CLASSIFICATION OF MINIMIZING HYPERSURFACES ASYMPTOTIC TO QUADRATIC CONES IN $\mathbb{R}^{n+1}$

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A celebrated theorem of S. Bernstein states that in $\mathbb{R}^3$, every entire minimal graph is an affine plane [B]. Nowadays, Bernstein's theorem can be understood as a corollary to a much broader result, sometimes called the *parametric Bernstein theorem*: *when* $n \leq 7$, *every area-minimizing hypersurface in* $\mathbb{R}^{n+1}$ *is an affine hyperplane.* Here, *hypersurface* means a current S of the form

$$S = \partial[\![U]\!],$$

where $[\![U]\!]$ denotes the current corresponding to oriented integration of $(n+1)$-forms over an open set $U \subset \mathbb{R}^{n+1}$. We say that S is *area-minimizing* if, whenever $r > 0$ and $B_r = \{x \in \mathbb{R}^{n+1} : |x| < r\}$, we have

$$\|S\|B_r \leq \|S+Z\|B_r$$

for all hypersurfaces Z supported within B_r. (i.e., inside B_r, S has less n-area than any other hypersurface which agrees with it outside B_r.) In particular, a Standard Stokes' Theorem argument (see e.g. [SL3]) immediately shows that entire minimal graphs are area-minimizing; this is why the original theorem of Bernstein follows from the later parametric version.

Here, we concern ourselves with the question of what happens when the parametric Bernstein theorem fails, as it does when $n \geq 8$. This failure was discovered in 1968 by Bombieri, DeGiorgi, & Giusti [BDG], and a considerable number of counterexamples in high dimensions are now known. Simplest among these are the hypercones $C^{p,q}$ with $\min\{p+q,pq\} > 5$, defined as follows.

$$C^{p,q} = \partial [\![U^{p,q}]\!]$$

$$U^{p,q} = \left\{(x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : q|x|^2 < p|y|^2\right\} .$$

As first noted by W.H Fleming [F], hypercones play a central role in the study of area-minimizing hypersurfaces. Indeed, if S is any area-minimizing hypersurface, then S is *asymptotic* to an area-minimizing hypercone near infinity. By this we mean there is a sequence $\{r_i\}$ of radii decreasing to zero, and an area-minimizing hypercone $C = C(S,\{r_i\})$ such that

$$\lim_{i\to\infty}(r_i)_\# S = C$$

Here (r_i) denotes the homothety $x \mapsto r_i x$ in $\mathbb{R}^{n+1}$, and $\lim_{i\to\infty}$ is taken in the appropriate weak sense (the integral flat topology [SL1]).

This asymptotically conical behaviour suggests that one should try to classify area-minimizing hypersurfaces according to their asymptotic limits. While such a classification appears formidably difficult in general, we can report here a complete solution of the problem for the particular cones $C^{p,q}$ defined above:

THEOREM: ([SS]): *If* S *is an area-minimizing hypersurface asymptotic to* $C^{p,q}$ *near infinity, then up to similarities of* $\mathbb{R}^{n+1}$, S *is either* $C^{p,q}$ *itself,* $T^{p,q}$, *or (if* $p \neq q$*)* $T^{q,p}$.

To clarify this statement, we should say that *similarity* means an isometry of $\mathbb{R}^{n+1}$ composed with a homothety. We also need to make the following definition.

DEFINITION: $T^{p,q}$ is the *unique area-minimizing* hypersurface in $U^{p,q}$ having unit distance to the origin $0 \in \mathbb{R}^{n+1}$.

The existence and uniqueness of $T^{p,q}$ is a special case of a result of Hardt & Simon [HS]. $T^{p,q}$ is also smooth, and its homothetic images $(r)_{\#}T^{p,q}$, $r > 0$, foliate $U^{p,q}$.

We now give an overview of the proof of our theorem.

PROOF: It is well-known that $C^{p,q} \cap S^n =: \Sigma^{p,q}$ is isometrically a product of spheres:

$$\Sigma^{p,q} = \left(\sqrt{\frac{p}{n-1}}\right) S^p \times \left(\sqrt{\frac{q}{n-1}}\right) S^q \subset S^n \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^{n+1}.$$

$\Sigma^{p,q}$ is of course a minimal hypersurface in S^n, and has constant length second fundamental form, so that its second variation operator $\mathcal{J}_\Sigma$ has the very simple expression

$$\mathcal{J}_\Sigma = \Delta_\Sigma + 2n - 2.$$

By Δ_Σ we denote the intrinsic covariant laplacian of $\Sigma^{p,q}$. Using the classical theory of spherical harmonics, it is possible to compute explicity the spectral decomposition of $L^2(\Sigma^{p,q})$ relative to the self-adjoint elliptic operator $\mathcal{J}_\Sigma$. In particular, we have the following proposition concerning the first three eigenspaces of $\mathcal{J}_\Sigma$. Below, the normal ν on $\Sigma^{p,q}$ is the restriction to $\Sigma^{p,q}$ of an orienting unit normal vectorfield

$$\nu : C^{p,q} \longrightarrow S^n$$

on $C^{p,q}$. Below $\mathfrak{so}(n+1)$ denotes the Lie algebra of the special orthogonal group SO(n+1); i.e. the $(n+1) \times (n+1)$ skew-symmetric matrices.

PROPOSITION: [SS]. *Let* V_1 , V_2 , V_3 *be the first three eigenspaces of* $\mathcal{J}_\Sigma$ *on* $\Sigma^{p,q}$. *if* $\varphi \in V_i$, *then for all* $\omega \in \Sigma^{p,q}$,

(i) φ is constant, when $i = 1$,

(ii) $\varphi(\omega) = \nu(\omega)\cdot z$ for some $z \in \mathbb{R}^{n+1}$ when $i = 2$,

(iii) $\varphi(\omega) = G\omega\cdot\nu(\omega)$ for some $G \in \mathfrak{so}(n+1)$ when $i = 3$.

It also happens that V_3 is the *kernel* of $\mathcal{J}_\Sigma$, otherwise known as the space of *Jacobi fields* on $\Sigma^{p,q}$. Conclusion (iii) above therefore shows that every Jacobi field is the initial normal velocity of a one-parameter family of rotations of $\Sigma^{p,q}$. This is a powerful fact. In particular, it implies that S decays to $C^{p,q}$ polynomially near infinity, by a theorem of Almgren & Allard ([AA] or [SL2]). More precisely, it means there is a function $u : C^{p,q} \longrightarrow \mathbb{R}$, such that (after rescaling if necessary)

$$\operatorname{spt}(S) \subset \{x+|x|u(x)\nu(x) \ : \ x \in C^{p,q}\} ,$$

and rather importantly, with $r := |x|$,

$$u = O(r^{-\alpha}) \quad \text{as} \quad r \to \infty$$

for some $\alpha > 0$. In other words, for sufficiently large $r > 0$, S is the graph over $C^{p,q}$ of a "rapidly" decaying function.

Since S is minimal, it now follows that for $r > 1$,

(1) $$\mathcal{M}u = 0 \ .$$

Here $\mathcal{M}$ is the mean curvature operator on $C^{p,q}$. On the other hand, since u decays to zero as $r \to \infty$, it turns out to be useful to write (1) as

(2) $$\mathcal{J}u = f,$$

where $\mathcal{J}$ is the linearization of $\mathcal{M}$ at $u \equiv 0$. More specifically,

$$\mathcal{J} = rD_{rr} + (n+1)D_r + \frac{1}{r}\mathcal{J}_\Sigma \ .$$

The function f in (2) is simply the nonlinearity $[\mathcal{M}-\mathcal{J}]u$. Note also that the spherical part of $\mathcal{J}$ is precisely the operator $\mathcal{J}_\Sigma$ discussed earlier.

Now, by studying (2), and using the decay estimate $u = O(r^{-\alpha})$ cited above, we derive an asymptotic expansion for u as $r \to \infty$, and find that exactly *one* of the following three alternatives applies.

(3) $\quad u(r\omega) = \frac{1}{r}\varphi(\omega) + o(\frac{1}{r}) \quad$ for some $0 \neq \varphi \in V_2$.

(4) $\quad u(r\omega) = \varphi(\omega)r^{-\gamma} + o(r^{-\gamma}) \quad$ for some $0 \neq \varphi \in V_1$.

(5) $\quad u(r\omega) = o(r^{-\gamma})$.

Here $r^{-\gamma}$ is the leading term in the analogous asymptotic expansion which one obtains for $T^{p,q}$ and $T^{q,p}$ as opposed to S.

Suppose (3) obtains. Using conclusion (ii) of our proposition, we rewrite $\varphi(\omega) = \nu(\omega)\cdot z$. But $\frac{1}{r}(\nu(\omega)\cdot z)$ agrees, to first order as $r \to \infty$, with the function on $C^{p,q}$ whose graph is merely the translate of $C^{p,q}$ by $z \in \mathbb{R}^{n+1}$. We then show that, by applying the translation $-z$ to our original hypersurface S , we reduce to the case in which either (4) or (5), but no longer (3), must obtain.

Consider then (4). Since every $\varphi \in V_1$ is a constant, (4) implies, in particular, that for sufficiently large $r > 0$, u has a fixed sign; i.e. S lies on one side of $C^{p,q}$ near infinity. A comparison argument based on [SL1, 37.10] then shows that S lies on one side of $C^{p,q}$ globally-- i.e. in $U^{p,q}$ or $U^{q,p}$. The uniqueness result of Hardt & Simon quoted after our definition of $T^{p,q}$ now shows that if (4) obtains, S is (up to similarity) either $T^{p,q}$ or $T^{q,p}$.

Finally, in case of (5), we see that S actually equals $C^{p,q}$. For, (5) says that S decays to $C^{p,q}$ *faster* than any homothety of $T^{p,q}$ or $T^{q,p}$ as $r \to \infty$. By the same type of comparison argument used above, it follows that S is then disjoint, globally, from every homothetic image of $T^{p,q}$ and $T^{q,p}$. But these homotheties of $T^{p,q}$ and $T^{q,p}$ actually foliate all of $\mathbb{R}^{n+1} \sim C^{p,q}$. S is therefore constrained within $C^{p,q}$, hence equals $C^{p,q}$ by the constancy theorem for currents [SL1] // .

For further discussion of this theorem, and some remarks about its generalizability to a wider class of minimizing hypercones, we refer the reader to [SS].

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