CAPILLARY SURFACE REGULARITY IN CORNER SUBDOMAINS OF R^n

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The nonparametric capillary problem is to find a surface S_u=graph(u) above a subdomain Ω of R^n so that S_u has prescribed mean curvature above Ω and makes prescribed angle of contact with the bounding cylinder $\Sigma=\partial\Omega \times R$. Letting υ be the downward normal to S_u (or its first n components when appropriate), and letting γ be the inner normal to Σ, this quasilinear elliptic boundary value problem can be written as

$$\text{CP} \qquad \begin{aligned} &\operatorname{div} \upsilon = \Psi \quad \text{in } \Omega, \text{ where } \Psi_u \geq 0. \\ &\upsilon \cdot \gamma = \Phi \quad \text{on } S_u \cap \Sigma, \text{ where } \Phi_u \geq 0 \text{ and } |\Phi| < 1-\delta. \end{aligned}$$

The capillary problem has been solved both variationally (using functions of bounded variation or geometric measure theory), and by using an elliptic partial differential equation approach that combines apriori estimates with the method of continuity. For smooth domains the solution u exists and is regular on the closed domain, at least in the case that one can prove an a priori height estimate $|u| \leq M$. (This is always the case if gravity is positive, $\Psi_u \geq \delta > 0$, but may not be the case in general. Without the assumtion of positive gravity the shape of Ω becomes important.)

The capillary problem makes sense even if $\partial\Omega$ has a compact (n-1)-dimensional singular set Γ. (The variational problem can still be solved, or alternately the P.D.E. approach can be combined with a domain approximation argument, to find a function that solves CP everywhere except on Γ.) In this case, at least for positive gravity, one knows that the solution is smooth away from Γ, and it is natural to study its behavior near Γ. For two-dimensional corner domains, where Γ is a point at which Ω has an interior angle θ, and where the contact angle is ϕ (i.e. $\upsilon \cdot (-\gamma) = \cos\phi$) the somewhat surprising results have been known for several years [1][6][2]:

(a) If $\theta < |\pi - 2\phi|$ the solution to CP is either unbounded at Γ or it doesn't exist (depending on whether gravity is positive or not).

(b) If $|\pi-2\phi|<\theta<\pi$ any solution to CP that is smooth except at Γ extends to be C^1 there.

(c) If $\pi<\theta$ there are domains Ω and bounded solutions to CP (for any $\phi\neq\pi/2$) that are regular away from Γ but which have jump discontinuities at Γ.

These results are related to peculiarities of the contact angle boundary condition (For the range of angles in *(a)* there are no functions that are C^1 at Γ and which can satisfy the contact angle boundary condition on both arcs of $\partial\Omega$ meeting there; for $\upsilon\cdot\gamma$ to attain the prescribed values, υ would have to be a vector with magnitude larger than 1.), and perhaps to the well known importance of domain convexity in solving the 2-dimensional Dirichlet problem for the prescribed mean curvature problem (non-convexity leads to problems for the contact angle problem as shown by *(c)*).

In light of these results a reasonable generalization would be:

CONJECTURE *Let Ω be compact in R^n. Let Γ be a compact subset of $\partial\Omega$, $H_{n-1}(\Gamma)=0$. Suppose $\partial\Omega\backslash\Gamma$ is smooth (C^3) and that there is a bounded solution u to CP, smooth on the closed domain, except possibly on Γ. Suppose*

(i) There exists a smooth function w on Ω that extends to be C^1 on the closed domain and that satisfies the boundary conditions of CP.

(ii) $\partial\Omega$ satisfies a uniform exterior sphere condition of radius $R>0$.

Then u extends to be C^1 on the closure of Ω.

In the paper [4] it is shown that with some modifications this conjecture can actually be proven: If condition *(i)* is replaced with the stronger requirement *(i')* below then one can conclude Lipschitz continuity for u on the closure of Ω. (For a corner subdomain of R^2, blow-up arguments show immediately that Lipschitz implies C^1 but this does not seem to be so immediate in higher dimensions.)

(i') There exists a "pseudo-distance" function $\rho\in C^3$ near Γ, $\rho|_\Gamma=0$, $\rho|_\Omega>0$, so that on $\partial\Omega\backslash\Gamma$ we have

$$-\Xi\nabla\rho\cdot\gamma=\Phi$$

where Ξ is smooth near Γ, $|\Xi|\leq 1-\delta$.

To understand *(i')* consider a corner in $\mathbb{R}^2$: If the symmetry axis of the interior angle is the x-axis and if the vertex Γ is 0 then $\rho(x,y)=x$ works. Furthermore this example has a natural generalization to higher dimensions, since a similar ρ can be constructed if $\partial\Omega$ consists of two smooth hypersurfaces meeting along some smooth, compact (n-2)-dimensional surface Γ, in such a way that the interior angle θ satisfies *(b)* at all points of Γ. If the codimension of Γ is larger than 2, for example the vertex of a cone in $\mathbb{R}^3$, then it is necessary to put more restrictions on the geometry of Ω near Γ. For the cone example the right cross sections must be circles in order for a ρ to exist (or for *(i)* to be satisfied).

The method used to prove the Lipschitz result involves approximating CP with capillary problems in smooth Ω_j near Ω (smoothed appropriately in a $^1/_j$ - neighborhood of Γ), and with positive gravity at least $^1/_k$:

$$CP_{j,k} \qquad \begin{array}{ll} \operatorname{div} \upsilon = \psi + {}^u/_k & \text{in } \Omega_j. \\ \upsilon\cdot\gamma = -\Xi\nabla\rho\cdot\gamma & \text{on } \partial\Omega_j. \end{array}$$

For the smooth solutions $u_{j,k}$ to $CP_{j,k}$ one can apply a maximum principle argument to derive bounds for $|\nabla u_{j,k}|$. The argument works because of the interplay between the boundary condition of $CP_{j,k}$ and $\nabla\rho$ that is a consequence of *(i')* (Note $\upsilon\cdot\gamma$ is not extended to be Φ on $\partial\Omega_j$, but rather in a manner using $\nabla\rho\cdot\gamma$. This extension is natural: for the corner in $\mathbb{R}^2$ the required contact angle is exactly the one attained by the hyperplane satisfying the original contact angle boundary condition along Σ as it contacts the tube above $\partial\Omega_j$.) Also important is the almost convex nature of the smoothed domains that is a consequence of *(ii)*. One derives bounds indepently of j (for k fixed), lets $j\to\infty$, uses the convergence properties of capillary surfaces and concludes a Lipschitz bound for u_k, the solution to the gravity capillary problem in Ω. After showing that this bound is actually independent of k, one lets $k\to\infty$ and concludes the desired Lipschitz bound for u.

In smooth domains and for interior estimates, maximum principle arguments of this type have been studied extensively by G. Lieberman and this author [3][4][5]. One way to understand them (but not the way they are explained in the previous work) is in terms of the intrinsic gradient and Laplacian of the surface S_u. One seeks to bound

$v=(1+|Du|^2)^{1/2}$, or some functional involving v. This is because $^1/_v = -\upsilon^{n+1} = -<\upsilon,e_{n+1}>$, so that $\Delta(^1/_v)$, hence Δv, is easy to compute for a surface of prescribed mean curvature: If an orthonormal frame $\{f_1,f_2,...,f_n\}$ is chosen on S_u so that at $P\in S_u$ the covariant derivative of f_i with respect to f_j is normal to S_u, if we use $[h_{ij}]$ and $|A|$ for the corresponding second fundamental form and its norm at P, and if Δ and ∇ are the surface Laplacian and gradient, then

$$\Delta(\frac{1}{v}) = f_i(f_i(\frac{1}{v})) = -f_i<\nabla_{f_i}\upsilon, e_{n+1}> = -f_i<h_{ij}f_j, e_{n+1}> = -<f_i(h_{ij})f_j + h_{ij}\nabla_{f_i}f_j, e_{n+1}>$$

$$= <-f_j(h_{ii})f_j, e_{n+1}> + <h_{ij}h_{ij}\upsilon, e_{n+1}> = <-\nabla\Psi, e_{n+1}> - |A|^2(\frac{1}{v}).$$

Note the use of the Codazzi formula to interchange derivatives of the second fundamental form, and then the use of the prescribing function Ψ.

In general for the capillary problem one actually studies expressions of the form $<Z,\upsilon>v$ on S_u, where Z is a vector field in (part of) R^{n+1}, with $<Z,\upsilon>\geq\delta>0$. (Z generally has the form $Z=\eta\upsilon+X$ where η and X are smooth functions in R^{n+1}, independent of S_u.) If one can show a bound for v at the maximum value of $<Z,\upsilon>v$ it follows that the expression is bounded in the entire domain. The strategy is to pick Z so that the maximum occurs in the interior of S, to use the fact that the gradient and Laplacian of $<Z,\upsilon>v$ are zero and non-positive there, and to conclude (for good Z and using calculations like the one above) that v is bounded there.

The way to force an interior maximum of $<Z,\upsilon>v$ is as follows: For $P\in S_u\cap\Sigma$ pick a local orthonormal frame $\{f_i\}$, $1\leq i\leq n$, on S_u so that for $1\leq i\leq n-1$, $f_i\in T_P(S_u\cap\Sigma)$, and with f_n pointing into the tube. Also complete the frame with f_0 so that $\{f_i\}$, $0\leq i\leq n-1$, is an orthonormal basis for $T_P(\Sigma)$. It suffices to force $f_n(<Z,\upsilon>v) > 0$. In computing what this expression is one gets a linear combination of terms involving h_{jn}, $1\leq j\leq n$. But by differentiating the boundary condition of CP one can control h_{in}, $1\leq i\leq n-1$, in terms of the second fundamental form $[k_{ij}]$ (actually the term k_{i0}) of Σ, and derivatives of Φ. The h_{nn} term cannot be bounded from the data, but its coefficient is a multiple of $<Z,\gamma>$. We require this to be zero. (Because of the boundary condition for CP this holds iff $\eta\Phi+X\bullet\gamma=0$ along the boundary.) By then adjusting the behavior of Z inside Ω one can force $f_n(<Z,\upsilon>v) >0$.

For the estimates in the n-dimensional corner problem one picks $\mathbb{Z}$ to be:

$$\mathbb{Z}=(\varepsilon+\rho)e^{Lz}(\upsilon+\Xi\nabla\rho), \quad \varepsilon \text{ small, L large.}$$

In computing whether $f_n(<\dot{\mathbb{Z}},\upsilon>v) >0$ one must eventually have a one-sided bound for $k_{ij}Z^iZ^j$, and that is where the almost convex nature of $\partial\Omega_j$, made possible by *(ii)*, is crucial (and the $\mathbb{R}^2$ examples show this is not just a technical requirement). The choices of ε and L are required to complete the maximum principle argument. The details are straightforward but necessarily technical. They are explained in some detail in [4], although as explained earlier the method of exposition is slightly different.

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