INTEGRATION OF LIE ALGEBRAS

Derek W. Robinson

1. Introduction

The principal question we wish to address can be informally phrased as follows:

When is a Lie algebra of closed operators on a Banach space the differential of a continuous representation of the corresponding Lie group?

An answer, expressed equally informally, can be given as follows:

Whenever an associated heat equation has a unique solution satisfying certain smoothness conditions.

The answer immediately raises a second question:

What are the minimal smoothness requirements?

The best response currently known to this latter problem is as follows:

For general group representations C_4*-conditions are sufficient but in special cases less is required, e.g. for unitary representations on Hilbert space* C_3*-conditions suffice.*

In order to pose these questions more precisely and to explain the answers more accurately we first introduce a number of formal definitions. Subsequently we outline the general strategies usually adopted to tackle such integrability problems. Finally we describe the various special techniques developed to solve the problems and survey various recent results in this area.

2. General Formalism

In this section we first formulate the integrability problem in detail. Second we discuss the general approaches to its solution.

Let $\mathcal{B}$ be a Banach space and V a collection of closed linear operators acting on $\mathcal{B}$. Further let $\mathcal{B}_n(V)$ denote the intersection of the domain of all monomials of order n in elements of V and $\mathcal{B}_\infty(V)$ the intersection of the $\mathcal{B}_n(V)$. If $\mathcal{B}$ is one of the usual function spaces over $\mathbb{R}^n$ and V consists of the operators of partial differentiation then $\mathcal{B}_n(V)$ corresponds to the subspace of n-times differentiable functions. Hence we refer to $\mathcal{B}_n(V)$ as the C_n-elements and the family of spaces as the C_n-structure.

Next we define a representation of the Lie algebra $\mathfrak{g}$ on the Banach space $\mathcal{B}$ as a collection of closed operators $V = \{V(x);\, x \in \mathfrak{g}\}$ indexed by the elements of $\mathfrak{g}$ with the two properties:

1. Density

$\mathcal{B}_\infty(V)$ is norm-dense in $\mathcal{B}$,

2. Structure relations

$$V(x+y)a = V(x)a + V(y)a$$

$$(\mathrm{ad}\ V(x))(V(y))a = V((\mathrm{ad}\ x)(y))a$$

for all $x,y \in \mathfrak{g}$ and $a \in \mathcal{B}_2(V)$.

There are a number of possible variants of this definition in which both conditions are weakened. For example one could assume $\mathcal{B}_n(V)$ is dense for some $n \in \langle 2,\infty\rangle$ and the structure relations hold on $\mathcal{B}_n(V)$. Alternatively one could assume the structure relations on a dense subspace $\mathcal{D} \subseteq \mathcal{B}_\infty(V)$ which is invariant under the action of the $V(x)$, $x \in \mathfrak{g}$. But for simplicity we adopt the foregoing definition. Nevertheless the weaker variations are commonly used and the precise formulation of many of the subsequent results is sensitive to these variations. This must be borne in mind when making comparisons with the literature.

Note that the C_n-elements $\mathcal{B}_n(V)$ of $(\mathcal{B},\mathfrak{g},V)$ form a Banach space with respect to the norm

$$\|a\|_n = \sup_{0\leq m\leq n} \rho_m(a)$$

where the seminorms ρ_m are defined recursively, with the aid of a basis $x_1,\ldots,x_d$ of $\mathfrak{g}$, by $\rho_0(a) = \|a\|$ and

$$\rho_m(a) = \sup_{1\leq i\leq d} \rho_{m-1}(V(x_i)a) .$$

If $(\mathcal{B},G,U)$ denotes a continuous representation of the connected Lie group G then for each $x \in \mathfrak{g}$, the Lie algebra of G, $t \in \mathbb{R} \mapsto U(\exp\{tx\})$ is a continuous one-parameter group. Let $dU(x)$ denote the generator of this group then the collection $dU = \{dU(x); x \in \mathfrak{g}\}$ forms a representation of $\mathfrak{g}$ in the foregoing sense. We refer to $(\mathcal{B},\mathfrak{g},dU)$ as the *differential* of $(\mathcal{B},G,U)$. Now we can give a precise formulation of the question posed in the introduction.

The representation $(\mathcal{B},\mathfrak{g},V)$ is defined to be *integrable* if there exists a continuous representation $(\mathcal{B},G,U)$ of the simply connected Lie group G which has $\mathfrak{g}$ as its Lie algebra such that $V(x) = dU(x)$ for each $x \in \mathfrak{g}$. The problem is to find simple, useful, and general, criteria for integrability. As a preliminary we describe some conditions which are necessary and others that are sufficient for integrability.

Let $x_1,\ldots,x_d$ denote a basis of $\mathfrak{g}$. Then integrability of $(\mathcal{B},\mathfrak{g},V)$ requires that each $V(x_i)$ generates a continuous one-parameter group. But then it follows from the Feller-Miyadera-Phillips theorem on generation of one-parameter groups that each $V(x_i)$ must satisfy appropriate dissipativity conditions. Hence we define V to be *weakly conservative* if there exists a basis

$x_1,\ldots,x_d$ of $\mathfrak{g}$, an $M \geq 1$, and an $\omega \geq 0$, such that

$$\|(I + \epsilon V(x_i))^n a\| \geq M^{-1}(1-|\epsilon|\omega)^n \ \|a\|$$

for all $a \in \mathcal{B}_n(V)$, $n \geq 1$, and all $\epsilon \in \langle -\omega,\omega\rangle$ and $i = 1,\ldots,d$. Moreover, V is defined to be *conservative* if there exists a basis $x_1,\ldots,x_d$ of $\mathfrak{g}$ such that

$$\|(I + \epsilon V(x_i))a\| \geq \|a\|$$

for all $a \in \mathcal{B}_1(V)$, and all $\epsilon \in \mathbb{R}$ and $i = 1,\ldots,d$. The latter condition is necessary for $(\mathcal{B},\mathfrak{g},V)$ to integrate to an isometric representation of G.

Although it is necessary for integrability that each $V(x_i)$ generates a one-parameter group this is not generally sufficient; the family of groups generated by the $V(x_i)$ do not automatically give a group representation. The general approach to integrability has been to consider the problem in two stages:

> First, establish conditions which ensure the $V(x_i)$ generate one-parameter groups V^{x_i},
>
> Second, find conditions which guarantee that the V^{x_i} patch together to give a group representation.

Successful accomplishment of the second stage requires smoothness of the action of the groups V^{x_i} with respect to the C_n-structure of the representation $(\mathcal{B},\mathfrak{g},V)$. The simplest result of this nature is the following:

Proposition 2.1. *Assume each* $V(x)$, $x \in \mathfrak{g}$, *generates a strongly continuous one-parameter group* V^x. *The following conditions are equivalent for each* $n = 1,2,\ldots$

1. $(\mathcal{B},\mathfrak{g},V)$ *is integrable.*

2. *For each* $x \in \mathfrak{g}$, *one has* $V^x\mathcal{B}_n = \mathcal{B}_n$ *and* V^x *restricted to* $\mathcal{B}_n$ *is*

$\|\cdot\|_n$*-continuous.*

Stronger versions of this proposition can be proved which only involve assumptions on the groups V^{x_i} associated with a basis of $\mathfrak{g}$. Alternatively the result can be used to establish an integrability result based on the analytic structure of the representation.

The analytic elements of an operator H on $\mathcal{B}$ are defined as the set of $a \in \cap_{n\geq 1} D(H^n)$ such that

$$\sum_{n\geq 1} \frac{t^n}{n!} \|H^n a\| < \infty$$

for some $t > 0$. The analytic elements form a subspace of $\mathcal{B}$ denoted by $\mathcal{B}_\omega(H)$. Similarly the analytic elements of the representation $(\mathcal{B},\mathfrak{g},V)$ are defined to be the subspace of $\mathcal{B}_\omega(V)$ of $a \in \mathcal{B}_\infty(V)$ such that

$$\sum_{n\geq 1} \frac{t^n}{n!} \|a\|_n < \infty$$

for some $t > 0$.

A basic result on one-parameter groups states that an operator H on $\mathcal{B}$ generates a strongly continuous group if, and only if, H is weakly conservative and $\mathcal{B}_\omega(H)$ is norm-dense. This result has a direct analogue for representation of Lie algebras.

Theorem 2.2. *The following conditions are equivalent:*

1. $(\mathcal{B},\mathfrak{g},V)$ *is integrable,*
2. a. V *is weakly conservative,*

   b. $\mathcal{B}_\omega(V)$ *is norm-dense.*

Although these results are useful as intermediate devices they are impractical in applications as they involve verification of conditions for a generating family of one-parameter subgroups. The striking feature of integration theory alluded to in the introduction is that one only needs information on one semigroup, the heat semigroup.

3. The Heat Semigroup

Let $x_1,\dots,x_d$ be a basis of $\mathfrak{g}$ and define the corresponding Laplacian Δ in the representation $(\mathcal{B},\mathfrak{g},V)$ by $D(\Delta) = \cap_{i=1}^{d} D(V(x_i)^2)$ and

$$\Delta = -\sum_{i=1}^{d} V(x_i)^2 .$$

If the representation is the differential of a group representation $(\mathcal{B},G,U)$ then Δ is closable and its closure $\bar{\Delta}$ generates a strongly continuous one-parameter semigroup S which we refer to as a heat semigroup. This semigroup gives the unique continuous solution $a_t = S_t a$ of the "heat equation"

$$\partial a_t/\partial t + \Delta a_t = 0 , \qquad a_0 = a ,$$

on $\mathcal{B}$. Moreover it has a representation

$$S_t = \int_G dg\, p_t(g)\, U(g)$$

where dg denotes left-invariant Haar measure and the heat kernel p is a positive solution of a heat equation on $L_1(G)$. Analytic properties of the heat kernel are reflected in the semigroup S which is holomorphic in the open right half-plane and also maps $\mathcal{B}$ into the analytic elements of $(\mathcal{B},\mathfrak{g},dU)$, i.e.

$$S_t\mathcal{B} \subseteq \mathcal{B}_\omega(dU)$$

for all $t > 0$. More specifically there exist $k,l > 0$ such that

$$\|S_t a\|_n \le k l^n \|a\| \, t^{-n/2}$$

for all $a \in \mathcal{B}$, $n = 1,2,\ldots$, and $t \in \langle 0,1\rangle$.

In fact these last bounds can be deduced from the n=1 bound if one knows in advance that $S_t\mathcal{B} \subseteq \mathcal{B}_\infty(dU)$ for $t > 0$. More generally the following conclusion holds.

Proposition 3.1. *Let Δ denote the Laplacian corresponding to the basis $x_1,\ldots,x_d$ of $\mathfrak{g}$ in the representation $(\mathcal{B}, \mathfrak{g}, V)$. Assume that*

1. *Δ is closable and its closure $\bar{\Delta}$ generates a strongly continuous semigroup S,*
2. $S_t\mathcal{B} \subseteq \mathcal{B}_\infty(V)\,, \qquad t > 0,$
3. *there is a $c > 0$ such that*

   $$\|S_t a\|_1 \le c\|a\| \, t^{-1/2}$$

   for all $a \in \mathcal{B}$ and $t \in \langle 0,1\rangle$.

It follows that there exist $k,l > 0$ such that

$$\|S_t a\|_n \le k l^n \, \|a\| \, t^{-n/2}$$

for all $a \in \mathcal{B}$, $n = 1,2,\ldots$, and $t \in \langle 0,1\rangle$. Consequently S is holomorphic $S_t\mathcal{B} \subseteq \mathcal{B}_\omega(V)$ for $t > 0$, and $\mathcal{B}_\omega(V)$ is norm-dense.

The idea behind the proof is very simple. If M_{n+1} is a monomial of order n+1 in the $V(x_i)$ and $M_{n+1} = V(x_i)\, M_n$ then

$$
\begin{aligned}
M_{n+1}S_t a &= V(x_i)\, S_{\lambda t} M_n S_{(1-\lambda)t} a + V(x_i)(\mathrm{ad}\, M_n)(S_{\lambda t})\, S_{(1-\lambda)t} a \\
&= V(x_i)\, S_{\lambda t}\, M_n\, S_{(1-\lambda)t} a \\
&\quad + t \int_0^\lambda d\mu\, V(x_i)\, S_{\mu t}\, (\mathrm{ad}\, \Delta)(M_n)\, S_{(2-\lambda-\mu)t} a\,.
\end{aligned}
$$

But $(\mathrm{ad}\,\Delta)(M_n)$ is a polynomial of order $n+1$ in the $V(x_i)$ and hence one readily obtains integral inequalities of the form

$$
\begin{aligned}
\|S_t a\|_{n+1} &\le c(\lambda t)^{-1/2}\, \|S_{(1-\lambda)t} a\|_n \\
&\quad + k\, t^{1/2} \int_0^\lambda d\mu\, \mu^{-1/2}\, \|S_{(2-\lambda-\mu)t} a\|_{n+1}\,.
\end{aligned}
$$

The proof then follows by "solving" these inequalities for small t with the special choice $\lambda = (n+1)^{-2}$, i.e. with $\lambda^{-1/2} = n+1$.

Holomorphy of S is a consequence of the estimates

$$
\|\Delta S_t a\| \le d\|S_t a\|_2 \le dkl^2\, \|a\|\, t^{-1}
$$

for $a \in \mathcal{B}$ and $t \in \langle 0,1\rangle$. The bounds of the proposition also imply immediately that $S_t\mathcal{B} \subseteq \mathcal{B}_\omega(V)$ for $t > 0$ and strong continuity of S implies norm-density of $\mathcal{B}_\omega(V)$.

Combination of this last observation with Theorem 2.2 and the comments at the beginning of the section gives the first heat semigroup integration theorem.

Theorem 3.2. *The following conditions are equivalent:*

1. $(\mathcal{B},\mathfrak{g},V)$ *is integrable.*
2. a. V *is weakly conservative,*

   b. *the Laplacian associated with some basis of* $\mathfrak{g}$ *is closable and* *its closure generates a strongly continuous semigroup* S,

c. $$S_t\mathcal{B} \subseteq \mathcal{B}_\infty(V)\,, \qquad t > 0\,,$$

d. *there is a* $c > 0$ *such that*

$$||S_t a||_1 \le c||a||\, t^{-1/2}$$

for all $a \in \mathcal{B}$ *and* $t \in \langle 0,1\rangle$.

The drawback with this integrability criterion is Condition 2c which in principle requires the verification of an infinite number of conditions. The next two sections will be devoted to the discussion of methods of weakening this condition but before passing to this topic we comment on the crucial estimate contained in Condition 2d.

Since the resolvent of $\bar{\Delta}$ is obtained by Laplace transformation of S bounds such as Condition 2d can be converted into bounds on the resolvent and these establish that the $V(x_i)$ are $\bar{\Delta}$-relatively bounded. But conversely if S is holomorphic, i.e. if $||\Delta S_t|| \le ct^{-1}$ for small $t > 0$ then these relative bounds can be converted into bounds on $||S_t a||_1$. In particular one has the following characterization.

Proposition 3.3. *If* $S_t = \exp\{-t\bar{\Delta}\}$ *is holomorphic then the following conditions are equivalent:*

1. *there is a* $c > 0$ *such that*

$$||S_t a||_1 \le c||a||\, t^{-1/2}$$

for all $a \in \mathcal{B}$ *and* $t \in \langle 0,1\rangle$.

2. *there are* $c',\epsilon' > 0$ *such that*

$$||(I+\epsilon\bar{\Delta})^{-1}a||_1 \le c'||a||\, \epsilon^{-1/2}$$

for all $a \in \mathcal{B}$ *and* $\epsilon \in <0,\epsilon'>$.

3. *there is a* $c'' > 0$ *such that*

$$\|a\|_1 \le \epsilon\, \|\Delta a\| + c''\, \epsilon^{-1}\, \|a\|$$

for all $a \in D(\Delta)$ *and* $\epsilon \in <0,1>$.

4. $\|\cdot\|_2$-estimates

Theorem 3.2 established integrability from smoothness properties of a heat semigroup S. Two types of smoothness were required, a range condition $S_t\mathcal{B} \subseteq \mathcal{B}_\infty$, and a bound on $\|S_t\|_1$. This bound is equivalent, by Proposition 3.3, to the bounds

$$\|a\|_1 \le \epsilon\, \|\Delta a\| + c\epsilon^{-1}\, \|a\|$$

for $a \in D(\Delta)$ and $\epsilon \in <0,1>$. Next we argue that additional bounds which we refer to as $\|\cdot\|_2$-estimates allow one to weaken the range condition. The $\|\cdot\|_2$-estimates state that $D(\Delta) \subseteq \mathcal{B}_2$ and

$$\|a\|_2 \le k(\|\Delta a\| + \|a\|)$$

for some $k > 0$ and all $a \in D(\Delta)$. Note that since $\mathcal{B}_2 \subseteq D(\Delta)$ and for all $a \in \mathcal{B}_2$ it follows from the $\|\cdot\|_2$-estimates that $\mathcal{B}_2 = D(\Delta)$ and Δ is closed.

It should be emphasized that the $\|\cdot\|_2$-estimates differ in one important respect from all bounds considered previously. They are not necessarily true for representations of $\mathfrak{g}$ obtained by differentiating a group representation. In particular they fail for the group $\mathbb{R}^d$ of translations acting on $C_0(\mathbb{R}^d)$ and the Laplacian defined with respect to the usual Cartesian basis. Nevertheless the

$\|\cdot\|_2$-estimates are true for translations on $L_p(\mathbb{R}^d)$ if $p \in \langle 1,\infty \rangle$, for unitary representation on Hilbert space, and they are "almost true" for general group representations in a sense we explain in the following section. Hence the subsequent discussion has a greater applicability than appears at first sight.

Proposition 4.1. *Let Δ denote a Laplacian associated with a representation $(\mathcal{B},\mathcal{G},V)$ and assume Δ satisfies $\|\cdot\|_2$-estimates. Further assume*

1. *Δ generates a strongly continuous semigroup* S,
2. $S_t\mathcal{B} \subseteq \mathcal{B}_4(V)\ , \qquad t > 0.$

It follows that

$$\mathcal{B}_\infty(V) = \cap_{n \geq 1} D(\Delta^n)$$

and consequently if S *is holomorphic then*

$$S_t\mathcal{B} \subseteq \mathcal{B}_\infty(V)\ , \qquad t > 0\ .$$

The second statement of the proposition is the one of most interest for the integration problem. It follows from the first statement because holomorphy of S implies that

$$S_t\mathcal{B} \subseteq \mathcal{B}_\omega(\Delta) \subseteq \mathcal{B}_\infty(\Delta) = \cap_{n \geq 1} D(\Delta^n)\ .$$

The proof of the first statement is straightforward but rather long. There are two ideas.

First let $V_i = V(x_i)$ and note that if $a \in D(\Delta^2)$ then since $S_ta \in \mathcal{B}_4(V) \subseteq \mathcal{B}_3(V)$ one has

$$\Delta V_i S_t a \qquad V_i S_t \Delta a + P_2(V) S_t a$$

where $P_2(V) = (\mathrm{ad}\ \Delta)(V_i)$ is a second-order polynomial in the V_i. It then follows in the limit $t \to 0$ by a closure argument, using the $\|\cdot\|_2$-estimates, that $V_i D(\Delta^2) \subseteq D(\Delta)$ and

$$\Delta V_i a = V_i\, \Delta a + P_2(V) a\ .$$

Consequently another application of the $\|\cdot\|_2$-estimates gives $D(\Delta^2) \subseteq \mathcal{B}_3(V)$ and a bound

$$\|a\|_3 \le k_3 \sum_{m=0}^{2} \|\Delta^m a\|\ .$$

Similarly

$$\Delta V_i V_j\, S_t a = V_i V_j\, S_t \Delta a + P_3(V)\, S_t a$$

and one concludes with the help of the previous argument that $V_i V_j D(\Delta^2) \subseteq D(\Delta)$ and hence $D(\Delta^2) \subseteq \mathcal{B}_4(V)$.

Now one uses the second idea.

Define $R = (I+\epsilon\Delta)^{-1}$ where $\epsilon > 0$ is small enough that the resolvent exists. Then $D(\Delta^3) = R^3\mathcal{B}$. But

$$V_i R^3 a = R^2 V_i R a - (\mathrm{ad}\ R)^2(V_i) R a + 2R(\mathrm{ad}\ R)(V_i) R a\ .$$

Now, however,

$$(\mathrm{ad}\ R)^2(V_i) R a = \epsilon R(\mathrm{ad}\ R)(\mathrm{ad}\ V_i)(\Delta) R^2 a$$

because $R^2 a \in D(\Delta^2) \subseteq \mathcal{B}_4(V) \subseteq \mathcal{B}_3(V)$. Therefore

$$\begin{aligned}(\mathrm{ad}\ R)^2(V_i) R a &= \epsilon R(\mathrm{ad}\ R)(P_2(V)) R^2 a \\ &= -\epsilon R^2(\mathrm{ad}\ \Delta)(P_3(V)) R^2 a \\ &= -\epsilon^2 R^2 P_3(V) R^3 a\end{aligned}$$

where we now use $R^3a \in D(\Delta^3) \subseteq D(\Delta^2) \subseteq \mathcal{B}_4(V) \subseteq \mathcal{B}_3(V)$. Similarly

$$R(\text{ad } R)(V_i)Ra = \epsilon R^2(P_2(V))R^2a$$

and one concludes that there is a $b \in \mathcal{B}$ such that

$$V_iR^3a = R^2b\ .$$

Consequently $V_iD(\Delta^3) \subseteq D(\Delta^2)$ and then by the previous argument $D(\Delta^3) \subseteq \mathcal{B}_5(V)$.

Next repeating this argument with V_i replaced by V_iV_j and using the fact that $D(\Delta^3) \subseteq \mathcal{B}_5(V)$ one concludes that $V_iV_jD(\Delta^3) \subseteq D(\Delta^3)$. Therefore $D(\Delta^3) \subseteq \mathcal{B}_6(V)$.

This argument extends to higher powers and one successively deduces that $D(\Delta^n) \subseteq \mathcal{B}_{2n-1}(V)$ and $D(\Delta^n) \subseteq \mathcal{B}_{2n}(V)$. In fact it is easiest to proceed by induction but we will not give any further details.

It should be emphasized that in the above proof it is not essential to assume that $\mathcal{B}_\infty(V)$ is norm-dense. It follows of course from strong continuity of S and the condition $S_t\mathcal{B} \subseteq \mathcal{B}_4(V)$ that $\mathcal{B}_4(V)$ must be norm-dense. Thus norm-density of $\mathcal{B}_\infty(V)$ is a consequence of density of $\mathcal{B}_4(V)$, the assumption of the theorem, and the structure relations of $(\mathcal{B},\mathfrak{g},V)$.

Straightforward combination of Theorem 3.2 and Proposition 4.1 gives an integrability criterion for systems with a Laplacian satisfying $\|\cdot\|_2$-estimates. A different criterion follows by noting that weak conservativeness and $\|\cdot\|_2$-estimates imply Condition 3 of Proposition 3.3. For example, if V is conservative

$$\begin{aligned}\epsilon\|V(x_i)a\| &\le \|(I-\epsilon V(x_i))a\| + \|a\| \\ &\le \|(I-\epsilon^2 V(x_i))a\| + \|a\| \\ &\le \epsilon^2\|V(x_i)^2 a\| + 2\|a\| \\ &\le \epsilon^2 k\|\Delta a\| + (2+k\epsilon^2)\|a\|\end{aligned}$$

for all $a \in \mathcal{B}_2 = D(\Delta)$. Therefore one has the following.

Theorem 4.2. *Let Δ denote a Laplacian associated with a representation $(\mathcal{B},\mathfrak{g},V)$ and assume Δ satisfies $\|\cdot\|_2$-estimates.*

The following conditions are equivalent;

1. $(\mathcal{B},\mathfrak{g},V)$ *is integrable,*
2. a. V *is weakly conservative,*

   b. Δ *generates a strongly continuous holomorphic semigroup,*

   c. $\cap_{n\geq 1} D(\Delta^n) \subseteq \mathcal{B}_4(V)$.

Our next aim is to explain how this result and the strengthened version of Theorem 3.2, can be proved without $\|\cdot\|_2$-estimates.

5. Lipschitz Spaces

Although $\|\cdot\|_2$-estimates are not generally true for Laplacians Δ associated with a group representation $(\mathcal{B},G,U)$ they are almost true in two different ways.

First, for each heat semigroup $S_t = \exp\{-t\bar{\Delta}\}$ one has estimates

$$\|dU(x_i)dU(x_j)\, S_t a\| \le c_2\|a\|\, t^{-1}$$

for all $a \in B$ and $t \in \langle 0,1\rangle$. Hence by the usual Laplace transformation arguments this gives bounds

$$\|a\|_2 \le k_{\epsilon,\delta}\, \|(I+\epsilon\bar{\Delta})^{1+\delta}a\|$$

for all small $\epsilon,\delta > 0$. The fractional power of $(I+\epsilon\bar{\Delta})$ can be chosen arbitrarily close to one but not actually equal to one which would be required for $\|\cdot\|_2$-estimates. Unfortunately these weakened $\|\cdot\|_2$-estimates have not appeared useful for integrability problems.

Second, one can consider the representation transported to Lipschitz spaces, spaces which are close to the original space, and on these spaces one has $\|\cdot\|_2$-estimates. These spaces can be defined directly in terms of the group representation but the important point is that they coincide with the Lipschitz spaces corresponding to each heat semigroup. Therefore if one has a Lie algebra representation for which a heat semigroup S exists one can hope to construct Lipschitz spaces on which one has representations satisfying $\|\cdot\|_2$-estimates. This is indeed the case if S satisfies certain smoothness conditions. Information about integrability of these representations can then be used to obtain information about the original representation.

The Lipschitz spaces $\mathcal{B}_{\alpha,q}$ that we need are defined for a heat semigroup S associated with $(\mathcal{B},\mathfrak{g},V)$ and two real parameters $\alpha \in \langle 0,1\rangle$ and $q \in [1,\infty\rangle$ by

$$\mathcal{B}_{\alpha,q} = \{a \in \mathcal{B};\ t \mapsto t^{-\alpha/2}\|(I-S_t)a\| \in L_q(dt/t;\ \langle 0,1\rangle)\}\ .$$

They form Banach subspaces of $\mathcal{B}$ with respect to the norms $\|\cdot\|_{\alpha,q}$ where

$$\|a\|_{\alpha,q} = \|a\| + \left(\int_0^1 \frac{dt}{t}\, (t^{-\alpha/2}\, \|(I-S_t)a\|)^q\right)^{1/q}.$$

In fact these spaces have been defined and analyzed extensively for a general semigroup S but they are of particular interest in the Lie algebra setting if the action of S relates the spaces to the C_n-structure of $(\mathcal{B},\mathfrak{g},V)$.

The key result is the following.

Theorem 5.1. *Assume there exists a heat semigroup* S *such that* $S_t\mathcal{B} \subseteq \mathcal{B}_{n+1}(V)$ *for some* $n \geq 2$ *and*

$$\|S_t a\|_1 \leq c_1\|a\|\, t^{-1/2}$$

for all $a \in \mathcal{B}$ *and* $t \in \langle 0,1\rangle$. *Then*

$$\mathcal{B}_{\alpha,q} = \{a;\ t \mapsto t^{(m-\alpha)/2}\ \|S_t a\|_m \in L_q(dt/t;\ \langle 0,1\rangle)\}$$

and $\|\cdot\|_{\alpha,q}$ *is equivalent to the norms*

$$a \mapsto \|a\|_{\alpha,q;m} = \left(\int_0^1 \frac{dt}{t}\,(t^{(m-\alpha)/2}\ \|S_t a\|_m)^q\right)^{1/q},$$

for $m = 1,2,\ldots,n$. *Moreover if* $n \geq 3$ *there is a* $k_{\alpha,q} > 0$ *such that*

$$\|V(x_i)V(x_j)a\|_{\alpha,q} \leq k_{\alpha,q}(\|\Delta a\|_{\alpha,q} + \|a\|_{\alpha,q})$$

for all $a \in \mathcal{B}_2 \cap \mathcal{B}_{\alpha,q}$ *such that* $\Delta a \in \mathcal{B}_{\alpha,q}$.

It follows directly from the definition of $\mathcal{B}_{\alpha,q}$ that it is an S-invariant subspace of $\mathcal{B}$ and S restricted to $\mathcal{B}_{\alpha,q}$ is $\|\cdot\|_{\alpha,q}$-continuous. Moreover $D(\bar{\Delta}) \subseteq \mathcal{B}_{\alpha,q}$ and hence $\mathcal{B}_m(V) \subseteq \mathcal{B}_{\alpha,q}$ for all $m \geq 2$. Thus if $S_t\mathcal{B} \subseteq \mathcal{B}_{n+1}(V)$ with $n \geq 2$ then $\mathcal{B}_{n+1}(V)$ is $\|\cdot\|_{\alpha,q}$-dense in $\mathcal{B}_{\alpha,q}$. Therefore if $n \geq 3$ one can define a representation $V_{\alpha,q}$ of $\mathfrak{g}$ on $\mathcal{B}_{\alpha,q}$ by restricting the $V(x)$ to $\mathcal{B}_3(V)$ and then taking their $\|\cdot\|_{\alpha,q}$-closures. Alternatively $V_{\alpha,q}(x)$ is the restriction of $V(x)$ to those $a \in \mathcal{B}_{\alpha,q}$ such that $V(x)a \in \mathcal{B}_{\alpha,q}$. One can then define the C_m-elements $\mathcal{B}_{m;\alpha,q}$ of $V_{\alpha,q}$ as before and it follows automatically that $V_{\alpha,q}$ satisfies the structure relations of $\mathfrak{g}$ on $\mathcal{B}_{2;\alpha,q}$. But there is no obvious reason for the $\mathcal{B}_{m;\alpha,q}$ to be $\|\cdot\|_{\alpha,q}$-dense for $m \geq 3$. Nevertheless Theorem 5.1 establishes that $\Delta_{\alpha,q}$, the generator of S restricted to $\mathcal{B}_{\alpha,q}$, satisfies $\|\cdot\|_{2;\alpha,q}$-estimates of the type discussed in Section 4. This observation together with the argument used to prove Proposition 4.1 then allows the following conclusion:

Corollary 5.2. *Adopt the assumptions of Theorem 5.1 with* $n = 3$. *Then*

$$\mathcal{B}_{\infty;\alpha,q} = \cap_{n\geq 1} D(\Delta^n_{\alpha,q})$$

and in particular $\mathcal{B}_{\infty;\alpha,q}$ *is* $\|\cdot\|_{\alpha,q}$*-dense in* $\mathcal{B}_{\alpha,q}$.

Thus on the Lipschitz spaces $\mathcal{B}_{\alpha,q}$ one has representations $V_{\alpha,q}$ of $\mathfrak{g}$ which satisfy $\|\cdot\|_{2;\alpha,q}$-estimates. But the conditions on S necessary for this conclusion, i.e. $S_t\mathcal{B} \subseteq \mathcal{B}_4(V)$ and $\|S_t\|_1 \leq c_1 t^{-1/2}$, suffice to imply that S is holomorphic and its restriction to $\mathcal{B}_{\alpha,q}$ is also holomorphic. Therefore.

$$S_t\mathcal{B}_{\alpha,q} \subseteq \cap_{n\geq 1} D(\Delta^n_{\alpha,q}) = \mathcal{B}_{\infty;\alpha,q} \, .$$

Consequently

$$\begin{aligned} S_t\mathcal{B} &= S_{t/2}(S_{t/2}\mathcal{B}) \subseteq S_{t/2}\, \mathcal{B}_4(V) \\ &\subseteq S_{t/2}\mathcal{B}_{\alpha,q} \subseteq \mathcal{B}_{\infty;\alpha,q} \subseteq \mathcal{B}_\infty(V) \, . \end{aligned}$$

Thus we have the following conclusions which is independent of the Lipschitz spaces.

Corollary 5.3. *If* $S_t\mathcal{B} \subseteq \mathcal{B}_4(V)$ *for* $t > 0$ *and* $\|S_ta\|_1 \leq c_1\|a\|\, t^{-1/2}$ *for all* $a \in \mathcal{B}$ *and* $t \in \langle 0,1\rangle$ *then* $S_t\mathcal{B} \subseteq \mathcal{B}_\infty(V)$ *for* $t > 0$.

This conclusion combined with the earlier results, notably Theorem 3.2 and Proposition 3.3, then gives the final general integrability theorem.

Theorem 5.4. *Let* $(\mathcal{B},\mathcal{g},\mathrm{V})$ *be a representation for which* V *is weakly conservative and let* Δ *be a Laplacian associated with the representation. Then the following conditions are equivalent:*

1. $(\mathcal{B},\mathcal{g},\mathrm{V})$ *is integrable,*

2. a. Δ *is closable and its closure generates a strongly continuous semigroup* S,

   b. *there is a* $c_1 > 0$ *such that*

$$\|S_t a\|_1 \leq c_1 \|a\| \, t^{-1/2}$$

   for all $a \in \mathcal{B}$ *and* $t \in \langle 0,1 \rangle$,

3. a. Δ *is closable and its closure* $\bar{\Delta}$ *generates a strongly continuous holomorphic semigroup,*

   b. $\cap_{n\geq 1} D(\bar{\Delta}^n) \subseteq \mathcal{B}_4(\mathrm{V})$,

   c. *there is a* $c_1 > 0$ *such that*

$$\|a\|_1 \leq \epsilon \, \|\Delta a\| + c_1 \epsilon^{-1} \|a\|$$

   for all $a \in D(\Delta)$ *and* $\epsilon \in \langle 0,1 \rangle$.

6. Commutator Theory

The foregoing integration results were derived by analytic element arguments In this section we briefly describe a completely different method of approach based on commutator theory. This method has the advantage that it only uses C_3-estimates, but has the disadvantage that it is restricted to the theory of isometric representations, and it requires $\|\cdot\|_2$-estimates.

The basic commutator result gives a criterion for a conservative operator to generate an isometric group with a smooth action.

Theorem 6.1. *Let* $S_t = \exp\{-tH\}$ *denote a strongly continuous contraction semigroup on* $\mathcal{B}$ *and let* K *be a closed conservative operator with the properties*

1. $D(H) \subset D(K)$ *and for each* $\epsilon \in \langle 0,1\rangle$ *there is a* $c_\epsilon > 0$ *such that*

$$\|Ka\| \le \epsilon\|Ha\| + c_\epsilon \|a\| , \qquad a \in D(H)$$

2. $KD(H^2) \subseteq D(H)$ *and there is a* $C > 0$ *such that*

$$\|(\mathrm{ad}\ K)(H)a\| \le C(\|Ha\| + \|a\|) , \qquad a \in D(H^2) .$$

It follows that K *generates a strongly continuous one-parameter group of isometries* T, $T_tD(H) \subseteq D(H)$ *for all* $t \in \mathbb{R}$, *and* $\|H(T_t\text{-}I)a\| \to 0$ *as* $t \to 0$ *for all* $a \in D(H)$.

The idea is to apply this result with $K = V(x)$ and $H = \bar{\Delta}$ in order to deduce that each $V(x)$ generates a group, and then to deduce integrability of $(\mathcal{B},\mathfrak{g},V)$ from Proposition 2.1. In order to follow this procedure one must first verify the assumptions of the theorem and the proposition. The main problem is to verify Condition 2 of Theorem 6.1. Since $(\mathrm{ad}\ V(x))(\Delta)$ is quadratic in the $V(x_i)$ this requires that Δ satifies the $\|\cdot\|_2$-estimates. These estimates then imply that $\mathcal{B}_2(V) = D(\Delta) = D(\bar{\Delta})$ and hence the last statement of Theorem 6.1 corresponds to invariance of $\mathcal{B}_n(V)$ under the groups generated by the $V(x)$ and $\|\cdot\|_2$-continuity of these groups. Therefore Proposition 2.1 applies and $(\mathcal{B},\mathfrak{g},V)$ is integrable.

Instead of continuing the discussion of the general situation we illustrate this method for unitary representations on Hilbert space.

First, an operator on a Hilbert space $\mathcal{H}$ is conservative if, and only if, it is skew-symmetric. Thus we consider a family $V = \{V(x); x \in \mathfrak{g}\}$ of closed skew-symmetric operators on $\mathcal{H}$. Further we assume that $\mathcal{H}_2(V)$ is norm-dense and the $V(x)$ satisfy the structure relations on $\mathcal{H}_2(V)$. We will not assume $\mathcal{H}_\infty(V)$ is norm-dense but it will be important that $\mathcal{H}_3(V)$ is $\|\cdot\|_2$-dense in $\mathcal{H}_2(V)$, or, alternatively, $\mathcal{H}_2(V)$ is $\|\cdot\|_1$-dense in $\mathcal{H}_1(V)$.

Second, if Δ is the positive symmetric Laplacian associated with the basis $x_1,\ldots,x_d$ of $\mathfrak{g}$ then

$$\|V(x_i)a\|^2 \le -(a,\Delta,a)$$

for all $a \in D(\Delta)$ and hence one has an estimate

$$\|V(x_i)a\| \le \epsilon\,\|\Delta a\| + (1/2\,\epsilon)\,\|a\|$$

for all $a \in D(\Delta)$ and $\epsilon > 0$. Moreover if $a \in \mathcal{H}_3(V)$ then

$$\begin{aligned}\|V(x_i)V(x_j)a\|^2 &\le (V(x_j)^2a, \Delta a) + (V(x_j)a, (\operatorname{ad}\Delta)(V(x_j))a)\\ &\le \|V(x_j)^2a\| \cdot \|\Delta a\| + \|V(x_j)a\| \cdot \|(\operatorname{ad}\Delta)(V(x_j))a\|\,.\end{aligned}$$

Since after use of the structure relations $(\operatorname{ad}\Delta)(V(x_j)$ is quadratic in the $V(x_i)$ this gives the estimate

$$\begin{aligned}\|V(x_j)V(x_j)a\| &\le \|\Delta a\| + k\,\sup_{1\le j\le d}\|V(x_j)a\|\\ &\le (1+\epsilon)\,\|\Delta a\| + (2\,k^2/\epsilon)\,\|a\|\end{aligned}$$

for all $a \in \mathcal{H}_3(V)$ and $\epsilon > 0$. Hence if $\mathcal{H}_3(V)$ is a core of $\bar{\Delta}$ then $\bar{\Delta}$ satisfies a $\|\cdot\|_2$-estimate, $D(\Delta) = D(\bar{\Delta}) = \mathcal{H}_2(V)$, and $\Delta = \bar{\Delta}$. Alternatively, if $\mathcal{H}_2(V)$ is $\|\cdot\|_1$-dense in $\mathcal{H}_1(V)$ then the structure relations extend to form relations on $\mathcal{H}_1(V) \times \mathcal{H}_1(V)$. But these form relations, together with the operator relations on $\mathcal{H}_2(V)$ allow one to make a similar estimate. Thus once again Δ satisfies a $\|\cdot\|_2$-estimate, $\Delta = \bar{\Delta}$, and $D(\Delta) = \mathcal{H}_2(V)$.

Third, if Δ is self-adjoint, $a \in \mathcal{H}_3(V)$, and $b \in D(\Delta^2) \subseteq D(\Delta) \subseteq \mathcal{H}_1(V)$ then

$$\begin{aligned}|(\Delta a, V(x)b) - (V(x)a, \Delta b)| &= |(P_2(V)a, b)| \\ &= |(a, P_2(V)b)| \\ &\le k'\|a\| \, (\|\Delta b\| + \|b\|)\end{aligned}$$

where $P_2(V)$ is the quadratic expression in the $V(x_i)$ corresponding to (ad $\Delta)(V(x))$, and the final step uses the $\|\cdot\|_2$-estimate. Thus if $\mathcal{H}_3(V)$ is a core of Δ one concludes that $V(x)\; D(\Delta^2) \subseteq D(\Delta)$ and

$$\|(\Delta V(x) - V(x)\Delta)b\| \le k'(\|\Delta b\| + \|b\|)$$

for all $b \in D(\Delta^2)$. Alternatively if $\mathcal{H}_2(V)$ is $\|\cdot\|_2$-dense in $\mathcal{H}_1(V)$ one can use the commutation relations as form relations on $\mathcal{H}_1(V) \times \mathcal{H}_1(V)$ to arrive at the same conclusion. One simply repeats the above calculation but with $a \in D(\Delta) = \mathcal{H}_2(V)$ and $b \in D(\Delta^2)$. But if Δ is self-adjoint one has

$$(a, \Delta a) = \sum_{i=1}^{d} \|V(x_i)a\|^2$$

for all $a \in \mathcal{H}_2(V)$ and hence $\|\cdot\|_1$-density of $\mathcal{H}_2(V)$ in $\mathcal{H}_1(V)$ is equivalent to the property that $\mathcal{H}_2(V)$ is a core of $\Delta^{1/2}$.

Thus the (essential) self-adjointness of Δ together with either of the density assumptions ensures the hypotheses of Theorem 6.1 are fulfilled, and $H = \Delta$ and $K = V(x)$. Then combination of Theorem 6.1 with Proposition 2.1 establishes the crucial statements of the following theorem.

Theorem 6.2. *Let* $V = \{V(x);\, x \in \mathfrak{g}\}$ *denote a family of closed skew-symmetric operators on the Hilbert space* $\mathcal{H}$ *satisfying the structure relations of the Lie algebra* $\mathfrak{g}$ *on the subspace* $\mathcal{H}_2(V)$ *and let* Δ *denote the*

Laplacian associated with some basis of $\mathfrak{g}$.

The following conditions are equivalent:

1. $(\mathcal{H},\mathfrak{g},V)$ *is integrable (to a unitary representation),*
2. Δ *is (essentially) self-adjoint and the semigroup* $S_t = \exp\{-t\bar{\Delta}\}$ *has the property*

$$S_t\mathcal{H} \subseteq \mathcal{H}_3(V)\,, \qquad t > 0\,,$$

3. Δ *is (essentially) self-adjoint and* $\mathcal{H}_3(V)$ *is a core of* $\bar{\Delta}$,
4. Δ *is (essentially) self-adjoint and* $\mathcal{H}_2(V)$ *is a core of* $\bar{\Delta}^{1/2}$.

The above discussion outlines the proof of $3 \Rightarrow 1$ and $4 \Rightarrow 1$. But the property $S_t\mathcal{H} \subseteq \mathcal{H}_2(V) \subseteq D(\bar{\Delta})$ ensures that $\mathcal{H}_3(V)$ is a core of $\bar{\Delta}$ and one has $2 \Rightarrow 3$. Moreover, Condition 3 implies $\|\cdot\|_2$-estimates for Δ which in turn imply $D(\Delta) = \mathcal{H}_2(V)$. Since $D(\Delta)$ is a core of $\Delta^{1/2}$ by general reasoning one has $3 \Rightarrow 4$. Finally it can be verified that the differential of a unitary group representation satisfies Conditions 2, 3 and 4.

To conclude we note that this last result indicates that the previous Banach space results might still be improved, with C_3-estimates replacing C_4-estimates at least for isometric group representation.

Notes and Remarks

The first general results on integrability of representation of general Lie algebras were given by Nelson, who also introduced many of the techniques

[Nel] Nelson, E.; Analytic Vectors. *Ann. Math.* **3** (1959) 572-615.

Nelson's paper was preceded, however, by related work of Rellich and others on representation of the Heisenberg group (see [Nel] Section 9).

Nelson developed the theory of analytic elements and showed by use of the heat kernel that any continuous representation of a Lie group on Banach space has a dense set of analytic elements. Then he proved a wide variety of results for representations of Lie algebras by skew-symmetric operators on Hilbert space. In particular he established $\|\cdot\|_2$-estimates of the Laplacian and integrability criteria in terms of essential self-adjointness of the Laplacian. His proof used analytic element techniques and was based on a Hilbert space version of Theorem 2.2.

Subsequently many other authors analyzed properties of analytic elements, both on Hilbert space and Banach space, with the aim of elucidation of the differential and integral structure of continuous representation of Lie groups. Most of this work, to 1982, is covered in the book by Jørgensen and Moore.

[JøM] Jørgensen, P.E.T. and R.T. Moore; *Operator Commutation Relations*, Reidel (1984).

In particular this book contains detailed proofs of Proposition 2.1 and its many variants. But the full Banach space version of Theorem 2.2 appeared later. It was proved independently by Goodman and Jørgensen, and by Rusinek.

[GoJ] Goodman, F.M. and P.E.T. Jørgensen; Lie algebras of unbounded derivations, *Journ. Func. Anal.* **52** (1983) 369-384.

[Rus] Rusinek, J.; On the integrability of representations of real Lie algebras in Banach space, preprint, Warsaw University (1981).

The results described in Sections 3-6 are even more recent and are extracted from a series of preprints by the Department of Mathematics, Institute of Advanced Studies, Australian National University, Canberra.

[Rob1] Robinson, D.W.; The differential and integral strcture of continuous representations of Lie Groups, Preprint No. 2 (1987), (to appear in the *Journal of Operator Theory*).

[Rob2] Robinson, D.W.; Lie groups and Lipschitz spaces, Preprint No. 16 (1987).

[Rob3] Robinson, D.W.; Lie algebras, the heat semigroup, and Lipschitz spaces (to appear).

[BGJR] Bratteli, O., Goodman, F.M., Jørgensen, P.E.T. and D.W. Robinson, The heat semigroup and integrability of Lie algebras, Preprint No. 17 (1987), (to appear in the *Journal of Functional Analysis*).

Proposition 3.1 and Theorem 3.2 occur in [BGJR] and the discussion of Sections 4 and 5 comes from [Rob2] and [Rob3]. Theorem 6.1 is a slight variant of a result in [Rob1] as is the proof of the Hilbert space application Theorem 6.2.

Finally we note that $3 \Rightarrow 1$ and $4 \Rightarrow 1$ are essentially reformulations of Nelson's Theorem 5 and Corollary 9.1. For example, to derive $4 \Rightarrow 1$ from Nelson's Corollary 9.1 one defines $V(x)$ as the closure of Nelson's $\rho(x)$ on the domain $\mathcal{D}$. Then the spaces $\mathcal{H}_n(V)$ are determined and $\mathcal{D}$, and hence $\mathcal{H}_2(V)$, is $\|\cdot\|_1$-dense in $\mathcal{H}_1(V)$. It is somewhat less evident that $\mathcal{D}$ is $\|\cdot\|_1$-dense in $\mathcal{H}_2(V)$ but this follows by uniqueness of self-adjoint extensions if Δ is essentially self-adjoint on $\mathcal{D}$.

Department of Mathematics
Institute of Advanced Studies
Australian National University
Canberra