A COUNTEREXAMPLE TO LOCALIZATION FOR MULTIPLE FOURIER SERIES

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ABSTRACT

An explicit description is given of a real-valued function f on $[-\pi,\pi]^2$ which is zero in a neighbourhood of 0 but for which the square partial Fourier sums $S_n f$ satisfy $\lim\sup_n S_n f(0,0) = \infty$. Furthermore, the function is infinitely differentiable everywhere except along the y-axis where it is continuous. Also its support is contained in a square at distance $\pi/2$ from 0 and the square may be chosen to have arbitrarily small sides. Finally, neither of the axes intersect the interior of the support of f.

Roughly speaking, localization for Fourier series means that the behaviour of a Fourier series at a point (or set) depends only on the function in a neighbourhood of that point (or set). For example, in one dimension if an integrable function f is zero in a neighbourhood U of 0, then its Fourier series converges uniformly to zero on every compact subset of U [6]. Igari [4] showed that this is not the case for square summation in dimensions exceeding 1. Analysis of his proof shows that he established the following: (by using the Banach-Steinhaus uniform boundedness principle and estimates of certain linear functionals):

1. THEOREM *There exists a continuous function* f *on* $C([-\pi,\pi]^d)$, $d \geq 2$, *which is zero in a neighbourhood of* 0 *such that*

(1) $$\limsup_N S_N f(0,\ldots,0) = \infty.$$

Here $S_N f(0,\ldots,0) = \Sigma_{|n_1|\leq N,\ldots,|n_d|\leq N} \hat{f}(n_1,\ldots,n_d)$, where the $\hat{f}(n_1,\ldots,n_d)$ make up the usual Fourier coefficients in d-dimensions, that is,

$$\hat{f}(n_1,\ldots,n_d) = (2\pi)^{-d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(x_1,\ldots,x_d)e^{-in_1x_1}\cdots e^{-in_dx_d}dx_1\ldots dx_d.$$

Since we are dealing with square partial sums,

$$S_N f(0,\ldots,0) = (2\pi)^{-d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(x_1,\ldots,x_d)D_N(x_1)\ldots D_N(x_d)dx_1\ldots dx_d,$$

where $D_N(x_k) = \sin(N+\frac{1}{2})x_k/\sin x_k/2$, the N-th Dirichlet kernel.

Using more constructive methods, Goffman and Liu [3] describe an everywhere differentiable function f of two variables which does not have the localization property. Since $D_N(x)D_N(y)$ is bounded as $N \to \infty$ except along the axes (and their 2π-translates), the criteria of whether or not a function possesses the localization property will necessarily involve the relationship between the support of the function and these axes. In the example of Goffman and Liu an axis passes through the interior of the support of f.

In this note we give a quite explicit method for constructing a function as described in the abstract. As shown by Proposition 5, this example is best possible in a number of senses.

For further information on multiple Fourier series and the localization property, see Ash [1] or Zhizhiashvili [5] and the references given there. We mention in particular the extension of the Carleson-Hunt theorem due to Fefferman [2] that if $f \in L^p([-\pi,\pi]^d)$, $p > 1$, then $S_N f \to f$ almost everywhere as $N \to \infty$.

2. THE FIRST EXAMPLE

For $m \in \mathbb{Z}^+ = \{1,2,\ldots\}$, let $(m) = 5(2^{2^m}-1)/2$. (Notice that the fractional part of (m) is always $\frac{1}{2}$). Choose $n \in \mathbb{Z}^+$ with $n \geq 10$ and define

$$f(x,y) = \sin\frac{y}{2}\,\chi(\tfrac{\pi}{2} \leq y \leq \tfrac{\pi}{2} + \tfrac{2\pi}{(n)})\ \Sigma_{m=n}^{\infty} \sin(m)y \times$$

$$m^{-1/2}\ \Sigma_{j=0}^{2^m}\ \chi(\frac{\pi/2+2\pi j}{(m)} \leq x \leq \frac{3\pi/4+2\pi j}{(m)}),$$

where $\chi(a \leq x \leq b)$ is the characteristic function of the interval $[a,b]$. Some immediate observations are:

(i) All the intervals in x are disjoint.

(ii) The period of $\sin(n)y$ is $2\pi/(n)$ and this is divisible by $2\pi/(m)$, the period of $\sin(m)y$, for each $m \geq n$. Hence the functions $\{\sin(m)y : m \geq n\}$ are orthogonal on $[\frac{\pi}{2}, \frac{\pi}{2} + \frac{2\pi}{(n)}]$. Furthermore, on this interval

$$\int \sin(m)y\ \sin(m)y\ dy = \pi/(n) \quad \text{for} \quad m \geq n.$$

(iii) The support of f is contained in a rectangle with sides $(3\pi/4 + 2\pi 2^n)/(n) \times 2\pi/(n)$, and so can be made as small as we please by taking n sufficiently large.

(iv) Because of the disjointness of the intervals in x and the factor $m^{-1/2}$, $f(x,y) \to 0$ as $x \to 0+$ for each value of y.

We now state and prove the crucial property of f:

3. PROPERTY $\lim S_{(N)-\frac{1}{2}} f(0,0) = \infty$ *as* $N \to \infty$.

Proof. Application of the definition of f gives

$$S_{(N)-\frac{1}{2}} f(0,0) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y) \frac{\sin(N)x}{\sin x/2} \frac{\sin(N)y}{\sin y/2} dxdy$$

$$= \frac{1}{(2\pi)^2} \Sigma_{m=n}^{\infty} m^{-1/2} \Sigma_{j=0}^{2^m} \int_{\frac{\pi/2+2\pi j}{(m)}}^{\frac{3\pi/4+2\pi j}{(m)}} \frac{\sin(N)x}{\sin x/2} dx \times$$

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{2\pi}{(n)}} \sin y/2 \sin(m)y \frac{\sin(N)y}{\sin y/2} dy.$$

After cancelling $\sin y/2$ and applying the orthogonality properties 1.2(ii), this expression becomes

$$\frac{\pi}{(n)} \frac{1}{(2\pi)^2} N^{-1/2} \Sigma_{j=0}^{2^N} \int_{\frac{\pi/2+2\pi j}{(N)}}^{\frac{3\pi/4+2\pi j}{(N)}} \frac{\sin(N)x}{\sin x/2} dx .$$

In each of the intervals of integration

$$\frac{\sin(N)x}{\sin x/2} \geq \frac{2\sin(N)x}{x} \geq \frac{2\sin 3\pi/4}{(3\pi/4+2\pi j)/(N)} = \frac{8(N)}{\sqrt{2}\pi(3+8j)}$$

and so each of the integrals is bounded below by

$$\frac{\pi}{4(N)} \times \frac{8(N)}{\sqrt{2}\pi(3+8j)} = \frac{2}{\sqrt{2}(3+8j)} .$$

Hence

$$S_{(N)-\frac{1}{2}} f(0,0) \geq \text{const. } N^{-1/2} \Sigma_{j=0}^{2^N} \frac{1}{3+8j}$$

$$\sim \text{const.}\ N^{-1/2} \log 2^N \to \infty \quad \text{as} \quad N \to \infty,$$

as asserted.

4. THE SECOND EXAMPLE According to properties 2(iii), 2(iv) and 3, the function f described above has all the properties described in the abstract except that it is not infinitely differentiable away from the y axis. This last requirement is achieved by defining $f^{\#}$ as f except that each of the functions

$$\sin\frac{y}{2}\ \sin(m)y\ \chi\left(\frac{\pi}{2} \le y \le \frac{\pi}{2} + \frac{2\pi}{(n)}\right)$$

and

$$\chi\left(\frac{\pi/2+2\pi j}{(m)} \le x \le \frac{3\pi/4+2\pi j}{(m)}\right)$$

are replaced by functions which equal these except that they are modified in small intervals within, but at the edges of, their supports. This is to be done in such a way as to make them C^{∞} but also the values of the integrals used in calculating $S_{(N)-1/2}f(0,0)$ are to be disturbed by such small amounts that $f^{\#}$ still has Property 3. This new function $f^{\#}$ has all the features described in the abstract.

The following proposition demonstrates ways in which the previous construction is best possible.

5. PROPOSITION *Suppose* f *is an integrable function on* $[-\pi,\pi]^2$ *with support in a rectangle* $[a,b] \times [c,d]$, *where* $0 \le a < b < \pi$ *and* $0 < c < d < \pi$. *Then*

$$S_N f(0,0) \to 0 \quad \textit{as} \quad N \to \infty$$

provided

(a) $a > 0$, *or*

(b) $a = 0$ *and* $\lim_{x \to 0} f(x,y)/x = 0$ *uniformly in* y. *(For example,* $\partial f/\partial x$ *exists along the* y*-axis and* f *is continuous in a neighbourhood of this axis.)*

Proof. The proof of part (a) follows from the Riemann Lebesgue lemma since $f(x,y)/(4 \sin x/2.\sin y/2)$ is integrable.

Under the hypotheses of part (b) for each $\varepsilon > 0$ there exists $\delta \in (0,\pi)$ such that

$$|f(x,y)/2\sin x/2| \le \varepsilon \quad \text{for} \quad |x| \le \delta.$$

Write $S_N f(0,0)$ as

$$S_N f(0,0) = \int_0^\delta dx \int_c^d dy\ f(x,y) D_N(x) D_N(y) + \int_\delta^b dx \int_c^d dy\ f(x,y) D_N(x) D_N(y).$$

The absolute value of the first double integral is bounded by

$$\int_c^d dy \int_0^\delta |f(x,y)/2\sin x/2| |\sin(N+\tfrac{1}{2})x| |D_N(y)| dx$$

$$\le \varepsilon\delta \int_c^d |D_N(y)| dy \le \text{const}.\varepsilon$$

since $c > 0$, while the second converges to 0 as $N \to \infty$ as in part (a). Hence $S_N f(0,0) \to 0$ as $N \to \infty$, as required.

6. HIGHER DIMENSIONS Similar procedures show that functions with properties analogous to those described in the abstract can be constructed for any dimension exceeding 1. They will be infinitely differentiable except along one of the axes.

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