Invariant differential operators in harmonic analysis on real hyperbolic space

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ABSTRACT We introduce specific first order differential operators that are invariant with respect to the isometry group of real hyperbolic space. They possess the fundamental properties of (i) injective principal symbol, (ii) non-trivial kernels in explicitly computable eigenspaces of the Casimir, and (iii) a multiplicity one lowest K-type. Identifications with restrictions of twisted Hodge-deRham (d, d^*)-systems are made. Using ideas from H^p-theory on Euclidean space we exhibit explicit Hilbert space realizations of unitarizable exceptional representations of the Lorentz group.

Section 1. Introduction

We continue our unified study of over-determined, elliptic differential operators (that is, first order systems with injective principal symbol, hereafter referred to as injective systems) arising in problems from classical analysis, geometry and representation theory associated with a Riemannian symmetric space (1), (2). In this paper the focus will be on n-dimensional real hyperbolic space H_n and the identity component G ($\sim SO_0(1,n)$) of its group of isometries. To each irreducible unitary representation $(\mathcal{H}_\tau, \tau)$ of the subgroup K ($\sim SO(n)$) of G leaving invariant a fixed point of H_n correspond a G-homogeneous vector bundle E_τ over H_n and G-invariant first order differential operator $\mathfrak{D}_\tau$ on the space $\mathcal{C}^\infty(E_\tau)$ of smooth sections of E_τ (section 2). In accordance with the program laid out in (1) and (2), we show that $\mathfrak{D}_\tau$ reflects the fundamental differential geometric, algebraic and analytic properties of H_n. This is accomplished by relating $\mathfrak{D}_\tau$ both with Hodge-deRham theory and with representation theory as applied to the induced representation π_τ of G on $\mathcal{C}^\infty(E_\tau)$. For instance, the G-invariance ensures that the restriction of π_τ to the kernel of $\mathfrak{d}_\tau$ defines a H (Hardy)-module representation $(\ker \mathfrak{d}_\tau, \pi_\tau)$ of G. Now various unitarizable exceptional representations of the Lorentz group that occur in widely different contexts are known on an ALGEBRAIC level to be equivalent. The representation $(\ker \mathfrak{d}_\tau, \pi_\tau)$ plays a pivotal role in that a natural ANALYTIC equivalence can be exhibited between each of these representations and $(\ker \mathfrak{d}_\tau, \pi_\tau)$. When τ is of class 1 each equivalence is the analogue of some aspect of the 'higher gradients' theory for H^p-spaces on Euclidean space (section 4). Equivalences analogous to the Euclidean H^p-theory as begun in (1) for arbitrary τ presumably will hold for all the unitarizable, exceptional representations of G. Precise conjectures to this effect are made in section 3. By reversing this point of view, however, we can regard the analytic concepts associated with these equivalences as the basic building blocks of harmonic analysis on H_n, using the links with representation theory to tie harmonic analysis on H_n with the isometry group of H_n just as Euclidean harmonic analysis is tied to the Euclidean motion group (cf. (2)). Full details, further results and different perspectives will be given elsewhere.

Section 2. The operator $\mathfrak{d}_\tau$

To define $\mathfrak{d}_\tau$ we identify H_n first with the coset-space $K \setminus G$ and use standard bundle-theoretic constructions (3). Let E_τ be the G-homogeneous vector bundle over H_n corresponding to any finite-dimensional representation $(\mathcal{H}_\tau, \tau)$ of K and $\mathcal{C}^\infty(E_\tau)$ the space of smooth sections. When $\mathrm{G} = \mathrm{k} \oplus \mathrm{p}$ is the Cartan decomposition determined by K, the co-tangent bundle T^*H_n, for example, arises from the co-adjoint representation (p^*, ρ) of K on the dual space p^* of p. On $\mathcal{C}^\infty(E_\tau)$ there is a representation π_τ of G, and a differential operator $\partial : \mathcal{C}^\infty(E_\tau) \to \mathcal{C}^\infty(E_\sigma)$ is said to be INVARIENT when $\partial \circ \pi_\tau(g) = \pi_\sigma(g) \circ \partial$, $g \in G$. For instance, the Riemannian connection on H_n lifts to a covariant derivative $\nabla : \mathcal{C}^\infty(E_\tau) \to \mathcal{C}^\infty(E_{\tau\otimes\rho})$ that is invariant in this sense. More generally, to each K-equivariant mapping $A : \mathcal{H}_\tau \otimes \mathrm{p}^* \to \mathcal{H}_\sigma$ corresponds an element, say A, in $\mathrm{Hom}(E_{\tau\otimes\rho}, E_\sigma)$ so that the composition $\partial_A = A \circ \nabla : \mathcal{C}^\infty(E_\tau) \to \mathcal{C}^\infty(E_\sigma)$ is an invariant first order differential operator. For unitary $(\mathcal{H}_\tau, \tau)$ we complexify p^* and assume $A : \mathcal{H}_\tau \otimes \mathrm{p}^*_{\mathbb{C}} \to \mathcal{H}_\sigma$ is K-equivariant.

Since $(\mathfrak{p}^*_{\mathbb{C}}, \rho)$ can be identified with the standard representation of $SO(n)$ on $\mathbb{C}^n$, we shall use the same choice of A_τ to define $\mathfrak{d}_\tau$ on H_n as was used in (1) to define $\mathfrak{d}_\tau$ on $\mathbb{R}^n$. Conceptually, this exploits the geometric relation between the isometry group $G \sim SO_0(1,n)$ of H_n and the Cartan motion group $K \circledS \mathfrak{p} \sim SO(n) \circledS \mathbb{R}^n$ on the tangent space $\mathfrak{p}$ ($\cong \mathbb{R}^n$) to H_n at the 'origin'. Both groups, for instance, have the same isotropy subgroup, K, at this point of tangency. For simplicity of exposition we assume from now on that $(\mathcal{H}_\tau, \tau)$ is an irreducible, single-valued unitary representation of $SO(n)$ with highest weight $\tau = (m_1, \ldots, m_r, 0, \ldots, 0)$, $m_r > 0$; any further restriction will be explicitly stated.

If $\varepsilon_1, \varepsilon_2, \ldots$ are the usual basis vectors of Euclidean space, then $\mathcal{H}_\tau \otimes \mathfrak{p}^*_{\mathbb{C}}$ admits the $SO(n)$-decomposition

$$\mathcal{H}_\tau \otimes \mathfrak{p}^*_{\mathbb{C}} \cong \oplus \sum_{j=1}^{r} \gamma_j \mathcal{H}_{\tau+\varepsilon_j} \oplus \cdots$$

where $\gamma_j = 1$ if $\tau + \varepsilon_j$ is dominant and is 0 otherwise. The highest weight space $\mathcal{H}_{\tau+\varepsilon_1}$ — the Cartan composition of $\mathcal{H}_\tau$ and $\mathfrak{p}^*_{\mathbb{C}}$ — always occurs.

Definition 1. *Let $A_\tau : \mathcal{H}_\tau \otimes \mathfrak{p}^*_{\mathbb{C}} \to \mathcal{H}_\tau \otimes \mathfrak{p}^*_{\mathbb{C}}$ be the orthogonal projection of $\mathcal{H}_\tau \otimes \mathfrak{p}^*_{\mathbb{C}}$ on the orthogonal* COMPLEMENT *of the subspace isomorphic to $\sum_{j=1}^{r} \gamma_j \mathcal{H}_{\tau+\varepsilon_j}$, and define $\mathfrak{d}_\tau : C^\infty(E_\tau) \to C^\infty(E_{\tau \otimes \rho})$ by $\mathfrak{d}_\tau = A_\tau \circ \nabla$.*

To exhibit $\mathfrak{d}_\tau$ explicitly as a non-constant coefficient partial differential operator, let $Y_1, \ldots, Y_{n-1}, Y$ be an orthonormal basis for $\mathfrak{p}$, take $\mathbf{A} = \mathbb{R}Y$ as maximal abelian subspace of $\mathfrak{p}$, and let $G = KAV$ be an Iwasawa decomposition. Then

$$H_{n+1} \cong AV \cong \mathbb{R}^n_+ = \left\{ z = (x, y) : x \in \mathbb{R}^{n-1} \ , \ y > 0 \right\}$$

provides a global coordinate structure for H_n with respect to which $C^\infty(E_\tau)$ is the space $C^\infty(\mathbb{R}^n_+, \mathcal{H}_\tau)$ of smooth $\mathcal{H}_\tau$-valued functions on $\mathbb{R}^n_+$. Define operators A_j, B from $\mathcal{H}_\tau$ to the range of A_τ by

$$A_j \xi = A_\tau(\xi \otimes Y_j) \ , \quad B\xi = A_\tau(\xi \otimes Y) \ , \qquad \xi \in \mathcal{H}_\tau \ .$$

Then on $C^\infty(\mathbb{R}^n_+, \mathcal{H}_\tau)$ the equation $\mathfrak{d}_\tau F = 0$ is simply

$$\sum_{j=1}^{n-1} A_j \left(y \, \frac{\partial F}{\partial x_j} - d\tau[Y_j, Y] F \right) + B \left(y \, \frac{\partial F}{\partial y} \right) = 0 \ ,$$

reflecting the hyperbolic metric on $\mathbb{R}^n_+$. For the Euclidean case, by contrast, $\mathfrak{D}_\tau F = 0$ is just

$$\sum_{j=1}^{n-1} A_j \frac{\partial F}{\partial x_j} + B \frac{\partial F}{\partial y} = 0 , \qquad F \in C^\infty(\mathbb{R}^n, \mathcal{H}_\tau) .$$

The zero order term $d\tau[Y_j, Y]$ present for H_n but absent for $\mathbb{R}^n$ arises because $\{0\} \neq [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ in the semi-simple case.

By a Weyl dimension formula argument we obtain

Theorem 1. (Over-Determinedness) *The operator $\mathfrak{D}_\tau$ is over-determined in the sense that*

$$\dim \mathcal{H}_\tau < \dim A_\tau \left(\mathcal{H}_\tau \otimes \mathfrak{p}^*_{\mathbb{C}}\right) .$$

Such basic operators in geometry and analysis on H_n as the Hodge-deRham (d, d^*)-systems arise as $\mathfrak{D}_\tau$ for the fundamental representations $\tau = \rho_r = (1, \ldots, 1, 0, \ldots, 0)$; the Dirac operator would have arisen from the spin representation had it been considered. More generally, in (1) we use classical polynomial invariant theory to embed $\mathcal{H}_\tau$ explicitly as the highest weight space in $\mathcal{H}_{\rho_r} \otimes \mathcal{H}_{\tau-\rho_r}$ $(\cong \Lambda^r(\mathbb{C}^n) \otimes \mathcal{H}_{\tau-\rho_r})$, $2r \leq n$, and so realize $C^\infty(E_\tau)$ as r-forms on H_n having coefficients in $C^\infty(E_{\tau-\rho_r})$. Results of (1) show

Theorem 2. (Geometric Identification) *If τ has highest weight $(m_1, \ldots, m_r, 0, \ldots, 0)$, $m_r > 0$ and $2r < n$, then $\mathfrak{D}_\tau$ can be identified with a restriction of the twisted (d, d^*)-system acting on r-forms having coefficients in $C^\infty(\mathcal{H}_{\tau-\rho_r})$.*

Even in the case excluded from theorem 2, $\ker \mathfrak{D}_\tau \subseteq \ker(d, d^*)$ for a suitable (d, d^*)-system. Hence

Theorem 3. (Ellipticity) *Each $\mathfrak{D}_\tau$ is a first order elliptic operator in the sense that $\xi \to A_\tau(\xi \otimes \alpha)$ is injective from $\mathcal{H}_\tau$ into $\mathcal{H}_\tau \otimes \mathfrak{p}^*_{\mathbb{C}}$ for each $\alpha \in \mathfrak{p}^*$, $\alpha \neq 0$.*

On the other hand, by realizing $C^\infty(E_\tau)$ as the space $C^\infty(G, \tau)$ of smooth, $\mathcal{H}_\tau$-valued covariant functions on G ((4), p. 93), we can regard the Casimir Ω as an invariant second order operator on $C^\infty(E_\tau)$ and establish a Bochner-Weitzenböck type result: if $\tau = (m_1, \ldots, m_r, 0, \ldots, 0)$, then

$$(dd^* + d^*d)f = (-\Omega + \lambda_\tau)f , \qquad f \in C^\infty(E_\tau) ,$$

where

$$\lambda_\tau = \langle \tau + 2\delta_k, \tau - \rho_r \rangle = \sum_{j=1}^{r} m_j(m_j + n - 1 - 2j) - r(n - r - 1)$$

and $2\delta_k$ is the sum of the positive compact roots. But $\ker \mathfrak{F}_\tau \subseteq \ker(d, d^*)$ always holds. Hence

Theorem 4. (EIGENSPACE PROPERTY) *Every solution of $\mathfrak{F}_\tau f = 0$ in $C^\infty(E_\tau)$ satisfies $\Omega f = \lambda_\tau f$.*

Now $\Omega = \Omega_{\mathbf{p}} + \Omega_{\mathbf{k}}$ while

$$\Omega_{\mathbf{p}} = \Delta_n - 2y \sum_{j=1}^{n-1} d\tau[Y_j, Y] \frac{\partial}{\partial x_j} + d\tau(\Omega_M)$$

where Δ_n is the Laplace-Beltrami operator on H_n and Ω_M is the Casimir of the centralizer M of $\mathbb{R}Y$ in K.

Corollary. *Every solution of $\mathfrak{F}_\tau F = 0$ in $C^\infty(\mathbb{R}^n_+, \mathcal{H}_\tau)$ satisfies the second order equation $\Omega_{\mathbf{p}} F = -\langle \tau + 2\delta_k, \rho_r \rangle$.*

Taking $\tau = \rho_r$ we deduce that any r-form solution of $\mathfrak{F}_\tau F = 0$, $\tau = \rho_r$, must satisfy

$$\Omega_{\mathbf{p}} F = -r(n - r)F \ ,$$

which is the usual Bochner-Weitzenböck formula for an n-dimensional Riemannian manifold of constant sectional curvature -1 ((5) p. 161). Already these explicit realizations of $\mathfrak{F}_\tau$ and $\Omega_{\mathbf{p}}$ suggest what analytic properties $\mathfrak{F}_\tau$ will have:

(i) both $\mathfrak{F}_\tau$ and $\Omega_{\mathbf{p}}$ degenerate as $y \to 0+$, so any boundary value theory for $\ker \mathfrak{F}_\tau$ on H_n will differ markedly from its counterpart on $\mathbb{R}^n_+$ in the Euclidean case;

(ii) since each $[Y_j, Y]$ belongs to the complement in $\mathbf{k}$ of the Lie algebra of M ($\sim SO(n-1)$), detailed analytic properties of $\ker \mathfrak{F}_\tau$ will depend on the M-invariant decomposition of $\mathcal{H}_\tau$.

Section 3. $\ker \mathfrak{F}_\tau$ as a representation space.

To derive the K-type theory of $\ker \mathfrak{F}_\tau$ we use the Cartan decomposition $G = KP = K\exp(\mathbf{p})$; for then

$$H_n \cong P \cong B_n = \{z \in \mathbb{R}^n : |z| < 1\} \ ,$$

and $\mathfrak{p} \cong \mathbb{R}^n$ can be identified with the tangent space at $z = 0$. Let U be an open ball in B_n centered at $z = 0$ and $C^\infty(U, \mathcal{H}_\tau)$ the smooth $\mathcal{H}_\tau$-valued functions on U. Although the representations of G and $E(n) = SO(n)Ⓢ\mathbb{R}^n$ induced from $(\mathcal{H}_\tau, \tau)$ do not leave $C^\infty(U, \mathcal{H}_\tau)$ invariant, their restrictions to K do and both coincide with

$$\pi_\tau(k) : f(z) \to \tau(k) f(z.k) \ , \qquad k \in K, \ f \in C^\infty(U, \mathcal{H}_\tau)$$

(cf. (1)). In addition, the corresponding derived representations of the Lie algebras of G and $E(n)$ are defined on $C^\infty(U, \mathcal{H}_\tau)$ and each acts compatibly with the common representation of K.

Theorem 5. *Both $C^\infty(U, \mathcal{H}_\tau)$ and $\ker \mathfrak{d}_\tau \cap C^\infty(U, \mathcal{H}_\tau)$ are $(\mathbf{G}, K)$-modules when*

(i) $\mathbf{G}$ is the Lie algebra of $SO_0(1, n)$ and $\mathfrak{d}_\tau$ is the invariant operator associated with hyperbolic space,

(ii) $\mathbf{G}$ is the Lie algebra of $SO(n)Ⓢ\mathbb{R}^n$ and $\mathfrak{d}_\tau$ is the invariant operator associated with Euclidean space.

A Taylor polynomial argument allows us to pass WITHIN $C^\infty(U, \mathcal{H}_\tau)$ from the hyperbolic theory to the Euclidean theory (6). For each $m \geq 0$ define the Taylor polynomial mapping $\mathcal{T}_m$ on $f \in C^\infty(U, \mathcal{H}_\tau)$ by

$$\mathcal{T}_m f(z) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha f(0) z^\alpha \ , \qquad z \in U \ ,$$

(usual multi-index notation). Then $\mathcal{T}_m$ is K-equivariant and

$$\mathcal{T}_{m-1}(\mathfrak{d}_\tau f) = \mathfrak{d}_\tau(\mathcal{T}_m f) \ , \qquad f \in \ker \mathcal{T}_{m-1} \ ,$$

using on the left hand side the hyperbolic space $\mathfrak{d}_\tau$ and on the right the Euclidean space $\mathfrak{d}_\tau$. Hence with this same convention,

$$\mathcal{T}_m(\ker \mathfrak{d}_\tau \cap \ker \mathcal{T}_{m-1}) \subseteq \ker \mathfrak{d}_\tau \cap \left(\mathcal{P}_m(\mathbb{R}^n) \otimes \mathcal{H}_\tau\right) \ , \quad m \geq 1 \ ,$$

where $\mathcal{P}_m(\mathbb{R}^n)$ is the space of polynomial functions homogeneous of degree m on $\mathbb{R}^n$. In (1) classical polynomial invariant theory was used to describe precisely the K-types occurring in the right hand side above. Together with ellipticity of $\mathfrak{d}_\tau$ this establishes necessity of

Theorem 6. (*K*-TYPE PROPERTY) *Let τ be a single-valued irreducible unitary representation of $K \cong SO(n)$ with highest weight $(m_1, \ldots, m_r, 0, \ldots, 0)$, $2r < n$, and $\mathfrak{D}_\tau$ the associated invariant operator for H_n. Then the K-types in $\ker \mathfrak{D}_\tau$ have multiplicity one and $\mu = (\mu_1, \ldots, \mu_p)$, $p = \operatorname{rank} K$, is such a K-type if and only if*

$$\mu_1 \geq m_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq m_r \ , \qquad \mu_j = 0 \ , \ j > r \ .$$

In particular, τ is the lowest K-type in $\ker \mathfrak{D}_\tau$.

To establish sufficiency the crucial link is made with non-unitary principal series representations $U(\sigma, \lambda)$ of G. The identification $H_n \cong \mathbb{R}^n_+$, with boundary $\mathbb{R}^{n-1}$, is used here. If $(\mathcal{V}_\sigma, \sigma)$ is a representation of M and $\lambda \in \mathbb{C}$, $U(\sigma, \lambda)$ is realized on the space $L^2(\sigma, \lambda; \mathbb{R}^{n-1})$ of $\mathcal{V}_\sigma$-valued functions f on $\mathbb{R}^{n-1}$ for which

$$\int_{\mathbb{R}^{n-1}} \|f(v)\|^2_\sigma (1 + |v|^2)^{2 \mathcal{R}\ell \lambda} dv$$

is finite. When σ occurs in $\tau|_M$ and $\mathcal{V}_\sigma \subseteq \mathcal{H}_\tau$, the associated CAUCHY-SZEGÖ TRANSFORM

$$\mathcal{S}_{\tau,\lambda} : f \to F(z) = \int_{\mathbb{R}^{n-1}} S_{\tau,\lambda}(z - v) f(v) \, dv$$

maps $L^2(\sigma, \lambda; \mathbb{R}^{n-1})$ into $C^\infty(\mathbb{R}^n_+, \mathcal{H}_\tau)$, intertwining $U(\sigma, \lambda)$ and π_τ (4). The first fundamental problem is the choice of (σ, λ) so that $\mathcal{S}_{\tau,\lambda}$ has range in $\ker \mathfrak{D}_\tau$, thus realizing $(\ker \mathfrak{D}_\tau, \pi_\tau)$ as the QUOTIENT of a non-unitary principal series representation. Now by the branching theorem, the K-types μ of theorem 6 label precisely those representations of $SO(n)$ which on restriction to $SO(n-1)$ contain both of the representations of $SO(n-1)$ with respective highest weights

$$\sigma_0 = (m_1, \ldots, m_{r-1}, 0, \ldots, 0) \ , \qquad \sigma_1 = (m_1, \ldots, m_r, 0, \ldots, 0) \ .$$

By Frobenius reciprocity, this suggests the choice of σ. On the other hand, the eigenvalue of the Casimir for $U(\sigma, \lambda)$ will coincide with λ_τ in theorem 4 when

$$\sigma = \sigma_0 \ , \quad \lambda = \lambda_0 = \pm(\rho + m_r - r) \ , \quad \text{or} \quad \sigma = \sigma_1 \ , \quad \lambda = \lambda_1 = \pm(\rho - r)$$

where $\rho = (1/2)(n-1)$ ((7), p.364). Highest weight vector arguments now establish

Theorem 7. (LANGLANDS' DATA CASE) *The Cauchy-Szegö Transform is a non-trivial G-equivariant mapping from $L^2(\sigma, \lambda; \mathbb{R}^{n-1})$ into* $\ker \eth_\tau$ *when* $\tau = (m_1, \ldots, m_r, 0, \ldots, 0)$ *and*

$$\sigma = (m_1, \ldots, m_r, 0, \ldots, 0)\ , \qquad \lambda = \rho - r\ .$$

Thus $\ker \eth_\tau$ is non-empty and END-POINT OF COMPLEMENTARY SERIES representations of $SO_0(1,n)$ are realized in $\ker \eth_\tau$ ((8), p.557). Known K-type results in (7) for such representations then complete the proof of theorem 6. On the other hand, for the case $r = 1$ when τ is of class 1 and σ_0 is the trivial representation of M, the following conjecture has been verified using recurrence formulae for ultra-spherical polynomials.

Conjecture 1. *The Cauchy-Szegö Transform is a non-trivial G-equivariant mapping from $L^2(\sigma, \lambda; \mathbb{R}^{n-1})$ into* $\ker \eth_\tau$ *when* $\tau = (m_1, \ldots, m_r, 0, \ldots, 0)$ *and*

$$\sigma = (m_1, \ldots, m_{r-1}, 0, \ldots, 0)\ , \qquad \lambda = -(\rho + m_r - r)\ .$$

Further relations between $\eth_\tau$ and representations of G can now be seen. The non-unitary principal series representations in theorem 7 and conjecture 1 are known to be reducible and to have infinitesimally equivalent, irreducible unitarizable quotients (cf. (7)). Thus reducibility accounts for the need for $\eth_\tau$ to single out an invariant subspace of the Casimir, and, granted the conjecture, $(\ker \eth_\tau, \pi_\tau)$ is then a simultaneous realization of the equivalent quotients. The choice of σ in conjecture 1 should prescribe the over-determinedness of $\eth_\tau$ more precisely than theorem 1 does. In addition, unitarizability ensures the existence of a Hilbert space in $\ker \eth_\tau$ on which $\pi_\tau(g)$ is unitary for each g in G. The deepest results in harmonic analysis on H_n will emerge through analysis of irreducible SUB-SPACE representations of the principal series representations in conjecture 1, since it is on such sub-space representations that the Cauchy-Szegö transform will be 1–1. All of this is summarized in a basic conjecture confirmed (for the most part) for τ of class 1. If $\tau = (m_1, \ldots, m_r, 0, \ldots, 0)$, let $\mathcal{H}_\tau = \mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_{m_r}$ be the orthogonal decomposition of $\mathcal{H}_\tau$ into M-invariant subspaces where each $\mathcal{V}_s$ is the sum of those M-invariant subspaces of $\mathcal{H}_\tau$ having highest weight $(\mu_1, \ldots, \mu_{r-1}, s, 0, \ldots, 0)$, $\mu_1 \geq m_1 \geq \cdots \geq \mu_{r-1} \geq m_r \geq s$. Denote by Π_τ the M-equivariant projection from $\mathcal{H}_\tau$ onto $\mathcal{V}_0$.

Conjecture 2. (A) (OVER-DETERMINEDNESS) *Each F in $\ker \eth_\tau$ with $\|F(x,y)\|_\tau = 0(1)$, $y \to \infty$, is uniquely determined by $\Pi_\tau F : \mathbb{R}^n_+ \to \mathcal{V}_0$.*

(B) (PALEY-WIENER) *Denote by $B_\tau(\mathbb{R}^n_+)$ the Bergman type space of those F in $\ker \eth_\tau$ for which $\|F(x,y)\|_\tau = 0(1)$ as $y \to \infty$ and*

$$\int_{\mathbb{R}^n_+} \|\Pi_\tau F(x,y)\|^2_\tau \, y^{-n} dx dy$$

is finite. Then, for all m_r sufficiently large, π_τ acts unitarily on B_τ and there is a Sobolev type space $\mathcal{B}(\sigma_0, \lambda_0; \mathbb{R}^{n-1})$, $\sigma_0 = (m_1, \dots, m_{r-1}, 0, \dots, 0)$ and $\lambda_0 = (\rho + m_r - r)$, such that

(i) $U(\sigma_0, \lambda_0)$ acts unitarily on $\mathcal{B}(\sigma_0, \lambda_0; \mathbb{R}^{n-1})$,

(ii) the Cauchy-Szegö transform $\mathcal{S}_{\tau,\lambda_0}$ is a G-equivariant isometry from $\mathcal{B}(\sigma_0, \lambda_0; \mathbb{R}^{n-1})$ onto B_τ.

(C) (BOUNDARY VALUES) *There is a Sobolev type space $\mathcal{B}(\sigma_1, \lambda_1; \mathbb{R}^{n-1})$, $\sigma_1 = (m_1, \dots, m_r, 0, \dots, 0)$ and $\lambda_1 = -(\rho - r)$, such that*

(i) $U(\sigma_1, \lambda_1)$ acts unitarily on $\mathcal{B}(\sigma_1, \lambda_1; \mathbb{R}^{n-1})$,

(ii) $F(x,y) \to \lim_{y \to 0} y^{-r} F(x,y)$ is a G-equivariant isometry from $B_\tau(\mathbb{R}^n_+)$ onto $\mathcal{B}(\sigma_1, \lambda_1; \mathbb{R}^{n-1})$.

The $\mathcal{B}(\sigma, \lambda; \mathbb{R}^n)$ will be derived from the corresponding non-unitary principal series representation on $L^2(\sigma, \lambda; \mathbb{R}^{n-1})$ using intertwining operators. As we shall see in the next section, motivation for conjectures 1 and 2 comes from classical H^p-theory for Euclidean space. Coupled with the geometric identification of $\eth_\tau$ as the restriction of a twisted (d, d^*)-system, conjecture 2(B) gives a very novel unitary structure on a cohomology group. It has its roots in 'Real' H^p-theory. Thus already for τ of class 1 the analytic properties (i) and (ii) of $\ker \eth_\tau$ as anticipated in the previous section have been confirmed. Although the restriction on τ excludes discrete series representations as well as limits of discrete series representations, conjecture 2(B) can be modified to accommodate these cases.

Section 4. Analysis for τ of class 1.

The class 1 representations of K have highest weight $\tau = \tau_m = (m, 0, \dots, 0)$, $m \geq 0$, and are realized on the space $\mathcal{H}_m = \mathcal{H}_m(\mathbb{R}^n)$ of harmonic polynomials in $\mathcal{P}_m(\mathbb{R}^n)$.

In addition, there exist constants $c_{sm}^{(\nu)}$ and *AXIAL* polynomials $R_{m-s}^{(\nu+s)}$, $\nu = \frac{1}{2}(n-2)$, so that every F in $C^\infty(\mathbb{R}^n_+, \mathcal{H}_m)$ can be written

$$f(z,\zeta) = \sum_{s=0}^{m} c_{sm}^{(\nu)}\, R_{m-s}^{(\nu+s)}\,(\zeta) F_s(z,\xi)\ , \qquad \zeta = (\xi,\eta) \in \mathbb{R}^n\ ,$$

where $F_s : \mathbb{R}^n_+ \to \mathcal{H}_s(\mathbb{R}^{n-1})$ (cf. (9)). Then $\mathbb{C} \to \mathbb{C}R_m^{(\nu)}$ is the M-equivariant embedding in $(\mathcal{H}_m, \tau_m)$ of the trivial representation σ_0 of M; also

$$(\Pi_\tau F)(z,\zeta) = c_{0m}^{(\nu)} F_0(z) R_m^{(\nu)}(\zeta)\ ,$$

with F_0 scalar-valued. Technically, this decomposition of F corresponding to the M-invariant decomposition of $\mathcal{H}_m$ is important because each $R_{m-s}^{(\nu+s)}(\zeta) = R_{m-s}^{(\nu+s)}(\xi,\eta)$ is *RADIAL* in ξ, whereas F_s is $\mathcal{H}_s(\mathbb{R}^{n-1})$-valued. Bochner's theorem then allows free use of Fourier Transform techniques.

Now in the Euclidean case $\mathfrak{d}_m$ $(= \mathfrak{d}_{\tau_m})$ coincides with the 'higher gradients' operator of Stein-Weiss (10), in which case each F in $\ker \mathfrak{d}_m \cap C^\infty(\mathbb{R}^n_+, \mathcal{H}_m)$ satisfies

$$F(z,\zeta) = \left(\zeta \cdot \frac{\partial}{\partial z}\right)^m \Phi(z)\ , \qquad (\Pi_\tau F)(z,\zeta) = \text{const.} \left(\frac{\partial}{\partial y}\right)^m \Phi_m(z) R_m^\nu(\zeta)$$

where Φ is a scalar-valued harmonic function. Hence $\Pi_\tau F$ uniquely determines F in that $F \equiv 0$ when $\Pi_\tau F = 0$ and $\|F(x,y)\| = o(1)$, $y \to \infty$; this is the basis for 'Real' H^p-theory in Euclidean analysis (11). We prove the following analogue for hyperbolic space.

Theorem 8. *For hyperbolic space each F in $\ker \mathfrak{d}_m \cap C^\infty(\mathbb{R}^n_+, \mathcal{H}_m)$ is uniquely determined by $\Pi_\tau F$ whenever $\|F(x,y)\|_\tau = O(1)$ as $y \to \infty$.*

Again in the Euclidean case, $\Phi(z) \to (\zeta \cdot \frac{\partial}{\partial z})^m \Phi(z)$ is a K-equivariant mapping from $\mathcal{H}_\mu(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n, \mathcal{H}_m)$ that annihilates $\mathcal{H}_\mu$, $\mu < m$, and is non-trivial on $\mathcal{H}_\mu$, $\mu \geq m$. Thus the K-types in $\ker \mathfrak{d}_m$ consist of the single ladder $\{(\mu, 0, \ldots, 0) : \mu \geq m\}$, identifying the representations $(\ker \mathfrak{d}_\tau, \pi_\tau)$ for τ of class 1 with the *LADDER REPRESENTATIONS* of G of importance in physics. On the other hand, the eigenspace representation of G on

$$\mathcal{E}_\lambda = \{f \in C^\infty(\mathbb{R}^n_+) : \Delta_n f = \lambda f\}$$

is reducible precisely for the eigenvalues $\lambda_\tau = (m-1)(n+m-2)$ of theorem 4 determined by $\tau = (m, 0, \ldots, 0)$, $m \geq 1$, while the space of harmonic functions on $\mathbb{R}^n$ is the only eigenspace of the Laplacian reducible under the Euclidean motion group $E(n)$ (12), (13). The $E(n)$-equivariant mapping $\Phi(z) \to (\zeta \cdot \frac{\partial}{\partial z})^m \Phi(z)$ not only exhibits this last reducibility but also suggests how the irreducible representations $(\ker \mathfrak{d}_\tau, \pi_\tau)$ of G for τ of class 1 are to be derived from REDUCIBLE EIGENSPACE REPRESENTATIONS OF G. Finally, the principal series representations $U(\sigma_0, \lambda_0)$ associated with τ of class 1 correspond to the trivial representation σ_0 of M and $\lambda_0 = \pm(\rho + m - 1)$, $m \geq 1$. But these are all the REDUCIBLE, SPHERICAL principal series of G (12). Hence in this case the G-equivariant mapping from reducible spherical principal series into $(\ker \mathfrak{d}_\tau, \pi_\tau)$ obtained from Cauchy-Szegö transforms is the analogue of the harmonic extension of 'Real' H^p-spaces on the boundary $\mathbb{R}^{n-1}$ of $\mathbb{R}^n_+$ to solutions of the Euclidean $\mathfrak{d}_m$ on $\mathbb{R}^n_+$. The components $F_s(z, \xi)$ in the M-invariant decomposition of F can thus be expected to be related to the determining component $F_0(z)$ by higher order Riesz Transforms as they are in the Euclidean case. On a qualitative level this is true, but on a quantitative level it fails in a very significant way. Let $\mathcal{N}(\sigma_0, \lambda_0)$ be the kernel

$$\left\{ f \in L^2(\sigma_0, \lambda_0, \mathbb{R}^{n-1}) : \int_{\mathbb{R}^{n-1}} |x - v|^{2(m-1)} f(v)\, dv = 0 \right\}$$

of the intertwining operator $A(\sigma_0, \lambda_0)$ from $L^2(\sigma_0, \lambda_0, \mathbb{R}^{n-1})$ into $L^2(\sigma_0, -\lambda_0, \mathbb{R}^{n-1})$; this is the analogue of the 'vanishing moments' conditions for atomic decompositions of 'Real' H^p-spaces. Then a fundamental step in the proof of (most of) conjectures 1 and 2 for τ of class 1 is

Theorem 9. *The Cauchy-Szegö transform $S_{\tau,\lambda}$, $\lambda = \lambda_0 = \rho + m - 1$, is a G-equivariant isomorphism from $\mathcal{N}(\sigma_0, \lambda_0)$ into $(\ker \mathfrak{d}_\tau, \pi_\tau)$, $\tau = \tau_m$; and for $F = S_{\tau,\lambda} f$, $f \in \mathcal{N}(\sigma_0, \lambda_0)$,*

(i) $\displaystyle \int_{\mathbb{R}^n_+} \|\Pi_\tau F(z, \zeta)\|^2\, y^{-n} dx\, dy$,

(ii) $\displaystyle \int_{\mathbb{R}^{n-1}} |\hat{f}(\xi)|^2\, |\xi|^{-(n+2m-3)} d\xi$,

(iii) $\displaystyle \left(\frac{d}{d\alpha} A(\sigma_0, \alpha) f|_{\alpha = \lambda_0} \,,\, f \right)$

define equivalent norms provided $2m > n$, where $\hat{f}$ is the Fourier Transform of f, and in (iii) $(\cdot,\cdot)$ is the dual pairing on $L^2(\sigma_0, \pm\lambda_0, \mathbb{R}^{n-1})$.

The norm in (iii) is just the usual norm obtained from intertwining operator theory, but modified to take account of reducibility; norm (ii) is a Sobolev-type norm analogous to the corresponding norm for complementary series representations; while (i) is the L^2-norm with respect to G-invariant measure on $\mathbb{R}^n_+$ of the determining component $\Pi_\tau F$ of F. In complete contrast to the Euclidean case, however, not all of the remaining components $F_s(z,\xi)$, $s = 1,\ldots,m$, of F have finite L^2-norm. Nor on representation-theoretic grounds could we expect them all to be finite; for otherwise, $F = \mathcal{S}_{\tau,\lambda} f$ would be square-integrable on $\mathbb{R}^n_+$, and hence $(\ker \mathfrak{d}_\tau, \pi_\tau)$ would be a discrete series representation of G. Thus the Euclidean 'Real' H^p-theory suggests how the unitary structure for discrete series representations has to be modified to include the other exceptional representation. Alternatively, we could just ignore the representation theory but utilize all these ideas to develop an 'H^p-theory' for real hyperbolic space as was envisaged in (2).

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