Momenta and Reduction for General Relativity II: the Level Sets.

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Abstract: The level sets of the momenta for GR are analyzed.

1. Introduction

This paper is a continuation of [2] where a proof of the slice theorem for the action of the group of diffeomorphisms $Diff(V)$ of a spacetime $V=\mathbb{R}^3\times\mathbb{R}^1$ on the space of asymptotically flat solutions to $Ein(V)$, the vacuum Einstein equations on V, was given. We work entirely in a spatial infinity setting, using a combination of the asymptotic conditions introduced by Regge and Teitelboim with the Quasi-Isotropic gauge conditions of York to separate out the Poincare' group as a subgroup of $Diff(V)$. For the definition of these concepts and notation we refer to [2], where it is shown that the group $Diff_P(V)$ of diffeomorphisms leaving invariant the set $Ein_{QI}(V)$ of solutions to the Einstein vacuum equations, satisfying the above mentioned asymptotic conditions, is of the form $Diff_P=Diff_I \circledS P$ where $Diff_I$ consists of diffeomorphisms tending to the identity at infinity and P is the Poincare' group.

The work contained herein is in a somewhat preliminary form, partly due to the fact that the asymptotic conditions assume only one degree of radial smoothness, in line with the presentation in [7]. Under these conditions there is, to the knowledge of the author, no proof in the literature that the momenta (in particular the boost momentum) of GR transform properly. Therefore it may be necessary to use two degrees of radial smoothness in accordance with the assumptions used by Regge and Teitelboim [15] which, without changing the overall picture, entails some additional technical difficulties which will be discussed in a future paper.

Let $M=\mathbb{R}^3$ and let $i:M\times\mathbb{R}\to V=\mathbb{R}^4$ denote a slicing of V such that the induced data on M satisfy the QI asymptotic conditions, i.e. are in $T^*\mathbf{M}_{QI}$. We will study the properties of the total momentum mapping

$$\Phi_E : T^*\mathbf{M}_{QI} \to \Lambda_d^0\times\Lambda_d^1\times\mathbf{p}^*$$

of GR, where $\Phi_E=\Phi+E$, $\Phi=(H,J)$ are the constraints and E are the momenta, consisting of certain integrals over spheres at infinity. As discussed in the introduction to [2] one expects from the general theory that the space of dynamical degrees of freedom, $\Phi^{-1}(0)/Diff_I$ is a symplectic ILH manifold and that for nonflat g, $\Phi_E^{-1}(0\times\xi)$ is an ILH variety with conical singularities corresponding to the elements in $\mathrm{C}=\Phi^{-1}(0)$ with exact rotational symmetries.

1.1. The 3+1 form of GR

As a preliminary we will introduce the notation for the Einstein equations that will be used in this paper. Details of a technical nature will be disregarded in this section. For $g\in Lor(V)$, the space of Lorenz metrics on V, the Hilbert Lagrangian for GR is $L_{GR}=\int_V R(g)\sqrt{-\det(g)}$. An application of Hamilton's principle: $\delta\int_V R(g)\sqrt{-\det(g)}=0$, with *compactly* supported variations gives after an integration by parts the Einstein equations $E(g)=0$ where $E(g)=Ric(g)-\frac{1}{2}R(g)g$ is the Einstein tensor of g. Note however, that the above is not true w.r.t. general variations. The necessary correction term, which corresponds to the second fundamental form of a cylinder at spatial infinity, gives rise to the mass in the Hamiltonian formulation of GR, see

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§1.2 below.

The phase space in the Hamiltonian formulation of Einsteins equations is $T^*\mathbf{M}$ where $\mathbf{M}$ denotes the space of Riemanninan metrics on M. An element in $T^*\mathbf{M}$ can be written as a pair (γ,π) where $\gamma\in\Gamma(S^2T^*(M))$ and $\pi\in\Gamma(S_d^2T(M))$, where $S_d^2T(M)=S^2T(M)\otimes\Omega(M)$ and $\Omega(M)$ denotes the space of scalar densities on M. For $\pi\in\Gamma(S_d^2T(M))$, we denote by π' the tensor $\pi/\mu(\gamma)$.

A Lorenz metric g on V such that $M_\lambda=i_\lambda(M)\subset V$ are spacelike for all $\lambda\in\mathbb{R}$ induces a curve in $T^*\mathbf{M}$ by $\gamma_\lambda=i_\lambda^*(g)$ and $\pi_\lambda=\mu(\gamma_\lambda)[(\mathrm{tr}k_\lambda)\gamma_\lambda-k_\lambda]^\#$, where $^\#$ denotes the operation of raising indices and k_λ denotes the second fundamental form of $i_\lambda(M)$. Denote the map which takes g into $(\gamma_\lambda,\pi_\lambda)$ by F_{i_λ}. Let

$$H(\gamma,\pi)=[\pi':\pi'-(tr\,\pi')^2-R(\gamma)]\mu(\gamma)$$

where $\pi':\pi'$ denotes the total contraction and let

$$J(\gamma,\pi)=-2\delta_\gamma\pi.$$

Then $H(\gamma,\pi)=0$ is the Hamiltonian constraint and $J(\gamma,\pi)=0$ is the momentum constraint of GR. Further, for a vectorfield Z on V, let $X_\lambda=i_\lambda^*(Z_{//})$ and $N_\lambda=i_\lambda^*(Z_\perp)$, where $//$ denotes the projection onto $Ti_\lambda(M)$ and $\perp$ denotes the projection onto $(Ti_\lambda(M))^\perp$. This gives a natural correspondence between curves (N_λ,X_λ) and elements of $Lie(Diff_V)$. If we put $Z=\partial i_\lambda/\partial\lambda$, then N_λ and X_λ are the lapse and shift of the curve of embeddings i_λ.

The operation of passing from 4-dimensional data g to 3+1 dimensional data $(\gamma_\lambda,\pi_\lambda)$, i_λ is called 3+1-ing and takes the Lagrangian form of Einsteins equations into the Hamiltonian form.

Each slicing $i:\mathbb{R}\times M\to V$ induces a map which takes elements of $Lie(Diff_V)$ into curves in $\mathbf{F}\times\Gamma(TM)$, where $\mathbf{F}$ denotes the space of functions on M. Let $\Phi:T^*\mathbf{M}\to\Lambda_d^0\times\Lambda_d^1$, where Λ_d^0 and Λ_d^1 denote the function and 1-form densities on M, respectively, be given by

$$\Phi(\gamma,\pi)=(H(\gamma,\pi),J(\gamma,\pi)).$$

By the above, there is a natural pairing between $\mathbf{F}\times\Gamma(TM)$ and $\Lambda_d^0\times\Lambda_d^1$, so we can consider Φ as a map from the phase space $T^*\mathbf{M}$ of GR to the 3+1 version of the dual of $Lie(Diff_V)$, the Lie algebra of the gauge group of GR, i.e. Φ is a momentum mapping for GR. This point of view has been developed in the work of J.E. Marsden and others[11].

In the noncompact case, for data admitting nontrivial solutions with finite energy, one finds that $\pi\in L^2$ while it is not the case that $\gamma-e\in L^2$. Therefore it is not natural in this case to use the weak L^2 metric which is commonly used in the spatially compact case in decompositions and to define the almost complex structure on $T^*\mathbf{M}$. Instead we will use the Riemannian structure defined by

$$\langle\langle(h_1,\omega_1),(h_2,\omega_2)\rangle\rangle_E=\int_M(C_\gamma h_1)h_2\mu(\gamma)+\int_M(C_\gamma^{-1}\omega'_1)\omega'_2\mu(\gamma) \tag{1.1}$$

(here $C_\gamma=(-\Delta_\gamma)^{1/2}$ and contractions are w.r.t. γ). A metric analogous to that defined in (1.1) was used in [14, p. 313] and [1, §3.2] in the study of the quantization of the Klein-Gordon field.

It can be shown that the asymptotic contribution of the gravitational field to the energy can be expressed in terms of the L^2 norm of $\nabla\gamma_{TT}$ and π_{TT}. This leads one to introduce the 'energy-metric'

$$\ll(h_1,\omega_1),(h_2,\omega_2)\gg=\int_M \nabla h_1\nabla h_2+\int_M \omega_1\omega_2$$

which was used in [9]. This metric does not appear to the natural choice in the present case, however.

Under the present asymptotic conditions (1.1) gives a nondegenerate weak C^∞ Riemannian structure which is invariant under the action of $Diff\,(M)$. In the following we will, unless otherwise stated work only with the $\ll\,,\gg_E$ Riemannian structure defined in (1.1) and statements about orthogonality etc. will refer to this.

Let $\mathrm{J}{:}T(T^*\mathrm{M})\rightarrow T(T^*\mathrm{M})$ denote the almost complex structure on $T(T^*\mathrm{M})$ defined by

$$\ll\mathrm{J}(h_1,\omega_1),(h_2,\omega_2)\gg_E=\Omega((h_1,\omega_1),(h_2,\omega_2)),$$

where Ω denotes the canonical symplectic structure on $T^*\mathrm{M}$, i.e.

$$\Omega((h_1,\omega_1),(h_2,\omega_2))=\int_M (h_1\omega_2-h_2\omega_1),$$

the standard L^2 symplectic structure. Here the pairing is the natural duality pairing between $S^2(TM)$ and $S^2_d(T^*M)$. Note that for the finite energy asymptotic conditions, the symplectic form *fails* to be well defined on the whole of $T^*\mathrm{M}$. In the present case it can be checked that Ω is in fact well defined on $T^*\mathrm{M}$. It would be interesting to know whether this is also the case for other weaker asymptotic conditions admitting well defined angular momentum.

The complex structure defined above is of the form

$$\mathrm{J}(h,\omega)=\begin{bmatrix}-(C_\gamma^{-1}\,\omega)^\flat/\mu(\gamma)\\ (C_\gamma h)^\#\,\mu(\gamma)\end{bmatrix}$$

(cf. [14, (1.14)]). It is important to note the difference from the complex structure defined w.r.t. the L^2 Riemannian structure [11].

One finds that for $(\gamma,\pi)\in T^*\mathrm{M}^{s+1}$, the generator of the dynamics, $\mathrm{J}\circ D\,\Phi_E^*$ is a map

$$\mathrm{J}\circ D\,\Phi_E^*{:}Lie\,(Diff_P^{\,s+1})\rightarrow T_{(\gamma,\pi)}T^*\mathrm{M}^s.$$

The ADM form of the evolution equations for GR is

$$\frac{\partial}{\partial\lambda}\begin{bmatrix}\gamma\\ \pi\end{bmatrix}=-\mathrm{J}\circ D\,\Phi_E^*(\gamma,\pi)\begin{bmatrix}N\\ X\end{bmatrix}.$$

Note here that we are using the $\ll,\gg_E$ Riemannian structure, so $D\,\Phi_E^*$ does not have the usual form, even though $J \circ D\Phi_E^*$ does. By definition,

$$\langle\langle D\Phi_E^*(N,X),(h,\omega)\rangle\rangle_E = \langle\langle D\Phi_{L^2}^*(N,X),(h,\omega)\rangle\rangle_{L^2}, \tag{1.3}$$

where $D\Phi_{L^2}^*$ is the form of $D\Phi^*$ given in eg. [11]. The form of $\langle\langle\,,\,\rangle\rangle_E$ implies that $D\Phi_E^*$ in the present case fails to be a differential operator, but is a pseudo-differential operator with injective symbol. In fact, we have that

$$D\Phi_E^* = \begin{bmatrix} C_\gamma^{-1} & 0 \\ 0 & C_\gamma \end{bmatrix} \circ D\Phi_{L^2}^*.$$

Further,

$$\langle\langle D\Phi_E^*(N,X), D\Phi_E^*(N',X')\rangle\rangle_E$$

is *finite* for (N,X) and (N',X') in $Lie(Diff_P)$, in contrast to the situation for $D\Phi_{L^2}^*$. In the following we will use the subindex E to make the context clear.

If we write the equations out using the notation introduced above, we get [6]

$$\frac{\partial}{\partial\lambda}\gamma = 2N[\pi'^{b} - ½(\mathrm{tr}\pi')\gamma] + (L_X\gamma) \tag{1.5.a}$$

and

$$\frac{\partial}{\partial\lambda}\pi = N[-½(\pi':\pi' - ½(\mathrm{tr}\pi')^2)\gamma^\# + 2(\pi'\times\pi' - ½(tr\,\pi')\pi')]\mu(\gamma)$$
$$-[HessN - (\Delta_\gamma N)\gamma - NEin(\gamma)]^\#\mu(\gamma) - L_X\pi. \tag{1.5.b}$$

Let $i:\mathbb{R}\to Emb(M,V)$ be a curve of embeddings with lapse and shift (N_λ,X_λ) and let $(\gamma_\lambda,\pi_\lambda):\mathbb{R}\to C\subset T^*\mathbf{M}$ be a solution to (1.5). Then the (unique) metric $g\in Lor(V)$ satisfying $F_{i_\lambda}(g)=(\gamma_\lambda,\pi_\lambda)$ for all $\lambda\in\mathbb{R}$ is a solution to Einsteins equations on V.

In terms of the quantities (γ,π) the Lagrangian for GR can be written as[11, p. 340]

$$\int_V R(g) = \int_{\mathbb{R}}\int_M (\pi\dot\gamma - \langle\Phi,(N,X)\rangle) d_\lambda + 2((\pi\cdot X - ½(tr\,\pi)X - N^{,i}\mu(\gamma))_{,i} - \frac{\partial}{\partial\lambda}\mathrm{tr}\pi)d_\lambda, \tag{1.6}$$

where $\langle\Phi,(N,X)\rangle = NH + XJ$. The above equation was derived by ADM using slightly different notation [6, p. 235]. It should be noted that (1.6) is an equality, which can be checked by (highly nontrivial) calculations, in contrast to the commonly used form which leaves out the divergence term at the end.

In the case where M is compact or $(N,X)\to 0$ at infinity, the divergence term in (1.6) integrates to zero by Stokes' theorem and can therefore be ignored, in which case the Hamiltonian of GR becomes

$$\mathrm{H} = \langle\langle\Phi,(N,X)\rangle\rangle = \int_M \langle\Phi,(N,X)\rangle. \tag{1.7}$$

1.2. The momentum mapping of GR

It was first noted by Dirac[10] that in order to get the correct evolution equations for GR on $\mathbb{R}^4$ from the Hamilton equations corresponding to the Hamiltonian (1.7) under asymptotic conditions admitting nontrivial solutions to the constraints and with $N \to 1$ at infinity, one must add a certain integral at infinity in order to compensate a term arising from a partial integration, a fact also noted by DeWitt[16]. This term is identical to the mass of the gravitational field in the rest frame defined by the slicing i.

In the study of ADM of the dynamics of GR, the form of the Hamiltonian for GR under the conditions that $(N,X)\to$constant at infinity was derived by an application of the Schwinger variational principle. The Hamiltonian they found corresponds to the momenta of the gravitational field dual to translations and time translations. The ADM momenta were by Regge and Teitelboim[15] noted to correspond to certain terms occurring in the derivation of the evolution equations of GR under the assumption that $(N,X)\to$ constant. Under certain asymptotic conditions RT also derived the correct form of the momenta corresponding to rotations and boosts, using the Noether principle, a method which closely parallels that used by ADM.

There is an important difference between the point of view of ADM and the later authors. Dirac and DeWitt in the case of the mass and RT in the case of the total momentum, view the correct Hamiltonian of GR in the asymptotically flat case as given by the *combination*

$$<(\Phi+E),(N,X)>$$

of the constraints and the surface integrals E defining the momenta. On the other hand, ADM view the momenta as the generator of the evolution arising after imposing the constraints.

We will now make a few remarks concerning the form of the momenta at spatial infinity. Assume that the QI conditions hold and let Z be a vectorfield on V asymptotic to the infinitesimal Poincare' transformation ξ. Then by the analysis in [2], Z is of the form $Z=\xi+Z_{(1)}$, where $Z_{(1)}\to 0$ at infinity (ξ contains an $O(1)$ part which comes from the QI gauge condition, see [2, §2.1]). In terms of a slicing i, Z corresponds to lapse and shift (N,X) such that $(N,X)=(N_\xi,X_\xi)+(N_{(1)},X_{(1)})$, where $(N_{(1)},X_{(1)})\to 0$ at infinity and (N_ξ,X_ξ) is the 3+1 version of ξ.

Now consider the process of deriving the evolution equations from the Hamiltonian (1.7). RT [15, Eq. (5.13)] computed the terms occurring in the partial integrations necessary for doing this. Let $(\gamma,\pi)\in T^*\mathrm{M}$ and let $(h,\omega)\in T_{(\gamma,\pi)}T^*\mathrm{M}$. In the present notation and under general asymptotic conditions we have (cf. [9, §7])

$$\begin{aligned} d\ll\Phi(\gamma,\pi),(N,X)\gg.(h,\omega) &= \ll D\,\Phi^*(\gamma,\pi).(N,X),(h,\omega)\gg_E \\ &\quad -\int_M \delta_\gamma(N(\delta_\gamma h+d\,\mathrm{tr}h))\mu(\gamma) \\ &\quad -\int_M \delta_\gamma(hdN-(\mathrm{tr}h)dN)\mu(\gamma) \end{aligned}$$

$$-\int_M \delta_\gamma (2X\omega + (2X^k\pi^{jl} - X^l\pi^{jk})h_{jk}).$$

By using the asymptotic conditions and the constraint equations we can get rid of some of the terms in the expression above. The nontrivial surface quantities corresponding to X are

$$-2\oint X\omega \mathrm{d}S \tag{1.8.a}$$

and corresponding to N are

$$-\oint N(\delta h - \mathrm{dtr}h)\mathrm{d}S - \oint N_{,l}\gamma^{ij}\gamma^{kl}h_{ik}\mathrm{d}S_j + \oint N_{,l}\gamma^{ij}\gamma^{kl}h_{ij}\mathrm{d}S_k. \tag{1.8.b}$$

Let E denote the collection of surface integrals defining the momenta and let Φ_E denote the combination $\Phi + E$. Then if we define

$$\ll J_E(\gamma,\pi), X \gg = 2\int_M (L_X\gamma)\pi \tag{1.9.a}$$

and

$$\begin{aligned}\ll H_E(\gamma,\pi), N \gg = \int_M NH(\gamma,\pi) + \oint_\infty \{ & N\gamma^{ij}\gamma^{kl}(\gamma_{ik,j} - \gamma_{ij,k}) - N_{,j}\gamma^{ij}\gamma^{kl}(\gamma_{ik} - \delta_{ik}) \\ & + N_{,k}\gamma^{ij}\gamma^{kl}(\gamma_{ij} - \delta_{ij})\}\mathrm{d}S^i,\end{aligned} \tag{1.9.b}$$

the variations of (1.9) give (1.8). The expression (1.9.b) for the momentum w.r.t. N which differs from that given by RT, was given by Beig and O-Murchadha[7, pp. 476,477, Appendix C] who also showed that under the present asymptotic conditions (see [2, Definition 2.3]) Φ_E is a smooth map

$$\Phi_E : T^*\mathrm{M} \to Lie^*_{3+1}(\mathbb{D})$$

where $Lie^*_{3+1}(\mathbb{D})$ denotes the 3+1 version of the dual of the Lie algebra to $\mathbb{D}$, i.e. the dual of the 'space of lapses and shifts' satisfying the correct asymptotic conditions, in this case given by [2, Theorem 2.2].

Remark 1.1:

1) The asymptotic conditions used by Regge and Teitelboim assumed two degrees of radial smoothness. This has the consequence that the boost momentum simplifies to the ordinary ADM form, i.e. the terms in (1.9.b) depending on dN vanish and (1.8.b) becomes

$$-\oint_\infty N(\delta h - \mathrm{dtr}h)\mathrm{d}S.$$

The results in this paper are essentially unchanged by the assumption of 2 degrees of radial smoothness.

2) We remark here that most parts of the expression (1.9) for the momenta given above can be found in the 3+1 form (1.6) of the Lagrangian.

□

From the explicit form of Φ_E it is clear that we should view Φ_E as a mapping

$$\Phi_E : T^* \mathbf{M}^{s+1} \rightarrow \Lambda_d^0 \times \Lambda_d^1 \times \mathbf{p}^* .$$

Here we have used the natural correspondence between asymptotically Poincare' vectorfields in the 3+1 and the 4-dimensional picture to identify E with an element in $\mathbf{p}^*$. However, it is important to note that off $\mathbf{C}=\Phi^{-1}(0)$, the surface integrals in Φ_E *diverge* even though the whole of Φ_E is well defined. Therefore, the identification of $Lie^*_{3+1}(\mathbf{D})$ with $\Lambda_d^0 \times \Lambda_d^1 \times \mathbf{p}^*$, while conceptually useful does not hold in any strict sense on the whole of $T^* \mathbf{M}$ but only on $\mathbf{C}$.

If we define $D\Phi_E^*$ by requiring that

$$\ll D\Phi_E^*(\gamma,\pi).(N,X),(h,\omega)\gg_E = \ll (N,X), D\Phi_E(\gamma,\pi).(h,\omega)\gg, \tag{1.10}$$

for any $(N,X) \in Lie_{\mathbf{D}}$ and $(h,\omega) \in T_{(\gamma,\pi)} T^* \mathbf{M}$, then $D\Phi_E^*$ coincides with the $D\Phi_E^*$ occurring in (1.3).

1.3. Statement of the Main Result.

As stated above, the aim of this paper is to complete the analysis started in [2] of the structure of the level sets $\Phi_E^{-1}(0\times\xi)$ and the reduced sets $\Phi_E^{-1}(0\times\xi)/\mathbf{D}_\xi$. With the above assumptions and notations, let $g \in Ein_{QI}(V)$ and let (γ,π) be data on M corresponding to g w.r.t. a slicing i.

Theorem 1.2(The Level Sets):

1) Assume that g is nonflat. The level set $\Phi_E^{-1}(0\times\xi)$ is a manifold at (γ,π) if and only if g has no nontrivial Killing fields. The singularities which occur are non-trivial and of conical type.

2) If g is flat then $I_g = P$, the Poincare' group. In this case, the momenta are all zero. The second order term giving the conical singularity is in fact weakly nondegenerate when restricted to a slice and the level set is equal to the orbit of g, i.e.

$$\Phi_E^{-1}(0\times 0) = \mathbf{O}(g) = \{\textit{all flat metrics}\}$$

□

Part 2) of the Theorem follows from the Positive Mass Theorem and computations similar to those in [9, §7] and will not be dealt with in this paper.

Acknowledgements

The technique used for the analysis is entirely due to Arms, Marsden and Moncrieff[4] and the present work should be seen as a step in the programme started in [12]. The author is grateful to Jerry Marsden for suggesting the problem which led to this work and for his hospitality during a visit to Berkeley where the work was started.

2. Structure of the Level Sets

In [4] a general analysis of the zero sets of momentum maps was presented. In this section we recall the main steps of the method used in [4]. In [5] these techniques were applied to analyze the structure of the space of solutions to the Einstein equations in the spatially compact case. In the case studied in [5] the interesting level set was $\Phi^{-1}(0)$. In this paper we are interested in the sets $\mathrm{C}_\xi=\Phi_E^{-1}(0\times\xi)$, i.e. the method used to study the zero set does not apply immediately. However, the following simple trick allows one to get from the general case to the case of the zero set. See [13, §6] for details.

Let (N,ω) be a symplectic manifold and let the group G act on N with a Hamiltonian action admitting a momentum mapping $\phi:N\rightarrow\mathbf{g}^*$ where $\mathbf{g}^*=Lie^*(G)$. For $\xi\in\mathbf{g}^*$, let O_ξ denote the orbit of ξ under the coadjoint action of G and let $\mathrm{C}_{O_\xi}=\phi^{-1}(O_\xi)$. Clearly C_{O_ξ} is invariant under the action of G.

Recall that the orbit O_ξ inherits a symplectic structure from the Poisson structure on $\mathbf{g}^*$. Now consider the symplectic manifold defined by $N\times O_\xi$ with O_ξ given minus its standard symplectic structure. Then G acts on $N\times O_\xi$ with momentum mapping ϕ_ξ given by

$$\phi_\xi(x,\mu)=\phi(x)-\mu.$$

Now the important fact to note is that $\phi_\xi=0$ implies $\mu=\phi(x)$ so we can make the identification

$$\phi_\xi^{-1}(0)=C_{O_\xi}$$

Further, $C_{O_\xi}=C_\xi\times O_\xi$ locally so we can analyze the structure of C_ξ by applying the techniques of [5] to the zero set of ϕ_ξ. Let G_ξ be the isotropy group of ξ under the coadjoint action of G. Then C_ξ is invariant under the action of G_ξ and it is clear from the above that

$$C_\xi/G_\xi=\phi_\xi^{-1}(0)/G.$$

Thus C_{O_ξ}/G is the Marsden-Weinstein reduction w.r.t. ξ.

Back to GR. In this section we will consider the structure of the level sets $\mathrm{C}_\xi=\Phi_E^{-1}(0\times\xi)$ and $\mathrm{C}_{O_\xi}=\Phi_E^{-1}(0\times O_\xi)$ where O_ξ denotes the orbit of ξ under the coadjoint action of the Poincare group. Let $\mathbf{D}$ denote the group $Diff_P=Diff_I\circledS P$. Here C_ξ should be interpreted as the set

$$\mathrm{C}_\xi=\{(\gamma,\pi)\in T^*\mathrm{M} \mid \Phi(\gamma,\pi)=0 \text{ and } E(\gamma,\pi)=\xi\}$$

and C_{O_ξ} similarly. Note that $\Phi(\gamma,\pi)=0$ implies that $E(\gamma,\pi)$ is well defined. For $\xi\in\mathbf{p}^*$, we let P_ξ and $\mathbf{D}_\xi$ denote the isotropy group of ξ by the coadjoint action of P and the group $Diff_I\circledS P_\xi$, respectively. Clearly, the action of $\mathbf{D}_\xi$ leaves the set $\mathrm{C}_\xi=\Phi_E^{-1}(0\times\xi)$ invariant.

As above, we give $T^*\mathrm{M}\times O_\xi$ the product of the canonical symplectic structure on $T^*\mathrm{M}$ and minus the symplectic structure of O_ξ as above. In the following, we will denote $T^*\mathrm{M}\times O_\xi$ with this symplectic structure by $T^*\mathrm{M}_\xi$ and for simplicity we will sometimes denote points $((\gamma,\pi),\mu)\in T^*\mathrm{M}_\xi$ by x. Now define the momentum mapping $\Phi_{E,\xi}:T^*\mathrm{M}_\xi\to\Lambda_d^0\times\Lambda_d^1\times\mathbf{p}^*$ by $\Phi_{E,\xi}((\gamma,\pi),\mu)=\Phi_E-\mu$. Similarly to (1.10), we have that at any $x\in T^*\mathrm{M}_\xi$

$$\ll D\,\Phi_{E,\xi}^*(N,X),z\gg_E=\ll(N,X),D\,\Phi_{E,\xi}.z\gg$$

for any $(N,X)\in Lie(\mathbb{D})$ and sufficiently nice $z\in T(T^*\mathrm{M}_\xi)$. Formally, $D\,\Phi_{E,\xi}^*$ is a map from $Lie(\mathbb{D})$ to $T(T^*\mathrm{M}_\xi)$. On O_ξ we choose an almost complex structure $\mathbb{I}$, tamed by minus the standard symplectic structure. Here we use $\ll\,,\gg_E$ to denote the weak Riemannian structure on $T^*\mathrm{M}_\xi$ defined by the $\ll\,,\gg_E$-metric on $T^*\mathrm{M}$ and the metric on O_ξ defined w.r.t. $\mathbb{I}$. One should think of $D\,\Phi_{E,\xi}^*$ as the mapping

$$(N,X)\to(D\,\Phi_E^*(N,X)\;,\;(N,X)_{\mathbf{p}^*})\in T(T^*\mathrm{M}_\xi)=T(T^*\mathrm{M})\times TO_\xi,$$

where $(N,X)_{\mathbf{p}^*}$ denotes the uniquely defined representative of (N,X) in TO_ξ defined by the $O(r)$ and $O(1)$ parts. Define a complex structure $\mathbb{J}_\xi$ on $T^*\mathrm{M}_\xi$ by $\mathbb{J}_\xi=\mathbb{J}\otimes\mathbb{I}$. Then the action of $\mathbb{D}$ on $T^*\mathrm{M}_\xi$ is generated by

$$\mathbb{J}_\xi\circ D\,\Phi_{E,\xi}^*(x).(N,X)\in T(T^*\mathrm{M}_\xi). \tag{2.1}$$

2.1. The Constraint set $\mathbb{C}=\Phi^{-1}(0)$

It is known that in the case where $M=\mathbb{R}^3$, the constraint set $\mathbb{C}=\Phi^{-1}(0)$ is a manifold. The proof of this in [8] (see also [9]) assumes that $\mathrm{tr}\pi=0$. This means that we will have to assume the global existence of maximal slicings. See [2,§ 2.3] for comments on this. The assumption of radial smoothness used here does not cause any additional difficulties in the proof. Given that $\mathrm{tr}\pi=0$, we have that $D\,\Phi$ is surjective as a mapping

$$D\,\Phi:T(T^*\mathrm{M}^s)\to RT^{s-2}_{3,e,\perp}(\Lambda_d^0\times\Lambda_d^1).$$

By a standard argument one gets

Theorem 2.1: The constraint set $\mathbb{C}=\Phi^{-1}(0)$ is a manifold.

□

2.2. Second Order Conditions

In contrast to the present case, in the spatially compact case, the constraint set has conical singularities at (γ,π) corresponding to metrics with nontrivial isometries. See [5] for a proof of this. The result extends to the case of relativity coupled to Yang-Mills and other matter fields. We will here show that $\mathbb{C}_{O_\xi}$ is smooth only at (γ,π) corresponding to metrics with trivial isotropy group.

Let $g\in Ein_{QI}$ be a solution to Einsteins equations which has nontrivial killing fields. Let $i\in\Sigma_{QI}$ be a slicing of V and let (N,X) be the lapse and shift corresponding to a Killing field ${}^{(4)}X\in Lie(I_g)$, the Lie algebra of the isometry group of g. If we let $\xi\in\mathbf{p}^*$ denote the momentum of g, then $I_g\subset\mathbb{D}_\xi$, since any element in I_g leaves g and hence ξ invariant. Thus if we let $x\in\Phi_{E,\xi}^{-1}(0)\subset T^*\mathrm{M}_\xi$ be a point corresponding to g, it is clear from (2.1) that $(N,X)\in\ker D\,\Phi_{E,\xi}^*$, since ${}^{(4)}X$ generates trivial dynamics.

Now, $(N,X)\in\ker D\,\Phi_E^*$ implies by (1.10) that $D\,\Phi_E(\gamma,\pi)$ is not surjective. A similar argument gives that $D\,\Phi_{E,\xi}(x)$ is not surjective. Recall that the linearization stability of the constraint set $\mathbb{C}$ is due to the fact that $D\,\Phi$ is surjective as a map onto $\Lambda_d^0\times\Lambda_d^1$, which might seem to contradict the statement above. However, the meaning of the statement that $D\,\Phi_E$ is not surjective is that in the case that (γ,π) corresponds to a metric g with nontrivial isotropy group, there is a correlation between the behaviour of the $D\,\Phi$ part and the boundary term in $D\,\Phi_E$.

We will now note the first consequence of the non-surjectivity of $D\,\Phi_{E,\xi}$. Let x_τ be a curve in $\Phi_{E,\xi}^{-1}(0)$ starting at $x=((\gamma,\pi),\xi)$ with $\frac{\partial}{\partial\tau}x_\tau|_{\tau=0}=\dot{x}$ and $\frac{\partial^2}{\partial\tau^2}x_\tau|_{\tau=0}=\ddot{x}$. The condition that x_τ is a curve in $\Phi_{E,\xi}^{-1}(0)$ gives

$$\frac{\partial}{\partial\tau}\Phi_{E,\xi}(x_\tau)|_{\tau=0}=D\,\Phi_{E,\xi}(x)\dot{x}=0 \tag{2.2}$$

Differentiating this expression w.r.t. τ gives

$$\frac{\partial^2}{\partial\tau^2}\Phi_{E,\xi}(x)|_{\tau=0}=D^2\Phi_{E,\xi}(x).(\dot{x},\dot{x})+D\,\Phi_{E,\xi}(x).\ddot{x}=0.$$

Pairing this expression with (N,X) and using the nonsurjectivity of $D\,\Phi_{E,\xi}$ gives by using the identification between $\mathbb{C}_{O_\xi}$ and $\Phi_{E,\xi}^{-1}(0)$ the following Lemma.

Lemma 2.2: Let $(\gamma,\pi)\in\mathbb{C}_{O_\xi}$ be data corresponding to a metric g with a nontrivial Killing field ${}^{(4)}X$ and let (N,X) be lapse and shift corresponding to ${}^{(4)}X$. Let (γ_τ,π_τ) be a curve in $\mathbb{C}_{O_\xi}$ starting at (γ,π) as above. Then

$$<(D^2\Phi_E(\gamma,\pi)(\dot{\gamma},\dot{\pi})(\dot{\gamma},\dot{\pi}),(N,X)>=0 \tag{2.3}$$

The second order condition on $(\dot{\gamma},\dot{\pi})$ defined by (2.3) is nontrivial.

□

The only part that is not clear from the above is the nontriviality of the second order condition. The proof of this in [3] works also for the present case with $M=\mathbb{R}^3$, see also [12, 4.6]. Thus we see that at (γ,π) corresponding to metrics with nontrivial Killing fields, the zero set $\mathbb{C}_{O_\xi}$ to $\Phi_{E,\xi}$ is not a manifold. It follows from this that also the level set $\mathbb{C}_\xi$ fails to be a manifold. In fact we have, similarly to the spatially compact case, the following result.

Theorem 2.3: The set C_ξ is a manifold near (γ,π) if and only if (γ,π) corresponds to a metric with no Killing fields.

Proof: What remains to check is that $\mathbb{C}_\xi$ is a manifold near (γ,π) corresponding to a metric g with no Killing fields. In particular we are assuming that g is nonflat. Consider the operator

$$A_{(\gamma,\pi)}=D\,\Phi_E(\gamma,\pi)\circ D\,\Phi_E^*(\gamma,\pi).$$

This is formally a mapping $A: Lie(\mathbb{D}) - Lie^*(\mathbb{D})$. W.r.t. the natural pairing we have that

$$\ll A.(N,X),(N',X')\gg = \ll D\Phi_E^*.(N,X), D\Phi_E^*.(N',X')\gg_E ,$$

so A is an isomorphism if and only if $\ker(D\Phi_E^*)=0$, i.e. when g lacks Killing fields. The result follows.

□

3. The Moncrieff Decomposition

To further analyze the local structure of $\mathbb{C}_{O_\xi}$ near a singular point we introduce, following [12] the Moncrieff decomposition.

The operator $D\Phi_E^*$ has an injective symbol and hence by the finite dimensionality of O_ξ, the ranges of $D\Phi_{E,\xi}^*$ and $\mathbb{J}_\xi \circ D\Phi_{E,\xi}^*$ are closed and splitting. Let $x=((\gamma,\pi),\mu)\in T^*\mathbb{M}_\xi$. We have the following two orthogonal decompositions of $T_xT^*\mathbb{M}_\xi$.

$$T(T^*\mathbb{M}_\xi)=R(D\Phi_{E,\xi}^*)\oplus\ker(D\Phi_{E,\xi}) \tag{3.1.a}$$

and

$$T(T^*\mathbb{M}_\xi)=R(\mathbb{J}_\xi\circ D\Phi_{E,\xi}^*)\oplus\ker(D\Phi_{E,\xi}\circ\mathbb{J}_\xi) \tag{3.1.b}$$

To see that the decompositions are orthogonal one uses the duality of $D\Phi_{E,\xi}^*$ to $D\Phi_{E,\xi}$.

By intersecting the splittings (3.1) and noting that the term $R(D\Phi_{E,\xi}^*)\cap R(\mathbb{J}_\xi\circ D\Phi_{E,\xi}^*)$ vanishes because $\mathbb{J}_\xi\circ D\Phi_{E,\xi}^*$ is the generator of the action of $\mathbb{D}$ and therefore takes values in $T(\Phi_{E,\xi}^{-1}(0))=\ker(D\Phi_{E,\xi})$, we get

Proposition 3.1: Let $x\in\Phi_{E,\xi}^{-1}(0)$. The tangent space $T_xT^*\mathbb{M}_\xi$ splits orthogonally as

$$T_x(T^*\mathbb{M}_\xi)=R(D\Phi_{E,\xi}^*)\oplus R(\mathbb{J}_\xi\circ D\Phi_{E,\xi}^*)\oplus\left[\ker(D\Phi_{E,\xi}\circ\mathbb{J}_\xi)\cap\ker(D\Phi_{E,\xi})\right].$$

□

The meaning of the various terms in the decomposition is the following: The first term is the orthogonal complement to $T(\Phi_{E,\xi}^{-1}(0))\cong T\mathbb{C}_{O_\xi}$, the second term is the tangent space to the orbit of $\mathbb{D}$ acting on x and the third term is the model for the quotient space $\mathbb{C}_{O_\xi}/\mathbb{D}$.

4. The Kuranishi Map

The tool which makes it possible to analyze the structure of the level sets near a singularity is the Kuranishi map. In [4] a general analysis of the level sets of momentum mappings was presented, which used the Kuranishi map in an essential way. This was in [5] applied to the constraint set for Einsteins equations in the spatially compact case. The presentation here follows closely that in [5]

Let $x_0=((\gamma_0,\pi_0),\mu_0)\in\Phi_{E,\xi}^{-1}(0)$. The operator $D\Phi$ is the formal adjoint to $D\Phi^*$ which has an injective symbol. Hence, the operator $D\Phi_E\circ D\Phi_E^*$ is elliptic and under the present asymptotic conditions Fredholm. By the finite dimensionality of O_ξ we find therefore that $D\Phi_{E,\xi}$ has closed range and splitting kernel and there is a pseudo inverse A_{x_0} to $D\Phi_{E,\xi}(x_0)$. The range of A_{x_0} is the orthogonal complement to $\ker(D\Phi_{E,\xi})$, i.e. by (3.1.a), $R(A_{x_0})=R(D\Phi^*_{E,\xi}x_0)$.

Let $h=x-x_0$. We define a remainder term $R_\xi(h)$ by

$$R_\xi(h)=\Phi_{E,\xi}(x)-D\Phi_{E,\xi}(x_0).h$$

Note that we have defined the remainder term $R_\xi(h)$ so that $R_\xi(0)=0$ and $DR_\xi(0)=0$. The Kuranishi map F_ξ can now be defined by

$$F_\xi(x)=x+A_{x_0}\circ R_\xi(h)$$

Then, $DF_\xi|_{x_0}=I+A_{x_0}\circ DR_\xi(0)=I$, so F_ξ is a diffeomorphism on a neighbourhood of x_0 in $T^*\mathrm{M}_\xi^{s+1}$.

A slice for the action of $\mathbb{D}$ is given by the set $S_{x_0}=\{x_0\}+U$ where U is a neighbourhood of 0 in $\ker D\Phi_{E,\xi}\circ \mathrm{J}_\xi$. Since $R(A_{x_0})=R(D\Phi^*_{E,\xi})$, it is clear from inspection that

$$F(x)=x+A_{x_0}\circ R(h)\in S_{x_0}.$$

Thus we have

Proposition 4.1: F_ξ maps S_{x_0} to itself.

□

Let $\mathbb{P}$ be the projection onto $R(D\Phi_{E,\xi}(x_0))$. The projected constraint set $\mathrm{C}_{\mathbb{P},\xi}$ is defined by

$$\mathrm{C}_{\mathbb{P},\xi}=\{(x)\,|\,\mathbb{P}(\Phi_{E,\xi}(x))=0\}.$$

This is a smooth manifold in a neighborhood of x_0 with tangent space given by $\ker D\Phi_E(x_0)$. We now prove that F is a local chart for $\mathrm{C}_{\mathbb{P},\xi}$.

Proposition 4.2: F maps a neighborhood of x_0 in $\mathrm{C}_{\mathbb{P},\xi}$ diffeomorphically onto a neighborhood of x_0 in $\{x_0\}+\ker D\Phi_{E,\xi}(x_0)$.

Proof: We have already noted that F_ξ is a diffeomorphism near x_0. Hence all we need to check is that for $x\in \mathrm{C}_{\mathbb{P},\xi}$, $F_\xi(x)\in\{x_0\}+\ker D\Phi_E(x_0)$. To see this we write

$$\begin{aligned}&D\Phi_{E,\xi}(x_0)\circ[F(x)-x_0]\\&\quad=D\Phi_{E,\xi}(x_0)h+\mathbb{P}\circ R_\xi(h)\\&\quad=\mathbb{P}[\Phi_{E,\xi}]=0.\end{aligned}$$

□

The following result corresponds to [5, Proposition 1.4].

Proposition 4.3: F_ξ is a local symplectic diffeomorphism of $\mathrm{C}_{\mathbb{P},\xi}\cap S_{x_0}$ to

$$\{x_0\}+[\ker D\Phi_{E,\xi}(x_0)\cap\ker D\Phi_{E,\xi}(x_0)\circ \mathrm{J}_\xi]. \tag{4.1}$$

Proof: It is clear from the above that F_ξ maps $\mathbf{C}_{\mathbf{P},\xi}$ to $\{x_0\}+\ker D\Phi_{E,\xi}(x_0)$ and S_{x_0} to itself. If we write the intersection of these spaces explicitely, we get (4.1). It is clear from §3 that (4.1) is a symplectic space. Therefore what remains to check is that F_ξ is a symplectic mapping.

Let Ω_ξ be the symplectic form on $T^*\mathbf{M}_\xi$. Then by definition, $\Omega_\xi(h_1,h_2)=\ll \mathbf{J}_\xi h_1,h_2 \gg$ and a calculation entirely similar to that in [5, p. 87] shows that F_ξ is symplectic.

□

The idea to complete the analysis of the constraint set $\mathbf{C}_{O_\xi}$ is to work by analogy with the Liapunov-Schmidt procedure and use the finite dimensional part $(I-\mathbf{P})(\Phi_{E,\xi})$ to study the second order conditions near x_0.

Recall that if g is not flat, then any Killing field is spacelike and there is a maximal slicing i of V which is invariant under the action of I_g. In terms of such a slicing, any element of $\ker D\Phi_E^*$ can be written on the form $(0,X)$. Let the dimension of $\ker D\Phi_E^*$ be l and let $\{X_i\}_{i=1}^{l}$ be a basis.

For $(\gamma,\pi)\in\mathbf{C}_\xi$ corresponding to $g\in Ein_\xi$, we have that $<[{}^{(4)}X],\xi>=0$, where $[{}^{(4)}X]$ denotes the Poincare' part of ${}^{(4)}X\in I_g$ and $<,>$ denotes the pairing between $\mathfrak{p}$ and $\mathfrak{p}^*$. In case g is flat, then $\xi=0$. Otherwise, in an appropriately chosen maximal slicing, ${}^{(4)}X$ corresponds to $(0,X)$ where X satisfies $L_X\gamma=0$ and $L_X\pi=0$ for (γ,π) corresponding to g and hence $\ll J_{E,\xi},X\gg=0$.

Define a mapping $T^*\mathbf{M}_\xi\to\mathbb{R}^l$ by

$$\mathbf{P}_J\Phi_{E,\xi}(\gamma,\pi)=\left[<X_1,J_{E,\xi}(\gamma,\pi)>,\cdots,<X_l,J_{E,\xi}(\gamma,\pi)>\right].$$

Then $\mathbf{P}_J$ corresponds to $\mathbf{P}_J$ defined in [5, p. 89]. By using the explicit form of $\Phi_{E,\xi}$ and the fact that X Killing implies that X annihilates O_ξ, we see that

$$\mathbf{P_J}\Phi_{E,\xi}=\mathbf{P_J}\Phi_E.$$

Let $\mathbf{C}_J$ denote the set

$$\mathbf{C}_J=\{x\in T^*\mathbf{M}_\xi \mid P_J\Phi_E(x)=0\}.$$

For $h=((k,\omega),\zeta)\in T_xT^*\mathbf{M}_\xi$, define the quadratic form $Q(h)\in\mathbb{R}^l$ by

$$Q_i(h)=\int_M (L_X)k\,\omega.$$

The following result corresponds to [5, Theorem 2.1].

Theorem 4.4: The Kuranishi map takes the set $\mathbf{C_P}\cap\mathbf{C}_{J_{E,\xi}}\cap S_{x_0}$ to the cone

$$C_{J_{E,\xi}}=\{x_0\}+\left\{h\in\ker D\Phi_{E,\xi}(x_0)\cap\ker(D\Phi_{E,\xi}(x_0)\circ\mathbf{J}_\xi)\ \mid\ Q(h)=0\right\}.$$

Proof: Let $\hat{\mathbf{P}}$ denote the projection onto $\ker(D\Phi_{E,\xi}(x_0))$. Then using the form of F_ξ we have that

$$F_\xi(x)-x_0=\hat{\mathbf{P}}(h)$$

for $x\in\mathbf{C_P}$. It follows that

$$Q\,(F_\xi(x)-x_0)=Q\,(\hat{\mathbf{P}}(h)).$$

Lemma 4.5: $Q\,(\hat{\mathbf{P}}h)=Q\,(h)$.

Proof: Let $\tilde{\mathbf{P}}$ denote the projection onto $R\,(D\,\Phi^*_{E,\xi})$ and let $\tilde{\mathbf{P}}h=\tilde{h}$. Then we can write $\hat{\mathbf{P}}=h-\tilde{h}$ which gives that

$$\begin{aligned}Q_i\,(\hat{\mathbf{P}}h)&=Q_i\,(h-\tilde{h}\,,h-\tilde{h}\,)\\&=\int(L_{X_i}k\,)\omega-\int(L_{X_i}k\,)\tilde{\omega}-\int(L_{X_i}\tilde{k}\,)\omega-\int(L_{X_i}\tilde{k}\,)\tilde{\omega}.\end{aligned}\tag{4.2}$$

From $x\in S_{x_0}$ we see that $\mathbf{J}_\xi h\in\ker D\,\Phi_{E,\xi}$ and from $\tilde{h}\in R\,(D\,\Phi^*_{E,\xi})$ we see that $\mathbf{J}_\xi\tilde{h}\in R\,(\mathbf{J}_\xi\circ D\,\Phi^*_{E,\xi})$, i.e. $\mathbf{J}_\xi\tilde{h}$ is pure gauge (w.r.t. $\mathbb{D}$). By differentiating (2.2) in the direction $\mathbf{J}_\xi h$ (with $\dot{x}=\mathbf{J}_\xi\tilde{h}$), and contracting with X_i we get that

$$\mathbf{P}_J D^2\Phi_{E,\xi}(x_0)(\mathbf{J}_\xi h\,,\mathbf{J}_\xi\tilde{h}\,)=0.\tag{4.3}$$

The O_ξ part of $\Phi_{E,\xi}$ is linear so the O_ξ part in the above expression is trivial. Let $\pi_{T^*\mathbf{M}}$ denote the projection from $T^*\,\mathbf{M}_\xi$ to $T^*\,\mathbf{M}$. Then

$$\pi_{T^*\mathbf{M}}\ker(D\,\Phi_{E,\xi})=D\,\Phi_E^{-1}(0\times TO_\xi)$$

and

$$\pi_{T^*\mathbf{M}}R\,(D\,\Phi^*_{E,\xi})=R\,(D\,\Phi^*_E).$$

On $T^*\,\mathbf{M}$, (4.3) reads

$$0=\ll X_i\,,D^2J_E\,(\gamma_0,\pi_0)(\mathbf{J}(k\,,\omega),\mathbf{J}(\tilde{k}\,,\tilde{\omega}))\gg$$

or, using the expression (1.9.a) for $\int XJ_E$,

$$\int(L_{X_i}(C_\gamma^{-1}\,\omega)^b/\mu(\gamma))((C_\gamma\tilde{k}\,)^\#\,\mu(\gamma))+\int(L_{X_i}(C_\gamma^{-1}\,\tilde{\omega})^b/\mu(\gamma))(C_\gamma k\,)^\#\,\mu(\gamma)=0\tag{4.4}$$

By assumption, X_i is a Killing field for γ so L_{X_i} commutes with C_γ^{-1}, $\mu(\gamma)$ and the operation of lowering indices. The C_γ^{-1} and C_γ terms cancel, so (4.4) reads

$$\int(L_{X_i}\omega)\tilde{k}+\int(L_{X_i}\tilde{\omega})k=0\tag{4.5}$$

We partially integrate the first term in (4.5) which gives

$$\int_M(L_{X_i}\omega)\tilde{k}=\oint(X\,\omega)\tilde{k}\,\mathrm{d}S-\int_M(L_{X_i}\tilde{k}\,)\omega.\tag{4.6}$$

Now, $(X\,\omega)\tilde{k}$ is $O\,(1/r^2)$ and even, so the surface term in (4.6) vanishes. Thus we find from (4.5) that

$$\int(L_{X_i}k\,)\tilde{\omega}+\int(L_{X_i}\tilde{k}\,)\omega=0$$

Applying the same argument with k replaced by $\tilde{k}$ brings (4.2) to the form $\int_M(L_{X_i}k\,)\omega$

which proves the Lemma.

□

The following result is easily proved in the same way as [5, Lemma 2.3.]:

Lemma 4.6: $\mathbf{P_J}\Phi_{E,\xi}(x)=Q(h)$.

□

We have now proved that

$$Q(F_\xi(x)-x_0)=\mathbf{P_J}\Phi_{E,\xi}(x)$$

which completes the proof of Theorem 4.4.

□

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