CLASSIFYING ACTIONS OF GROUPS ON VON NEUMANN ALGEBRAS

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The objective of this paper is to provide an overview of results obtained recently by the author and M. Takesaki on the classification, up to cocycle conjugacy, of actions of discrete amenable groups on injective factor von Neumann algebras of type III_λ, $\lambda \neq 1$. Details will appear elsewhere, [14]. We also describe the conclusions of preceding work of Oceanu, [10] extending previous work of Connes [1],[2] and Jones [7], and of Sutherland and Takesaki, [13], on actions of discrete amenable groups and groupoids on injective semifinite von Neumann algebras.

§0 NOTATION

Throughout, G will denote a locally compact second countable group, and M will denote a von Neumann algebra with separable predual M_*; when required, we will regard M as being realized as a weakly closed, unital, self-adjoint subalgebra of the bounded operators $B(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$. An <u>action</u> α of G on M means a homomorphism $\alpha: G \to \mathrm{Aut}(M)$ which is strongly continuous i.e. for which $g \in G \to \omega(\alpha_g(x))$ is continuous for each $x \in M$ and $\omega \in M_*$. For a unitary $u \in M$, Adu will denote the inner automorphism, $\mathrm{Adu}(x) = uxu^*$.

§1 COCYLE CONJUGACY

Actions α and β of G on M are said to be cocycle conjugate if there exists $\theta \in \mathrm{Aut}(M)$ and a unitary valued function $g \in G \to u(g) \in M$ such that

1) $\theta \circ \alpha_g \circ \theta^{-1} = \mathrm{Ad}\, u(g) \circ \beta_g$ for $g \in G$;

2) $g \to u(g)$ is continuous, and

$$u(g)\beta_g(u(h)) = u(gh) \quad \text{for} \quad g,h \in G.$$

Functions $g \to u(g)$ satisfying 2) above are called 1-cocycles for β, and the space of all such 1-cocycles is denoted $Z^1_\beta(G,U(M))$.

It is routine to check that cocycle conjugacy is an equivalence relation on actions of G on M. To see what is involved in this relation, we look at some examples.

EXAMPLE a) $M = L^\infty(X,\mu)$, where (X,μ) is a standard measure space. Since M is abelian, cocycle conjugacy and conjugacy coincide, and the cocycle conjugacy classes of actions on M correspond bijectively which conjugacy classes of non-singular actions of G on (X,μ). Even for $G = \mathbb{Z}$, there is no sense in which there is any reasonable classification of such actions

b) $M = B(\mathcal{H})$. If $\alpha : G \to \mathrm{Aut}(B(\mathcal{H}))$ is an action, then, since every automorphism of $B(\mathcal{H})$ is inner, we may choose a (Borel) function $g \in G \to a(g) \in U(\mathcal{H})$ such that $\alpha_g = \mathrm{Ad}\, a(g)$ for each $g \in G$. Since $\alpha_g \circ \alpha_h = \alpha_{gh}$, we obtain

$$a(g)a(h) = \mu_a(g,h)a(gh)$$

for some Borel function $\mu_a : G \times G \to \mathbb{T}$. It is routine to check that μ_a satisfies the functional equation

$$\mu_a(h,k)\,\mu_a(g,hk) = \mu_a(g,h)\,\mu_a(gh,k)$$

for $g,h,k \in G$, and that if $a(g)$ is replaced by $b(g) = c(g)\,a(g)$ with $c(g) \in \mathbb{T}$, then

$$\mu_b(g,h) = \mu_a(g,h)c(g)^{-1}c(h)^{-1}c(gh).$$

In the language of cohomology theory, μ_a defines a (Borel) 2-cocycle on G, whose class in the corresponding cohomology group $H^2(G,\mathbb{T})$ depends only on the action α. More is true; if $\mu_\alpha = [\mu_a] \in H^2(G,\mathbb{T})$, then μ_α depends only on the cocycle conjugacy class of α, and actions α,β are conjugate if and only if $\mu_\alpha = \mu_\beta$.

These two examples suggest that, in general, the cocycle conjugacy classification problem will be controlled by a mixture of invariants coming from ergodic theory and cohomology. We will see that this is indeed the case. One more example is instructive.

c) Let M be a factor and let α be an automorphism such that α^n is inner (for some $n \in \mathbb{N}$), but α^k is outer for $1 \le k < n$. Choose a unitary $a \in M$ such that $\alpha^n = \mathrm{Ad}\, a$, and observe that

$$\mathrm{Ad}\, \alpha(a) = \alpha \circ \mathrm{Ad} a \circ \alpha^{-1} = \alpha^n = \mathrm{Ad}\, a$$

so that $\alpha(a) = \gamma a$ for some $\gamma \in \mathbb{T}$. Note that

$$a = \mathrm{Ad} a(a) = \alpha^n(a) = \gamma^n a,$$

so that $\gamma^n = 1$, and that γ is a cocycle conjugacy invariant of the action $n \in \mathbb{Z} \to \alpha^n$.

The following result is one of the prototypes for results in this area.

THEOREM [1,2] *Let* M *be the hyperfinite factor* R *of type* II_1, *and let* $\alpha,\beta \in \mathrm{Aut}(M)$ *be such that* $n_\alpha = n_\beta$ *and* $\gamma_\alpha = \gamma_\beta$*; then the actions* $n \to \alpha^n$ *and* $n \to \beta^n$ *are cocycle conjugate. The same is true for automorphisms* α *for which* α^n *is outer for all* $n \neq 0$.

The hyperfinite factor R of type II_1 of the theorem is ubiquitous; it may be constructed as the von Neumann algebra generated by the left regular representation of any discrete amenable infinite conjugacy class group.

§2 THE GROUP $\Lambda_\alpha(G,N,A)$

As example c) above shows, classifying actions up to cocycle conjugacy requires a group more sensitive than an ordinary second cohomology group. Our purpose here is to introduce the groups which are relevant to the problem.

Let G be a locally compact group, N a closed normal subgroup, and let A be an abelian Polish G-module as in [9], so A is a complete separable metric abelian group carrying a jointly continuous action α of G. We consider the group $Z_\alpha(G,N,A)$ of pairs (λ,μ) of functions $\lambda : N \times G \to A$, $\mu : N \times N \to A$ such that

1) $\mu \in Z^2_\alpha(N,A)$ i.e.

$$\alpha_m(\mu(n,p))\mu(m,np) = \mu(m,n)\mu(mn,p);$$

2) $\alpha_g(\mu(g^{-1}mg,\ g^{-1}ng))\mu(m,n)^{-1} = \lambda(m,g)\alpha_m(\lambda(n,g))\lambda(mn,g)^{-1}$.

3) $\lambda(m,gh) = \alpha_g(\lambda(g^{-1}mg,h))\lambda(m,g)$

4) $\lambda(n,m) = \mu(m,m^{-1}nm)\mu(m,n)^{-1}$

5) $\mu(m,n) = 1$ and $\lambda(n,g) = 1$ if any of m,n or g is the identity.

(Here $m,n,p \in N$ and $g,h \in G$)

We also define $B_\alpha(G,N,A)$ as the subgroup of $Z_\alpha(G,N,A)$ consisting of pairs $(\partial_1 c,\ \partial_2 c)$, where $c : N \to A$ is Borel, and

$$(\partial_1 c)(n,g) = \alpha_g(c(g^{-1}ng))c(n)^{-1}$$

and

$$(\partial_2 c)(m,n) = c(m)\alpha_m(c(n))c(mn)^{-1}$$

The quotient group $Z_\alpha(G,N,A)/B_\alpha(G,N,A)$ is denoted $\Lambda_\alpha(G,N,A)$. This group was introduced by Jones [7] and modified to the context of measured groupoids in [8] and [13]. Algebraically, it is an interesting cohomological object, since it parametrizes extensions of the form

$$1 \to A \to H \to N \to 1$$

in the category of G modules, where the action on N is by conjugation, and the action on A is the given one.

§3 ACTIONS ON SEMIFINITE ALGEBRAS

Consider an action α of G on a semifinite factor M, with trace Tr. Put $N(\alpha) = \alpha^{-1}(\mathrm{Int}\, M)$, where $\mathrm{Int}\, M$ denotes the inner automorphism group, and choose a Borel map $n \in N \to a(n) \in U(M)$ with $\alpha_n = \mathrm{Ad}\, a(n)$. Since $\mathrm{Int}(M)$ is normal in $\mathrm{Aut}(M)$, $N(\alpha)$ is normal in G; since $\alpha_m \circ \alpha_n = \alpha_{mn}$ and $\alpha_g \circ \alpha_{g^{-1}ng} \circ \alpha_g^{-1} = \alpha_n$, we find

$$a(m)a(n) = \mu_a(m,n)a(mn)$$

and

$$\alpha_g(a(g^{-1}ng)) = \lambda_a(n,g)a(n)$$

for $m,n \in N(\alpha)$ and $g \in G$. It is routine to check that $(\lambda_a,\mu_a) \in Z_\alpha(G,N,\mathbb{T})$, and that the class $\chi_\alpha = [(\lambda_\alpha,\mu_\alpha)]$ of this element in $\Lambda_\alpha(G,N,\mathbb{T})$ is a cocycle conjugacy invariant of α.

We may also obtain another cocycle conjugacy invariant $\delta_\alpha \in \mathrm{Hom}(G,\mathbb{R}_+^\times)$ by using essential uniqueness of the trace thus; for each $g \in G$, then is a constant $\delta_\alpha(g) > 0$ with

$$\mathrm{Tr} \circ \alpha_g = \delta_\alpha(g)\ \mathrm{Tr}.$$

Note that if M is finite (i.e. $\mathrm{Tr}(1_M) < \infty$), δ_α is automatically trivial.

THEOREM *Let* $M = R$ *or* $M = R_{0,1} = R \otimes B(\mathcal{H})$ *be a semifinite injective factor, and let* α,β *be actions of a discrete amenable group* G *on* M. *Then the actions* α,β *are cocycle conjugate if and only if*

i) $N(\alpha) = N(\beta)$, and

ii) $\chi_\alpha = \chi_\beta$ and $\delta_\alpha = \delta_\beta$.

With the obvious restrictions, all possible values of the invariants can occur.

In the form stated, the theorem is due to Ocneanu [11], although it had previously been proven for $G = \mathbb{Z}$ by Connes [1], [2] and for G finite, by Jones, [7].

It is relatively easy to construct model actions realizing a given set of invariants. For simplicity, suppose G is infinite, and realize R as the infinite tensor product $\otimes_G M_2(\mathbf{C})$ of 2×2 matrix algebras (with respect to the trace) indexed over G. We let G act on R via (left) translation of the indices, say τ, and consider the twisted crossed product $R \times_{\tau,\mu} N = P$. Thus P is generated by an isomorphic copy $\pi(R)$ of R and a (particular) family $\{\rho^\mu(n) : n \in N\}$ of unitaries satisfying

$$\rho^\mu(m)\ \rho^\mu(n) = \mu(m,n)\rho^\mu(mn),$$

and

$$\rho^\mu(m)\pi(x)\rho^\mu(m)^{-1} = \pi(\tau_m(x))$$

for $m,n \in N$ and $x \in R$. If $(\lambda,\mu) \in Z(G,N,\mathbb{T})$, there is an action α of G on P determined by

$$\alpha_g(\pi(x)) = \pi(\tau_g(x))$$

and

$$\alpha_g(\rho^\mu(g^{-1}ng)) = \lambda(n,g)\rho^\mu(n)$$

P is in fact isomorphic to R if N is amenable, and the action α described above has invariants $[(\lambda,\mu)] \in \Lambda(G,N,\mathbb{T})$.

Ocneanu's results may be generalized to actions of discrete amenable groups G on semifinite injective algebras $M = \quad \bar{\otimes} R$ or $\bar{\otimes} R_{0,1}$ where is abelian. This was done in [13], following a strategy devised by Jones and Takesaki in [8]. The extension is non-trivial, since one must incorporate the action of G on the centre of M; the best way to formulate the problem is using groupoids.

In general, if $\alpha : G \to \mathrm{Aut}(\,\bar{\otimes}\,P)$ is an action, with P a factor, we may realize as $L^{\infty}(X,\mu)$ for some standard measure space (X,μ), and the action of G on a as being of the form

$$(\alpha_g\phi)(x) = \phi(xg)$$

for $\phi \in L^{\infty}(X,\mu)$, where $(x,g) \in X \times G \to xg \in X$ is some non-singular action of G on (X,μ). If we view M as consisting of (classes of) bounded measurable functions from X to P, the action of G on M is of the form

$$(\alpha_g T)(x) = \bar{\alpha}_{(x,g)}(T(xg))$$

for $T \in L^{\infty}(X,\mu,M)$, where $\bar{\alpha} : X \times G \to \mathrm{Aut}(P)$ is a measurable map. If we endow $X \times G$ with the partially defined "product"

$$(x,g)(xg,h) = (x,gh),$$

$X \times G$ becomes a groupoid, denoted $X \times G$, and the map $\bar{\alpha}$ becomes a "homomorphism" from $X \times G$ to $\mathrm{Aut}(P)$ i.e.

$$\bar{\alpha}_{(x,g)}\,\bar{\alpha}_{(xg,h)} = \bar{\alpha}_{(x,gh)}$$

a.e.(μ) in x for each $g,h \in G$. The results of Ramsay [12] ensure that we may assume the above identity holds for every x,g,h.

One way now repeat Ocneanu's program for actions of (measured) groupoids such as $\mathcal{G} = X \times G$ on R or $R_{0,1}$. To describe the invariants, we let

$$\mathcal{H} = \{(x,g) \in \mathcal{G} : xg = x\},$$

the "isotropy part" of $\mathcal{G}$, and let

$$\mathcal{N}_{\alpha} = \{(x,g) \in \mathcal{H} : \bar{\alpha}_{(x,g)} \in \mathrm{Int}(P)\}$$

The group $\lambda(\mathcal{G},\mathcal{N},\mathbb{T})$ may be defined in the same way as in §2, and choice of implementing unitaries on $\mathcal{N}_{\alpha}$ yields a cocycle conjugacy invariant $\chi_{\alpha} \in \Lambda(\mathcal{G},\mathcal{N}_{\alpha},\mathbb{T})$ for α, as above. Choosing a trace Tr on P yields a homomorphism $\delta : \mathcal{G} \to \mathbb{R}_{+}^{\times}$ satisfying

$$\mathrm{Tr} \circ \bar{\alpha}_{(x,g)} = \delta(x,g)\mathrm{Tr};$$

The cohomology class $\delta_\alpha = [\delta] \in H^1(\mathcal{G},\mathbb{R}_+^\times)$ is readily seen to be a cocycle conjugacy invariant of α.

THEOREM ([13] and [14]) *Let* α,β *be actions of* G *on* $\bar{\otimes}$ P, *where* P = R *or* $R_{0,1}$, *and suppose that either*

a) G *is discrete amenable, or*

b) G = H × R *where* H *is discrete amenable, that the actions* α β *of* G *admit traces* τ_α, τ_β *with* $\tau_\alpha \circ \alpha_{(h,s)} = e^{-s}\tau_\alpha$ *and* $\tau_\beta \circ \beta_{(h,s)} = c^{-s}\tau_\beta$, *and that the* R*-action on* $\neq$ **C** *is ergodic.*

Then α,β *are cocycly conjugate if and only if* $\mathcal{N}_\alpha = \mathcal{N}_\beta$, $\chi_\alpha = \chi_\beta$ *and* $\delta_\alpha = \delta_\beta$ *modulo an automorphism of* (X,μ) *intertwining the respective* G*-actions on* . *Further, with the obvious exceptions, all possible values of* $\mathcal{N}_\alpha$, χ_α, δ_α *occur.*

In the case where G is discrete, there are three main ingredients for the proof. The first is that the groupoid $\mathcal{G}$ may be split as a semi-direct product $\mathcal{G} = \mathcal{H} \times_s \mathcal{K}$, where $\mathcal{H}$ is as above, and $\mathcal{K} = \{(xg,x) : x \in X, x \in G\}$ is the "principal part", or equivalence relation since by [3], $\mathcal{K}$ is hyperfinite and hence singly generated. This allows us to split the classification problem in two parts; the part involving $\mathcal{H}$ is handled using Ocneanu's results (since $\mathcal{H}$ is a "bundle of amenable groups"), and the part involving $\mathcal{K}$ is handled using the third main ingredient, the "cohomology lemma" of [12]. Technically, this is by far the most intricate part of the whole argument.

In the case where G = H × $\mathbb{R}$, only one new ingredient is necessary.One replaces the (continuous) groupoid $\mathcal{G}$ by its (discrete) restriction $\mathcal{G}_B$ to a subset $B \subseteq X$ which meets each $\mathbb{R}$-orbit in a discrete set, and establishes a natural isomorphism of $\Lambda(\mathcal{G},\mathcal{N},\mathbb{T})$ with

$\Lambda(\mathcal{G}_B, \mathcal{N}_B, \mathbb{T})$; this requires care since B is a null set, and uses the "discrete reduction" techniques of [5].

§4 ACTIONS ON TYPE III FACTORS

We now turn to classification of actions on a type III injective factor M. The strategy for the classification is reasonably clear; one should use the duality theory of Takesaki, [15] to reduce the problem to one of actions on a II_∞ algebra, and to use the results described in §3. To achieve this, we need the following

PROPOSITION [14] *If* $\alpha : G \to \mathrm{Aut}(M)$ *is an action of a locally compact separable group* G *on a type III factor* M, *then* α *is cocycle conjugate to an action* β *such that*

i) *there is a dominant weight* ϕ *on* M *which is invariant under* $\{\beta_g : g \in G\}$

ii) *if* $s \in \mathbb{R} \to u(s)$ *is a 1-parameter unitary group in* M *with* $\sigma_t^\phi(u(s)) = e^{ist}u(s)$, *then* $\beta_g(u(s)) = u(s)$ *for all* $g \in G$, $s \in \mathbb{R}$.

Here $\{\sigma_t^\phi : t \in \mathbb{R}\}$ is the modular automorphism group of ϕ. The effect of the proposition is that the map $(g,s) \in G \times \mathbb{R} \to \beta_g \circ \mathrm{Ad}u(s)$ provides an action of $G \times \mathbb{R}$ on the centralizer $M_\phi = \{x \in M: \sigma_t^\phi(x) = x$ for all $t \in \mathbb{R}\}$; since M_ϕ is semifinite, and injective if M is injective, and since cocycle conjugacy of the $G \times \mathbb{R}$ actions implies cocycle conjugacy of the original actions, the problem has been reduced to the semifinite case. Further, the results of §3 are applicable since the $G \times \mathbb{R}$ action satisfies the hypothesis of the second theorem of §3, provided M is not a III_1 factor.

One impediment to applying §3 is the possibility that different invariant dominant weights may give rise to different centralizers, and in particular, to different $\mathbb{R}$-actions on the centres of these centralizers. This in fact does not happen, since this flow is intrinsic to the factor M - it is usually called the flow of weights $\mathcal{F}(M)$ of M, [4] - and the homomorphism $g \in G \to \mathrm{Aut}(\mathcal{F}(M))$ = {automorphisms commuting with the flow} is similarly intrinsic, and usually called the module, mod α, of the action α.

The only remaining difficulty is to identify the invariants for the $G \times \mathbb{R}$ actions on M_ϕ intrinsically in terms of the original action α. To do this, let $\mathrm{Cnt}(M)$ denote the centrally trivial automorphisms of M - for our purposes, these are the automorphisms of the form $\alpha = \mathrm{Ad}u \circ \sigma_c^\phi$ where $u \in M$ is unitary, ϕ is a dominant weight on M, $c \in Z^1(\mathcal{F}(M))$ is a unitary cocycle on the flow of weights, and σ_c^ϕ is the extended modular automorphism of [4]. If $\alpha : G \to \mathrm{Aut}(M)$ is an action, put

$$N(\alpha) = \alpha^{-1}(\mathrm{Cnt}(M)),$$

and for $n \in N(\alpha)$, choose unitaries $a(n) \in M$ and cocycles $c(n) \in Z^1(\mathcal{F}(M))$ such that

$$\alpha_n = \mathrm{Ad}\, a(n) \circ \sigma^\phi_{c(n)} .$$

As in §3, the unitaries $\{a(n) : n \in N\}$ give rise to a Λ-invariant χ_α, this time in $\Lambda_\alpha(G,N,U\mathcal{F}(M))$; the projection $\pi : Z^1(\mathcal{F}(M)) \to H^1(\mathcal{F}(M))$ gives rise to a homomorphism $\nu_\alpha : n \in N \to \pi(c(n)) \in H^1(\mathcal{F}(M))$. Both χ_α and ν_α are cocycle conjugacy invariants for α.

THEOREM [1] *Let* α,β *be actions of a discrete amenable group* G *on an injective factor* M *of type* III_λ, $\lambda \neq 1$. *Then* α *and* β *are cocycle conjugate if and only if,* $N(\alpha) = N(\beta)$ *and, up to an automorphism of* $\mathcal{F}(M)$, mod α = mod β, $\chi_\alpha = \chi_\beta$, *and* $\nu_\alpha = \nu_\beta$. *Further, if we fix a homomorphism* $\gamma : G \to \mathrm{Aut}(\mathcal{F}(M))$ *and a normal subgroup* $N \subseteq \ker\gamma$, *then if* $(\chi,\nu) \in \Lambda(G,N,U(\mathcal{F}(M)) \times \mathrm{Hom}(N,H^1(\mathcal{F}(M)),$ *there is an action*

α *of* G on M *such that* $(\mathrm{mod}\ \alpha, \chi_\alpha, \nu_\alpha) = (\gamma, \chi, \nu)$ *if and only if* $(\chi, \nu) \in \ker \delta$, *where* δ *is a natural homomorphism from* $\Lambda(G,N,U(\mathcal{F}(M)) \times \mathrm{Hom}(N,H^1(\mathcal{F}(M)))$ to $H^1(R,B_\alpha(G,N,U(\mathcal{F}(M))))$.

In conclusion, we note one structural consequence of the classification; if a discrete amenable group acts via α on von Neumann algebra M which is injective and either semifinite, or a factor of type III_λ, $\lambda \neq 1$, then α is cocycle conjugate to an action β which admits an invariant Cartan subalgebra in the sense of [5]. A direct proof of this fact would be of considerable value, since it would allow one to use conceptually simpler ideas from ergodic theory to effect the classification.

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