DEFORMING RIEMANNIAN METRICS ON COMPLEX PROJECTIVE SPACES

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0. INTRODUCTION

Hamilton [Hm1,Hm2] and Huisken [Hs] have given conditions on the curvature of a compact n-dimensional Riemannian manifold M under which the metric may be deformed to one of constant positive curvature. Their method was to allow the metric to evolve according to the equation

$$(0.1) \qquad \frac{\partial}{\partial t} g_{ij} = \frac{2}{n} r g_{ij} - 2R_{ij} \,,$$

where $r = \int_M R d\mu \,/\, \int_M d\mu$ is the average of the scalar curvature, and study its behaviour as $t \to \infty$. They proved the following

THEOREM *If (a) $n = 3$ and M has positive Ricci curvature [Hm1], (b) $n = 4$ and M has positive curvature operator [Hm2] or (c) $n \geq 4$, M has positive scalar curvature and*

$$(0.2) \qquad |W|^2 + |V|^2 < \delta_n |U|^2 \,,$$

where W, V and U are the Weyl part, the traceless Ricci part and *the scalar curvature part of the Riemann curvature tensor and*

$\delta_4 = 1/5$, $\delta_5 = 1/10$ *and* $\delta_n = 2/(n-2)(n+1)$ *for* $n > 5$, *[Hs]*

then the equation (0.1) has a unique solution for all $t \geq 0$ *which converges in the* C^∞*-topology as* $t \to \infty$ *to a metric of constant positive curvature. Furthermore, any isometries of M are preserved as the metric evolves.*

The aim of this paper is to prove a similar theorem giving conditions under which a metric will evolve, according to a system of equations related to (0.1), to a Kähler metric of constant positive holomorphic sectional curvature (i.e. a multiple of the Fubini-Study metric on $\mathbb{C}P(n)$).

We start with a compact, simply connected Riemannian manifold M of dimension 2n. We then show that under certain conditions on M there exists a principal S^1-bundle over M with a Riemannian metric which satisfies Huisken's condition (i.e. condition (c) above) such that the projection map is a Riemannian submersion. We allow the metric on this bundle to evolve according to (0.1); by the theorem above it evolves to a metric of constant positive curvature. As all isometries are preserved, this induces an evolution of the metric on M, by maintaining the projection map as a Riemannian submersion, under which it evolves to a multiple of the Fubini-Study metric.

The theorem we prove is

MAIN THEOREM *Let M be a compact, simply connected Riemannian manifold of dimension $2n \geq 4$. If there exists a closed 2-form Φ on M and a function $F > 0$ on M such that*

$$A(F,\Phi) = R - F^2|\Phi|^2 - (2/F)\Delta F > 0 \text{ and}$$

$$\begin{aligned} B(F,\Phi) = {} & \varepsilon_n A(F,\Phi)^2 - |R_m|^2 + 6F^2 R_{ijkl}\Phi^{ij}\Phi^{kl} - 6F^4|\Phi|^4 - 10F^4|\Phi^2|^2 \\ & - 24|\nabla F|^2|\Phi|^2 + 24\Phi^2{}_{ij}\nabla^i F\nabla^j F - 4F^2|\nabla\Phi|^2 \\ & - 24F\nabla_i\Phi_{jk}\nabla^i F\Phi^{jk} - (4/F^2)|\nabla\nabla F|^2 - 8F\Phi^2{}_{ij}\nabla^i\nabla^j F > 0 , \end{aligned}$$

where R_m denotes the Riemann curvature tensor, Φ^2 is the symmetric tensor given by $\Phi^2{}_{ij} = \Phi_{ik}\Phi^k{}_j$, $\varepsilon_2 = 11/100$ and $\varepsilon_n = \delta_{2n+1}$ for $n > 2$, then M is diffeomorphic to $\mathbb{C}P(n)$ and the system

$$(0.3)\qquad \begin{aligned} \frac{\partial}{\partial t}g_{ij} &= \frac{2}{2n+1}kg_{ij} - 2R_{ij} - 4F^2\Phi^2_{ij} + \frac{2}{F}\nabla_i\nabla_j F \\ \frac{\partial}{\partial t}F &= \frac{1}{2n+1}kF + \Delta F - F^3|\Phi|^2 \\ \frac{\partial}{\partial t}\Phi_{ij} &= \nabla_i\nabla^k\Phi_{kj} - \nabla_j\nabla^k\Phi_{ki} + \frac{3}{F^2}\nabla_i F\nabla^k F\Phi_{jk} - \frac{3}{F^2}\nabla_j F\nabla^k F\Phi_{ik} \\ &\quad - \frac{3}{F}\nabla^k F\nabla_i\Phi_{jk} + \frac{3}{F}\nabla^k F\nabla_j\Phi_{ik} - \frac{3}{F}\nabla_i\nabla^k F\Phi_{jk} + \frac{3}{F}\nabla_j\nabla^k F\Phi_{ik} , \end{aligned}$$

where $k = \int_M FA(F,\Phi)\,d\mu \,/\, \int_M F d\mu$, has a unique solution for all $t \geq 0$ and g_{ij} converges in the C^∞-topology as $t \to \infty$ to a multiple of the Fubini-Study metric.

The conditions of the theorem are fairly unpleasant, however we have

THEOREM *There exist constants $\delta < 1$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that if M is an almost Hermitian manifold whose holomorphic sectional curvature is δ-pinched and whose curvature and almost complex structure J satisfy $|\nabla R_m| < \varepsilon_1$, $|\nabla J| < \varepsilon_2$, and $|\nabla\nabla J| < \varepsilon_3$ then M satisfies the conditions of the theorem above with F a constant function and Φ a closed 2-form representing the Chern class of M. If M is Kähler we need only assume the holomorphic sectional curvature is δ-pinched.*

1. CURVATURE OF S^1-BUNDLES

Let P be a principal S^1-bundle over M with projection map π. Let ω and Ω be the connection form and curvature form of a connection Γ in the bundle P.

By choosing once and for all an isomorphism between $\mathbb{R}$ and the Lie algebra of S^1, we may consider ω and Ω to be real valued forms on P. Ω is invariant, as S^1 is abelian, and horizontal, thus $\Omega = \pi^*(\gamma)$ for some 2-form γ on M.

Let V be the fundamental vertical vector field corresponding to 1.

Given a positive function f on M, we may define an invariant metric on P by

(1.1) $$\langle u,v\rangle_P = \langle \pi^*u,\pi^*v\rangle_M + f^2(\pi(p))\omega(u)\omega(v) \text{ for } u,v\in T_p(P).$$

Note that all invariant metrics on P have this form; in fact, given such a metric, we may recover the connection Γ, and therefore the curvature form $\Omega = \pi^*(\gamma)$, by defining the horizontal space to be the orthogonal complement of V, we may recover the metric on M from $\pi^*(\langle X,Y\rangle_M) = \langle X^*,Y^*\rangle_P$, where X^* and Y^* are the horizontal lifts of X and Y with respect to the connection just defined, and we may recover f from $\pi^*(f^2) = \langle V,V\rangle_P$.

Let K and K_m denote the scalar curvature and the Riemann curvature tensor of P. Using the fact that $|U|^2 + |V|^2 + |W|^2 = |K_m|^2$ and $|U|^2 = K^2/n(2n+1)$ for a $2n+1$-dimensional manifold, we see that (0.2) for an S^1-bundle over M is equivalent to

$$\varepsilon_n K^2 - |K_m|^2 > 0.$$

A tedious but straightforward calculation shows that

$K = \pi^*(R - f^2|\gamma|^2 - (2/f)\Delta f)$ and

$$|K_m|^2 = \pi^*(|R_m|^2 - 6f^2R_{ijkl}\gamma^{ij}\gamma^{kl} + 6f^4|\gamma|^4 + 10f^4|\gamma^2|^2 + 24|\nabla f|^2|\gamma|^2$$
$$- 24\gamma^2{}_{ij}\nabla^i f\nabla^j f + 4f^2|\nabla\gamma|^2 + 24f\nabla_i\gamma_{jk}\nabla^i f\gamma^{jk} + (4/f^2)|\nabla\nabla f|^2$$
$$+ 8f\gamma^2{}_{ij}\nabla^i\nabla^j f).$$

2. CONSTRUCTION OF P

We use the following theorem of Kostant [Ks] (see also [T]).

THEOREM *Let γ be a closed, integral 2-form on M. There exists a principal S^1-bundle P over M with projection map π and a connection in P such that the curvature form of Γ is $\pi^*(\gamma)$.*

We also use the following slight generalization of a lemma of Kobayashi [Kb], the proof of which is nearly identical to Kobayashi's.

LEMMA *Let $\delta > 0$ and let β be a harmonic 2-form on M. There exists a real number c and an integral 2-form α on M such that $|\beta - c\alpha|^2 < \delta$ and $|\nabla(\beta - c\alpha)|^2 < \delta$.*

Clearly if $H^2(M;\mathbb{R}) \cong \mathbb{R}$ then we can choose $\delta = 0$, i.e. $\beta = c\alpha$.

Writing Φ as the sum of an exact 2-form and a harmonic 2-form and combining the above results gives

LEMMA *Let $\delta > 0$. There exists a principal S^1-bundle P over M with projection map π, a connection Γ in P and a real number c such that the curvature form of Γ is $\pi^*(\gamma)$ where $|\Phi - c\gamma|^2 < \delta$ and $|\nabla(\Phi - c\gamma)|^2 < \delta$. Furthermore, if $H^2(M;\mathbb{R}) \cong \mathbb{R}$ then we can choose $\delta = 0$, i.e. $\Phi = c\gamma$.*

We have assumed that $A(F,\Phi) > 0$ and $B(F,\Phi) > 0$. A and B vary continuously with Φ and $\nabla\Phi$ thus there exist P,Γ,c and γ, as given by the above lemma, such that $A(F,c\gamma)$ and $B(F,c\gamma)$ are arbitrarily close to $A(F,\Phi)$ and $B(F,\Phi)$. We choose P,Γ,c and γ such that $A(F,c\gamma) > 0$ and $B(F,c\gamma) > 0$, which is possible as M is compact. Now $A(cF,\gamma) = A(F,c\gamma)$ and $B(cF,\gamma) = B(F,c\gamma)$ so we also have $A(cF,\gamma) > 0$ and $B(cF,\gamma) > 0$.

Let P have the Riemannian metric determined in (1.1) by the function cF, the connection Γ and the metric on M. Then $K = \pi^*(A(cF,\gamma)) > 0$ and $\varepsilon_n K^2 - |K_m|^2 = \pi^*(B(cF,\gamma)) > 0$ so P, with this metric, satisfies Huisken's condition.

3. EVOLUTION OF THE METRIC

We allow the metric on P to evolve according to (0.1). By Huisken's theorem, it converges to a metric of constant positive curvature and furthermore, since the metric was initially invariant it remains so for all t and thus induces a function f, a closed 2-form γ and a Riemannian metric on M such that these determine the metric on P by (1.1). Thus f,γ and the metric on M all evolve as the metric on P does.

In the limit, P has constant positive curvature and is still a principal S^1-bundle over M. Let $\tilde{P}$ be the universal cover of P and endow $\tilde{P}$ with the metric induced from the limit metric on P. As $\pi_1(P)$ is finite, by using the homotopy sequence of the bundle, we can show that $\tilde{P}$ is also a principal S^1-bundle over M and that the projection map $\tilde{\pi}:\tilde{P} \rightarrow M$ is a Riemannian submersion. Of course $\tilde{P}$ is isometric to S^{2n+1} with a metric of constant curvature.

In the limit, M is the quotient of S^{2n+1} by an orthogonal S^1-action and so must be $\mathbb{C}P(n)$ with a multiple of the Fubini-Study metric.

As M is diffeomorphic to $\mathbb{C}P(n)$, $H^2(M;\mathbb{R}) \cong \mathbb{R}$, thus we may assume that when we chose c and γ earlier we chose them such that

$\Phi = c\gamma$.

The evolution of the metric on M depends on f and γ; however, if we define $F = f/c$ and $\Phi = c\gamma$ for all $t \geq 0$ (which agrees with their initial values), a calculation shows that the evolution of the metric on M may be described by (0.3).

REFERENCES

[Hm1] R.S. Hamilton, *Three-Manifolds with Positive Ricci Curvature,* J. Differential Geom. **17** (1982), 255-306.

[Hm2] ______________, *Four-Manifolds with Positive Curvature Operator,* Preprint.

[Hs] G. Huisken, *Ricci Deformation of the Metric on a Riemannian Manifold,* J. Differential Geom. **21** (1985), 47-62.

[Kb] S. Kobayashi, *Topology of Positively Pinched Kaehler Manifolds,* Tôhoku Math. J.(2) **15** (1963), 121-139.

[Ks] B. Kostant, *Quantization and Unitary Representations,* Lectures in Modern Analysis and Applications III, pp.87-208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[T] D. Tischler, *Closed 2-Forms and an Embedding Theorem for Symplectic Manifolds,* J. Differential Geom. **12** (1977), 229-235.

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