \frac{1}{x},$$

then the catenoid generated by the catenary $y = a \cosh(\frac{x}{a})$ intersects the unit sphere around the origin at a right angle.

We may similarly ask: *What are the disk-type minimal surfaces that intersect a cylinder* $S = C \times \mathbb{R}$ *at a right angle*? Here C denotes a closed curve contained in a plane E with the normal vector e. As is to be expected, the only continuous disk-type solutions are the planar surfaces obtained by orthogonally intersecting S by a plane parallel to E. In this case, the proof is very easy. In fact, if $X^*(r,\theta)$ is the adjoint of a solution $X(r,\theta)$ for S, we infer from the equations

$X_r(1,\theta) = X^*_\theta(1,\theta)$ and $X_r(1,\theta)\cdot e = 0$ that $X^*_\theta(1,\theta)\cdot e = 0$. Thus $X^*(1,\theta)$ is a plane curve, and $X^*(r,\theta)$ turns out to be planar as well. Correspondingly, X^* has a constant surface normal whence the surface normal of X is constant as well, and the assertion follows at once.

There is still another case where we can make good use of the adjoint surface. Consider the so called *thread problem*, where one asks for area minimizing minimal surfaces bounded by a configuration $\langle\Gamma,S\rangle$ consisting of a fixed arc Γ and a moveable arc Σ of prescribed length $L(\Sigma)$ connecting the two end points P_1 and P_2 of Γ. If the length $L(\Sigma)$ of Σ satisfies

$$|P_1-P_2| < L(\Sigma) < L(\Gamma),$$

the existence of nontrivial solutions is to be expected, the "open parts" of which, however, will in general be disconnected, as simple thought-experiments show. H.W. Alt [1] has established an existence theorem. One expects that those parts of Σ which are not attached to Γ will be regular real analytic curves, and it will then also turn out that their curvature is constant. J.C.C.Nitsche [15] proved that they have a $C^{2,\alpha}$-parameter representation, but had to allow for possible branch points. Between branch points, the parameter representation was shown to be of class C^∞ . U.Dierkes, S.Hildebrandt, and H.Lewy [3] recently proved that the *non-attaching and non-selfintersecting parts of Σ are regular, real-analytic curves, with a parameter representation by the solution of the thread problem that is free of boundary branch points.*

The proof of this fact, given in [3], is based on the fact that if $X : B \to \mathbb{R}^3$ is an area minimizing solution of the thread problem which maps an arc C of ∂B into a non-attaching part of the thread, then the adjoint surface X^* of X maps C into a curve Σ^* that lies on a sphere S, and X^* meets S along Σ^* at a right angle. (Note that

X^* need not stay strictly on one side of S). Regularity then follows from the results of [6].

Let us return to the free boundary problem for configurations $\langle\Gamma,S\rangle$. The following finiteness result has been proved by Alt and Tomi [2]:

Let S *be a compact, embedded, analytic surface in* $\mathbb{R}^3$ *and* Γ *be a homotopically non-trivial Jordan curve in the unbounded component of* $\mathbb{R}^3$-S. *Then there are only finitely many (geometrically different) surfaces of disk-type which minimize Dirichlet's integral (and hence area) among all surfaces bounded by the configuration* $\langle S\rangle$.

To our knowledge, the following result due to Hildebrandt and Nitsche [11] is the only nontrivial *uniqueness theorem* for minimal surfaces stationary within a configuration $\langle\Gamma,S\rangle$ which cannot (by an obvious reflection argument) be directly derived from a uniqueness theorem for a single Jordan curve. To formulate this result, we make the following *assumptions*:

Let S be the half plane $\{(x,y,z) : x \geq 0 \ , \ y = 0\}$ in $\mathbb{R}^3$, and let Γ be a regular curve of class $C^{1,\alpha}$, $0 < \alpha < 1$, which does not meet S except for its end points P_1 and P_2, where Γ issues from S at right angles. Suppose also, that P_1 and P_2 are different and lie in the interior of S. Finally, assume that Γ is symmetric with respect to the x-axis and the orthogonal projection γ of Γ onto the (x,y)-plane is a closed, strictly convex and regular curve of class $C^{1,\alpha}$, and that the projection map is 1-1 , except for P_1 and P_2 which are projected at the same point of γ . Then the following holds:

There exists exactly one disk-type solution of the free boundary problem for $\langle\Gamma,S\rangle$, *the free trace of which on* S *is touching* ∂S . *This uniquely determined solution is area minimizing and can be written as graph over a planar slit domain.* (There may, still, be other solutions whose boundary does not touch ∂S. They are, however, not interesting since their interior will intersect S .)

We will close our report by mentioning an estimate for the length $L(\Sigma)$ of the free trace of a solution $X : \bar{\Omega} \to \mathbb{R}^3$ for the free boundary problem to the contour $\langle \Gamma, S \rangle$. In fact, we have

$$L(\Sigma) \le L(\Gamma) + \frac{2}{R} A(X) \tag{3}$$

provided that S satisifies a two-sided sphere condition with spheres of radius R.

If Γ is empty, formula (3) reduces to

$$L(\Sigma) \le \frac{2}{R} A(X) \ . \tag{4}$$

Here, $A(X)$ denotes the area of X which is given by the Dirichlet integral of X.

These formulas had first been proved by Hildebrandt and Nitsche [12] with a worse constant than 2, assuming also that X had no boundary branch points of odd order. Küster [13] proved the result with the optimal constant 2, and finally Dziuk [5] showed the assumption on the boundary branch points to be superfluous.

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