Induced Representations of Crossed Products by Coactions

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§0 Introduction

Let $\delta : A \to \tilde{M}(A \otimes C_r^*(G))$ be a coaction of a locally compact group G on a C^*-algebra A. Then for any closed normal amenable subgroup H of G we define a coaction $\delta| : A \to \tilde{M}(A \otimes C_r^*(G/H))$ of G/H on A. We present dense $*$-subalgebras of the crossed products $A \times_\delta G$ and $A \times_{\delta|} (G/H)$ and use these to obtain a process whereby representations of $A \times_\delta G$ may be constructed from those of $A \times_{\delta|} (G/H)$. We then classify those representations of $A \times_\delta G$ which can be obtained in this way. In other words we exhibit an induction process and formulate an imprimitivity theorem for it. Finally we examine an elegant reformulation of Green's imprimitivity theorem suggested by the above results.

Proof of these results is to be found in my doctoral thesis, [8].

§1 Background

Firstly we establish some notation. $B(\mathcal{H})$ will denote the bounded linear operators on the Hilbert space $\mathcal{H}$ and $K(\mathcal{H})$ the closed ideal of compact operators on $\mathcal{H}$.

G will be a locally compact group with λ_G and ρ_G the left, respectively right regular representations of G and $L^1(G)$ on $B(L^2(G))$. $C^*(G)$ will denote the group C^*-algebra of G and $C_r^*(G)$ the closure of $\lambda_G(L^1(G))$ in $B(L^2(G))$. For a C^*-algebra A, $C_b(G,A)$, $C_o(G,A)$ and $C_c(G,A)$ will denote the continuous functions from G to A which (i) are bounded, (ii) vanish at infinity and (iii) have compact support. If A is the complex numbers we will denote the above simply as $C_b(G)$, $C_o(G)$, $C_c(G)$. Finally, the multipler algebra of a C^*-algebra A will be denoted $M(A)$.

Let $\beta : G \to \mathrm{Aut}A$ be an action of a locally compact abelian group G on a C^*-algebra A. Then there is a natural action $\beta' : \widehat{G} \to \mathrm{Aut}A \times_\beta G$, called the *dual action*, of the dual group $\widehat{G}$ on the crossed product $A \times_\beta G$. Regarding $C_c(G,A)$ as a subalgebra of $A \times_\beta G$ in the usual way, this action is determined by

$$(\beta'_\sigma(y))(s) = \sigma(s)y(s) \qquad \sigma \in \widehat{G},\ y \in C_c(G,A),\ s \in G\ ,$$

and we have the following duality theorem :

Theorem (Takai [14]) *Let $\beta : G \to \mathrm{Aut}A$ be an action of a locally compact abelian group G on a C^*-algebra A. Then there exists an isomorphism*

$$(A \times_\beta G) \times_{\beta'} \widehat{G} \cong A \otimes K(L^2(G))$$

such that the second dual action β'' of $\widehat{\widehat{G}} = G$ on $(A \times_\beta G) \times_{\beta'} \widehat{G}$ is carried to the action $\beta \otimes Ad\,\rho_G$.

Takai's theorem enables us to recover the action β from $A \times_\beta G$ and establishes an important duality between abelian group actions and their crossed products. This duality has been of fundamental importance in the study of abelian group actions on C^*-algebras and the theory of coactions and their crossed products was motivated by a desire to realize a similar duality for non-abelian groups. Before giving the definition of a coaction we need to introduce the algebra

$$\tilde{M}(A \otimes B) = \{x \in M(A \otimes B) : x(1 \otimes z),\ (1 \otimes z)x \in A \otimes B \quad \forall\, z \in B\}$$

and the comultiplication map

$$\delta_G : C_r^*(G) \to M(C_r^*(G) \otimes C_r^*(G))\ ,$$

which is the homomorphism determined by : $\lambda_G(s) \to \lambda_G(s) \otimes \lambda_G(s)$. Spatially δ_G is given by

$$\delta_G(z) = W_G(z \otimes 1)W_G^*\ , \tag{1}$$

where $W_G \in UB(L^2(G \times G))$ is defined by

$$(W_G\xi)(s,t) = \xi(s, s^{-1}t) \qquad \xi \in C_c(G \times G)\ .$$

Definition A *coaction* of a locally compact group G on a C^*-algebra A is an injective homomorphism $\delta : A \to \tilde{M}(A \otimes C_r^*(G))$ such that

(i) there is an approximate identity $(e_j)_{j \in J}$ of A such that $\delta(e_j) \to 1$ strictly in $\tilde{M}(A \otimes C_r^*(G))$,

(ii) $(\delta \otimes i) \circ \delta = (i \otimes \delta_G) \circ \delta$.

We say δ is *non-degenerate* if in addition

(iii) for each $\zeta \in A^*$ there exists $\psi \in C_r^*(G)^*$ such that $(\zeta \otimes \psi) \circ \delta \neq 0$.

Condition (i) ensures $\delta \otimes i$ extends to the multiplier algebra $M(A \otimes C_r^*(G))$. $i \otimes \delta_G$ also extends and hence condition (ii) makes sence. The essence of the definition resides in condition (ii). To see why, suppose G is abelian, $\alpha : \widehat{G} \to \mathrm{Aut}A$ is an action of the dual group $\widehat{G}$ on A and that the multiplication map

$$\alpha_G : C_b(\widehat{G}) \cong M(C_o(\widehat{G})) \to C_b(\widehat{G} \times \widehat{G}) \cong M(C_o(\widehat{G}) \otimes C_o(\widehat{G}))$$

is defined by $(\alpha_G(f))(\sigma,\tau) = f(\sigma\tau)$. Spatially α_G is given by

$$\alpha_G(z) = V_G(z \otimes 1)V_G^* \,, \tag{2}$$

where $V_G \in UB(L^2(\widehat{G} \times \widehat{G}))$ is defined by

$$(V_G\xi)(\sigma,\tau) = \xi(\sigma\tau^{-1},\tau) \qquad \xi \in C_c(\widehat{G} \times \widehat{G}) \,.$$

Then if we define $\tilde{\alpha} : A \to C_b(\widehat{G}, A) \cong \tilde{M}(A \otimes C_o(\widehat{G}))$ by $(\tilde{\alpha}(a))(\sigma) = \alpha_\sigma(a)$ we have that

$$\begin{aligned}(((\tilde{\alpha} \otimes i) \circ \tilde{\alpha})(a))(\sigma,\tau) &= \alpha_{\sigma\tau}(a) \\ &= \alpha_\sigma(\alpha_\tau(a)) \\ &= (((i \otimes \alpha_G) \circ \tilde{\alpha})(a))(\sigma,\tau) \quad \forall\, \sigma,\tau \in \widehat{G},\ a \in A.\end{aligned}$$

i.e.
$$(\tilde{\alpha} \otimes i) \circ \tilde{\alpha} = (i \otimes \alpha_G) \circ \tilde{\alpha} \,, \tag{3}$$

and a moment's reflection shows that this equation captures the multiplicative nature of the action.

Now for any locally compact abelian group G, let

$$F_G : C_o(G) \to C^*(\widehat{G})$$

be the inverse Gelfand transform and let

$$\delta = (i \otimes F_{\widehat{G}}) \circ \tilde{\alpha} : A \to \tilde{M}(A \otimes C^*(G)) \,. \tag{4}$$

Then from (1) and (2) it can be checked that

$$\delta_G = (F_{\widehat{G}} \otimes F_{\widehat{G}}) \circ \alpha_G \circ F_{\widehat{G}}^{-1} \,. \tag{5}$$

So
$$\begin{aligned}(\delta\otimes i)\circ\delta &= \big(((i\otimes F_{\widehat{G}})\circ\tilde{\alpha})\otimes i\big)\circ\big((i\otimes F_{\widehat{G}})\circ\tilde{\alpha}\big)\\ &= (i\otimes F_{\widehat{G}}\otimes F_{\widehat{G}})\circ(\tilde{\alpha}\otimes i)\circ\tilde{\alpha}\\ &= (i\otimes F_{\widehat{G}}\otimes F_{\widehat{G}})\circ(i\otimes\alpha_G)\circ\tilde{\alpha} && \text{(by 3)}\\ &= \big(i\otimes((F_{\widehat{G}}\otimes F_{\widehat{G}})\circ\alpha_G\circ F_{\widehat{G}}^{-1})\big)\circ\big((i\otimes F_{\widehat{G}})\circ\tilde{\alpha}\big)\\ &= (i\otimes\delta_G)\circ\delta && \text{(by 5)}\end{aligned}$$

showing δ satisfies condition (ii) of the definition of a coaction. It can be shown that the technical condition (i) is also satisfied so that δ is a coaction of G on A. Hence every action of the dual group $\widehat{G}$ on A gives a coaction of G on A. Further it can be shown this correspondence is bijective.

As was mentioned earlier one of the main purposes of introducing coactions is in the hope that they will pave the way to a generalisation of the duality theorem. But before this hope can be realised it is necessary to have a non-abelian analogue of a crossed product by an action of the dual group.

Definition Suppose $\pi : A \to B(\mathcal{H})$ is a faithful representation of A on the Hilbert space $\mathcal{H}$, M_G is the representation of $C_o(G)$ on $L^2(G)$ by multiplication operators and $\delta : A \to \tilde{M}(A\otimes C_r^*(G))$ is a coaction of G on A. Then the *crossed product* $A\times_\delta G$ of A by δ is the C^*-subalgebra of $B(\mathcal{H}\otimes L^2(G))$ generated by the elements $(\pi\otimes i)(\delta(a))(1\otimes M_G(f))$, $a\in A$, $f\in C_o(G)$.

It can be shown that $A\times_\delta G$ is independent of the choice of π. In fact it can be given a non-spatial definition [7 def. 2.4]. To motivate the definition, suppose G is abelian, $\alpha : \widehat{G}\to \mathrm{Aut}A$ is an action of the dual group $\widehat{G}$ on A, δ is the corresponding coaction of G on A and $\mathcal{F}: L^2(G) \to L^2(\widehat{G})$ is the Fourier transform, then the isomorphism

$$Ad(1 \otimes \mathcal{F}): B(\mathcal{H} \otimes L^2(G)) \to B(\mathcal{H} \otimes L^2(\widehat{G}))$$

maps the generators $(\pi \otimes i)(\delta(a))(1 \otimes M_G(f))$ of the crossed product $A \times_\delta G$ to the elements $(\pi \otimes M_G)(\tilde{\alpha}(a))(1 \otimes \lambda_{\widehat{G}}(\hat{f}))$. Now it can be shown that elements of the form $(\pi \otimes M)(\tilde{\alpha}(a))(1 \otimes \lambda_{\widehat{G}}(g))$, $a \in A$, $g \in C^*(G)$ generate $\pi \times \lambda_{\widehat{G}}(A \times_\alpha \widehat{G}) \subset B(\mathcal{H} \otimes L^2(\widehat{G}))$ and hence that

$$A \times_\delta G \cong A \times_\alpha \widehat{G} .$$

So in the abelian case crossed products by coactions correspond to crossed products by actions of the dual group.

Now if $\beta : G \to \mathrm{Aut} A$ is an action of G on A then there is a natural coaction

$$\hat{\beta} : A \times_\beta G \to M((A \times_\beta G) \otimes C_r^*(G)),$$

called the *dual coaction* of G on $A \times_\beta G$, determined by $\hat{\beta}(i_G(s)) = i_G(s) \otimes \lambda_G(s)$ and $\hat{\beta}(i_A(a)) = i_A(a) \otimes 1$, where i_G and i_A are the natural inclusions of G and A in $M(A \times_\beta G)$. Also if δ is a coaction of G on A then there is a natural action, the *dual action*, of G on $A \times_\delta G$ defined by

$$\hat{\delta}_s = Ad(1 \otimes \rho_G(s)) \qquad s \in G ,$$

and we have the following duality theorems, the first of which generalises Takai's theorem.

Theorem (Imai-Takai [3]) *Suppose $\alpha : G \to \mathrm{Aut} A$ is an action of a locally compact group G on A. Then there exists an isomorphism*

$$(A \times_\alpha G) \times_{\hat{\alpha}} G \cong A \otimes K(L^2(G)) ,$$

which carries $\hat{\alpha}$ to the action $\alpha \otimes Ad\, \rho_G$.

Theorem (Katayama [4]) *Suppose* $\delta : A \to \tilde{M}(A \otimes C_r^*(G))$ *is a non-degenerate coaction of* G *on* A. *Then there exists an isomorphism*

$$(A \times_\delta G) \times_{\hat{\delta}} G \cong A \otimes K(L^2(G)) .$$

which carries $\hat{\hat{\delta}}$ *to* $\tilde{\delta}$ *where*

$$\tilde{\delta}(x) = (1 \otimes W_G^*)\big((i \otimes \Sigma)((\delta \otimes i)(x))\big)(1 \otimes W_G)$$

and Σ *is the flip map of* $C_r^*(G) \otimes K(L^2(G))$ *onto* $K(L^2(G)) \otimes C_r^*(G)$.

To exploit the duality theorems it is essential to have an understanding of the representation theory of the crossed products. Such a theory was investigated and determined by Landstad-Phillips-Raeburn-Sutherland [7] in terms of "covariant representations" and updated somewhat by Raeburn [10]. To make this precise we give the following definitions.

Let G be any locally compact group. Then we define a unitary element ϖ_G of $C_b^s(G, M(C_r^*(G)))$, the bounded strictly continuous maps from G to $M(C^*(G))$, by $\varpi_G(s) = \lambda_G(s)$ for all $s \in G$. For future reference we note that :

(i) if $\mu : C_o(G) \to B(\mathcal{H})$ is a representation of $C_o(G)$ and f is an element of the Fourier algebra $A(G)$ [1] considered as an element of $vN(G)_*$ then it can be shown that

$$(1 \otimes f)(\mu \otimes i(\varpi_G)) = \mu(f) . \qquad (6)$$

(ii) if G is abelian, F_G is as above and

$$F_G \otimes F_{\widehat{G}}^{-1} : M(C_o(G) \otimes C^*(G)) \cong C_b^s(G, M(C^*(G))) \to$$

$$M(C^*(\widehat{G}) \otimes C_o(\widehat{G})) \cong C_b^s(\widehat{G}, M(C^*(\widehat{G})))$$

then $$F_G \otimes F_{\widehat{G}}^{-1}(\varpi_G) = \varpi_{\widehat{G}}\,. \tag{7}$$

Note : ϖ_G is being considered an element of $M(C_o(G) \otimes C^*(G))$, and $\varpi_{\widehat{G}}$ as an element of $M(C^*(\widehat{G}) \otimes C_o(\widehat{G}))$.

Definition Let $\delta : A \to \tilde{M}(A \otimes C_r^*(G))$ be a coaction of G on A. Then a *covariant representation* for the system (A, G, δ) on $\mathcal{H}$ is a pair (π, μ) of non-degenerate representations of A and $C_o(G)$ on the Hilbert space $\mathcal{H}$ such that

$$(\pi \otimes i)(\delta(a)) = ((\mu \otimes i)(\varpi_G))(\pi(a) \otimes 1)((\mu \otimes i)(\varpi_G{}^*)) \quad \forall\, a \in A$$

Theorem (Landstad-Phillips-Raeburn-Sutherland [7]) *The non-degenerate representations of $A \times_\delta G$ on $\mathcal{H}$ correspond bijectively to the covariant representations of (A, G, δ) on $\mathcal{H}$.*

Once again we turn to the abelian case for movitation. Let δ be a coaction of an abelian group G on A with α the corresponding action of $\widehat{G}$, then $A \times_\delta G \cong A \times_\alpha \widehat{G}$ and thus the representation theory of $A \times_\delta G$ is already understood in terms of covariant pairs (π, U) of the covariant system $(A, \widehat{G}, \alpha)$. That is in terms of a non-degenerate representation π of A on $\mathcal{H}$ and a unitary representation U of $\widehat{G}$ on $\mathcal{H}$ such that

$$\pi(\alpha_\sigma(a)) = U_\sigma \pi(a) U_\sigma^*\,. \tag{8}$$

Now U determines a representation μ of $C_o(G)$ on $\mathcal{H}$ by $\mu = \mathcal{U} \circ F_G$ where $\mathcal{U}$ is the integrated form of U. Now if $\tilde{\alpha}$ is as above and $\sigma \in \widehat{G}$ then

$$\begin{aligned} ((\pi \otimes i)(\tilde{\alpha}(a)))(\sigma) &= \pi(\alpha_\sigma(a)) \\ &= U_\sigma\, \pi(a)\, U_\sigma^* && \text{(by 8)} \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{U}(\lambda_{\widehat{G}}(\sigma))\,\pi(a)\,\mathcal{U}(\lambda^*_{\widehat{G}}(\sigma)) \\
&= \mathcal{U}(\varpi_{\widehat{G}}(\sigma))\,\pi(a)\,\mathcal{U}(\varpi_{\widehat{G}}{}^*(\sigma)) \\
&= ((\mathcal{U}\otimes i)(\varpi_{\widehat{G}}))(\sigma)\,\pi(a)\,((\mathcal{U}\otimes i)(\varpi_{\widehat{G}}))(\sigma) \\
&= \big(((\mathcal{U}\otimes i)(\varpi_{\widehat{G}}))\cdot(\pi(a)\otimes 1)\cdot((\mathcal{U}\otimes i)(\varpi_{\widehat{G}}{}^*))\big)(\sigma)\,,
\end{aligned}
$$

where $\pi(a)\otimes 1$ is the constant function with value $\pi(a)$ and $\cdot$ denotes pointwise multiplication. So

$$
(\pi\otimes i)(\tilde{\alpha}(a)) = ((\mathcal{U}\otimes i)(\varpi_{\widehat{G}}))\cdot(\pi(a)\otimes 1)\cdot((\mathcal{U}\otimes i)(\varpi_{\widehat{G}}{}^*))
$$

and we have that,

$$
\begin{aligned}
(\pi\otimes i)(\delta(a)) &= (\pi\otimes F_{\widehat{G}})(\tilde{\alpha}(a)) && \text{(by 4)} \\
&= (i\otimes F_{\widehat{G}})\big(((\mathcal{U}\otimes i)(\varpi_{\widehat{G}}))\cdot(\pi(a)\otimes 1)\cdot((\mathcal{U}\otimes i)(\varpi_{\widehat{G}}{}^*))\big) \\
&= (((\mathcal{U}\circ F_G)\otimes i)(\varpi_G))(\pi(a)\otimes 1)(((\mathcal{U}\circ F_G)\otimes i)(\varpi_G{}^*)) && \text{(by 7)} \\
&= ((\mu\otimes i)(\varpi_G))(\pi(a)\otimes 1)((\mu\otimes i)(\varpi_G{}^*))\,,
\end{aligned}
$$

so each covariant representation (π, U) of the system $(A, \widehat{G}, \alpha)$ corresponds to a covariant representation (π, μ) of $(A, G, \delta\,)$. Further this correspondence is bijective.

The duality theory presented above has its roots in a similar theory for von Neumann algebras. The original duality theorem, analogous to Takai's, being due to Takesaki [15]. The von Neumann algebra counterparts which led to Imai-Takai and Katayama's theorems were shown independently by Landstad [5,6], Nakagami [9] and Strătilă-Voiculescu-Zsidó [13].

§2 Induced Representations of Crossed Products by Coactions

Given an action $\beta : G \to \mathrm{Aut}A$ of G on A it is possible to define an action of any closed subgroup H of G on A by restriction. The analogous result for coactions is ;

Definition Suppose $\delta : A \to \tilde{M}(A \otimes C_r^*(G))$ is a coaction and H is a closed normal amenable subgroup of G. Then one can define a coaction $\delta| : A \to \tilde{M}(A \otimes C_r^*(G/H))$ of G/H on A by

$$\delta|(a) = (i \otimes \Phi)(\delta(a)) ,$$

where $\Phi : C_r^*(G) \to C_r^*(G/H)$ is the map obtained by lifting the integrated form of $: s \to \lambda_{sH}$ to the quotient $C_r^*(G)$.

When δ is a coaction of an abelian group G and α is the corresponding action of $\widehat{G}$ we have that

$$A \times_{\delta|} (G/H) \cong A \times_\alpha H^\perp \quad \text{and} \quad A \times_\alpha H \cong A \times_{\delta|} (\widehat{G}/H^\perp) . \tag{9}$$

One of the reasons the development of the theory of coactions has been less tractible than that of actions is that hitherto there has been no dense $*$-subalgebra of $A \times_\delta G$ analogous to the subalgebra $C_c(G, A)$ of $A \times_\alpha G$. I now present such an analogue which I hope, despite its complexity, will facilitate research in the area.

Theorem 1 *Suppose $\delta : A \to \tilde{M}(A \otimes C_r^*(G))$ is a non-degenerate coaction of G on A, H is a closed normal amenable subgroup of G, $\delta|$ is as above and $\mathcal{D}_H$ is the set of norm limits of sequences $(x_j)_{j=1}^\infty$ in $B(\mathcal{H} \otimes L^2(G/H))$, of the form*

$$x_j = \sum_{i=1}^{n_j} (\pi \otimes i)(\delta|((1 \otimes u)(\delta(a_{ij}))))(1 \otimes M_{G/H}(f_{ij})) ,$$

where $a_{ij} \in A$, $f_{ij} \in C_c(G/H)$ with the support of the f_{ij} contained in some fixed compact subset of G/H for all i, j and u is a fixed element of $A(G) \cap C_c(G)$ considered as an element of $vN(G)_*$. Then $\mathcal{D}_H$ is a dense $*$-subalgebra of $A \times_{\delta|} (G/H)$. In particular if H is the trivial subgroup 1, $\mathcal{D}_1 = \mathcal{D}$ is a dense $*$-subalgebra of $A \times_\delta G$.

Now we wish to consider the question of how the representations of $A \times_{\delta|} (G/H)$ relate to those of $A \times_\delta G$. Once again direction is given by investigating the abelian case. In this case $A \times_{\delta|} (G/H) \cong A \times_\alpha H^\perp$ and $A \times_\delta G \cong A \times_\alpha \widehat{G}$, so a representation of $A \times_{\delta|} (G/H)$ (i.e. of $A \times_\alpha H^\perp$) can be induced, by the theory of Green [2], to a representation of $A \times_\delta G$ (i.e. of $A \times_\alpha \widehat{G}$). So for G abelian we have an induction process. It turns out that this induction process is still possible when G is any locally compact group and H is a closed normal amenable subgroup of G. The induction process is given by the following theorem.

Theorem 2 *$\mathcal{D}$ is a $\mathcal{D}_H$-rigged $\mathcal{D}$ module with the $\mathcal{D}$ action being by bounded operators.*

That is $\mathcal{D}$ has a $\mathcal{D}_H$-valued inner product $\langle \cdot , \cdot \rangle_{\mathcal{D}}$ which satisfies certain compatibility conditions regarding the actions. For a fuller account see Rieffel [11].

Recall that if $\nu : A \times_{\delta|} (G/H) \to B(\mathcal{H})$ is a representation of $A \times_{\delta|} (G/H)$ then given a left $\mathcal{D}$, right $\mathcal{D}_H$ bimodule which is $\mathcal{D}_H$-rigged, such as say $\mathcal{D}$, we can equip the tensor product $\mathcal{D} \otimes_{\mathcal{D}_H} \mathcal{H}$ with the pre-inner product

$$\langle x \otimes \xi , y \otimes \eta \rangle_{\mathcal{D} \otimes_{\mathcal{D}_H} \mathcal{H}} = \langle \langle x , y \rangle_{\mathcal{D}} \cdot \xi , \eta \rangle_{\mathcal{H}}$$

and obtain a Hilbert space $\mathcal{R}$ by factoring out by the vectors of length zero and completing. The Hilbert space $\mathcal{R}$ comes equipped with a natural action of $A \times_\delta G$, that is we obtain a representation indν of $A \times_\delta G$ on $\mathcal{R}$, determined by

$$(\mathrm{ind}\nu(x))\Big(\sum_{i=1}^{n} y_i \otimes \xi_i\Big) = \sum_{i=1}^{n} (xy_i) \otimes \xi_i \quad x, y_i \in \mathcal{D},\ \xi_i \in \mathcal{H}.$$

Of course we call the representation indν obtained in this way, the representation induced from ν.

In the action case one has Green's imprimitivity theorem to characterise those representations of $A \times_\beta G$ which are induced from representations of $A \times_\beta H$. A natural question is ; Is there an imprimitivity theorem for the induction of representations of crossed products by coactions? If δ is the trivial coaction of G on the complex numbers, i.e. $\delta(z) = z \otimes 1$ for all complex numbers z, so that $A \times_\delta G \cong C_o(G)$, then the question has an affirmative answer in the form of the following strong Morita equivalence due to Rieffel [12]

$$C_o(G) \times_\sigma H \approx C_o(G/H),$$

where $\approx$ denotes a strong Morita equivalence and σ is the right translation action. From this it is easy to guess at a strong Morita equivalence that will determine an imprimitivity theorem for crossed products by coactions.

Theorem 3 *Suppose $\delta : A \to \tilde{M}(A \otimes C_r^*(G))$ is a non-degenerate coaction of a locally compact group G on a C^*-algebra A and H is a closed normal amenable subgroup of G. Then*

$$(A \times_\delta G) \times_{\hat{\delta}} H \approx A \times_{\delta|} (G/H)\,.$$

Corollary *The map*

$$: \mathrm{Prim}\,(A \times_{\delta|} (G/H)) \to \mathrm{Prim}\,(A \times_\delta G) : \ker\nu \to \ker\,(\mathrm{ind}\nu)\;,$$

is continuous.

The imprimitivity theorem which can be read off from the strong Morita equivalence of the theorem says that a representation ν of $A \times_\delta G$ is induced from a representation of $A \times_{\delta|} (G/H)$ if and only if there exists a unitary representation $U : H \to UB(\mathcal{H})$ of H on $\mathcal{H}$ such that

$$\nu(\hat{\delta}_h(x)) = U_h\,\nu(x)\,U_h^* \quad \forall\, h \in H,\; x \in A \times_\delta G\,.$$

As is usual an investigation of the abelian case is extremely instructive. Suppose $\delta : A \to M(A \otimes C^*(\widehat{G}))$ is a coaction of the dual group $\widehat{G}$ on A, α is the corresponding action of G, and H is a closed subgroup of G. Then by theorem 3

$$(A \times_\delta \widehat{G}) \times_{\hat{\delta}} H^\perp \approx A \times_{\delta|} (\widehat{G}/H^\perp) ,$$

and hence by (9),

$$(A \times_\alpha G) \times_{\hat{\alpha}|} (G/H) \approx A \times_\alpha H .$$

Now it turns out that this is true more generally and we have the following theorem.

Theorem 4 *Suppose $\alpha : G \to \mathrm{Aut}A$ is an action of G on A, $\hat{\alpha}$ is the dual coaction on $A \times_\alpha G$ and H is a closed normal amenable subgroup of G. Then*

$$(A \times_\alpha G) \times_{\hat{\alpha}|} (G/H) \approx A \times_\alpha H .$$

The corresponding imprimitivity theorem concerning the induction of representations of $A \times_\alpha H$ to those of $A \times_\alpha G$, says that a representation μ of $A \times_\alpha G$ is induced from a representation of $A \times_\alpha H$ if and only if there exists a non-degenerate representation ζ of $C_o(G/H)$ such that

$$(\mu \otimes i)(\hat{\alpha}|(b)) = ((\zeta \otimes i)(\varpi_{G/H}))(\mu(b) \otimes 1)(((\zeta \otimes i)(\varpi_{G/H}{}^*)) \quad \forall\, b \in \; A \times_\alpha G \qquad (10)$$

As you may have guessed by now, this is a reformulation of the following theorem due to Green.

Theorem (Green [2]) *Suppose $\alpha : G \to \mathrm{Aut}A$ is an action of G on A, H is a closed subgroup of G, τ is the left translation action of G on $C_o(G/H)$ and $\alpha \otimes \tau$ is the action of G on $A \otimes C_o(G/H)$ defined on elementary tensors by $(\alpha \otimes \tau)_s(a \otimes f) = \alpha_s(a) \otimes \tau_s(f)$ then*

$$(A \otimes C_o(G/H)) \times_{\alpha \otimes \tau} G \approx A \times_\alpha H .$$

The corresponding imprimitivity theorem is ; Let μ be a representation of $A \times_\alpha G$ on $\mathcal{H}$. Split μ up, i.e. find the covariant representation (π, U) corresponding to μ. Then μ is induced from a representation of $A \times_\alpha H$ if and only if there exits a non-degenerate representation ζ of $C_o(G/H)$ such that

(i) $\pi(a)\zeta(f) = \zeta(f)\pi(a) \quad \forall\, a \in A,\ f \in C_o(G/H)$.

(ii) $(\pi \otimes \zeta)((\alpha \otimes \tau)_s(w)) = U_s \pi \otimes \zeta(w) U_s^* \quad \forall\, s \in G,\ w \in A \otimes C_o(G/H)$.

To see that the two formulations are equivalent, let $f \in A(G/H) \subset vN(G/H)_*$ and $a \in A$. Then

$$\begin{aligned}
\zeta(f)\pi(a) &= ((1 \otimes f)((\zeta \otimes i)(\varpi_{G/H})))\mu(i_A(a)) && \text{(by 6)}\\
&= (1 \otimes f)\big(((\zeta \otimes i)(\varpi_{G/H}))(\mu(i_A(a)) \otimes 1)\big)\\
&= (1 \otimes f)\big(((\mu \otimes i)(\hat{\alpha}|(i_A(a))))((\zeta \otimes i)(\varpi_{G/H}))\big) && \text{(by 10)}\\
&= (1 \otimes f)\big(((\mu \otimes i)(i_A(a) \otimes 1))((\zeta \otimes i)(\varpi_{G/H}))\big)\\
&= (1 \otimes f)\big((\pi(a) \otimes 1)((\zeta \otimes i)(\varpi_{G/H}))\big)\\
&= \pi(a)((1 \otimes f)((\zeta \otimes i)(\varpi_{G/H})))\\
&= \pi(a)\zeta(f) && \text{(by 6)}
\end{aligned}$$

which gives (i) and ensures $\pi \otimes \zeta$ is a well defined representation of $A \otimes C_o(G/H)$ on $\mathcal{H}$. To obtain (ii), let $s \in G$ then

$$\begin{aligned}
\zeta(f)U_s &= ((1 \otimes f)((\zeta \otimes i)(\varpi_{G/H})))\,U_s && \text{(by 6)}\\
&= (1 \otimes f)\big(((\zeta \otimes i)(\varpi_{G/H}))(U_s \otimes 1)\big)\\
&= (1 \otimes f)\big(((\zeta \otimes i)(\varpi_{G/H}))(\mu(i_G(s)) \otimes 1)\big)\\
&= (1 \otimes f)\big(((\mu \otimes i)(\hat{\alpha}|(i_G(s))))((\zeta \otimes i)(\varpi_{G/H}))\big) && \text{(by 10)}\\
&= (1 \otimes f)\big(((\mu \otimes i)(i_G(s) \otimes \lambda_{G/H}(sH)))((\zeta \otimes i)(\varpi_{G/H}))\big)
\end{aligned}$$

$$= (1 \otimes f)\big((U_s \otimes \lambda_{G/H}(sH))((\zeta \otimes i)(\varpi_{G/H}))\big)$$

$$= U_s((1 \otimes f)\big((1 \otimes \lambda_{G/H}(sH))((\zeta \otimes i)(\varpi_{G/H}))\big))$$

$$= U_s((1 \otimes \tau_s(f))((\zeta \otimes i)(\varpi_{G/H}))$$

$$= U_s\zeta(\tau_s(f)) \qquad \text{(by 6)}$$

This and the fact that (π, U) is a covariant pair, i.e. that $\pi(\alpha_s(a)) = U_s\, \pi(a)\, U_s^*$ gives (ii). Similarly (i) and (ii) imply (10).

A major advantage of the first formulation is that to check whether or not a representation μ of $A \times_\alpha G$ is induced, one uses μ directly, whereas in the second, it is necessary to split μ up. i.e. it is necessary to find the covariant representation corresponding to μ.

The conceptual simplification of Green's theorem resulting from the introduction of coactions suggests that they are in some sense natural, and may eventually play a pivotal role in the study of dynamical systems.

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