ON RADIATIVE SPACE - TIMES

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The Regular Conformal Field Equations

The general object of the work I want to talk about, is the analysis of the "conformal structure of Einstein equations" and the investigation of its consequences for the conformal or, equivalently, the causal structure of their solutions.

The motivation for this is the fact, that in classical general relativity some of the most important open problems of the structure of space-times in the large are essentially questions about their causal behaviour. These are concerned with the nature of singularities, the formation of horizons which possibly hide the singularities from distant observers, with the existence of "naked singularities", and, seemingly completely unrelated to these subjects, with the asymptotic behaviour of fields of isolated gravitating systems and of radiative solutions.

To ask concrete questions about the conformal properties of the equations and their solutions I have concentrated on the analysis of the asymptotic behaviour of the solutions, for which Penrose [15,16] made a suggestion how it may possibly be described in terms of the extensibility of the "physical" conformal structure beyond "infinity". In the following I shall discuss some resent progress and point out the decisive open problem in this field. Since in this article I shall try mainly to present the general ideas, I refer the reader to the literature for details.

By the "conformal structure of the field equations" I understand in a loose way the properties of the equations which are obtained from Einstein's equations

$$(1) \qquad \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\,\tilde{g}_{\mu\nu} + \lambda\,\tilde{g}_{\mu\nu} = \kappa\,\tilde{T}_{\mu\nu} ,$$

where λ denotes the cosmological constant, $\tilde{g}_{\mu\nu}$ the "physical metric", $\tilde{T}_{\mu\nu}$

the "physical energy–momentum tensor", and $\kappa = 8\pi G$ with G the gravitational constant, by rewriting them in terms of the "unphysical metric" $g_{\mu\nu}$ which is related to the physical metric by a conformal rescaling

$$(2) \qquad g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$$

with a conformal factor Ω and by expressing the left hand side of the equation by fields derived from $g_{\mu\nu}$ and Ω.

Under the rescaling (2) the Ricci–tensor transforms in four dimensions according to

$$(3) \quad \tilde{R}_{\mu\nu} = R_{\mu\nu} + 2\,\Omega^{-1}\,\nabla_\mu\nabla_\nu\Omega + g_{\mu\nu}\{\Omega^{-1}\nabla_\lambda\nabla_\rho\Omega - 3\,\Omega^{-2}\nabla_\lambda\Omega\nabla_\rho\Omega\}g^{\lambda\rho},$$

which shows formally that Einstein's field equations are not conformally invariant. In Penrose's definition of the conformal boundary at infinity this boundary is characterized by the fact that a suitably chosen conformal factor vanishes there. If we want to study the field equations (1) near conformal infinity by writing them in terms of the rescaled metric (2), we see that by (3) we obtain equations which are singular at the boundary because of the occurrence of the factors Ω^{-k}, $k=1,2$, on the right hand side of (3). Somewhat unexpectedly it turned out, that one can proof a

Regularity theorem [8,13]:

For conformally invariant source fields Einstein's field equations (1) *can be represented by an equivalent system of "regular conformal field equations" for the matter fields, the unphysical metric, the conformal factor, and fields which are derived from these, which is regular and has a principal part which is independent of the conformal factor.*

To simplify the presentation, I shall consider in the following only the source–free case. More general situations, in particular the coupled system of the Einstein–Maxwell–Yang–Mills field equation, have been discussed in detail in [13].

The regular conformal field equations are given by

(4) $$R_{\mu\nu\lambda\rho} = \Omega\, d_{\mu\nu\lambda\rho} + 2\{g_{\mu[\lambda}L_{\rho]\nu} + L_{\mu[\lambda}g_{\rho]\nu}\}$$

(5) $$\nabla_\mu \nabla_\nu\, \Omega = -\,\Omega\, s_{\mu\nu} + s\, g_{\mu\nu}$$

(6) $$\nabla_\mu\, s = -\, s_{\mu\nu}\, \nabla^\nu \Omega - 2\Lambda\, \nabla_\mu \Omega - \Omega\, \nabla_\mu\, \Lambda$$

(7) $$2\, \nabla_{[\lambda} L_{\rho]\nu} = \nabla_\mu\, \Omega\, d^\mu{}_{\nu\lambda\rho}$$

(8) $$\nabla_\mu\, d^\mu{}_{\nu\lambda\rho} = 0$$

(9) $$6\,\Omega s - 3\nabla_\mu \Omega\, \nabla^\mu \Omega + 6\Omega^2 \Lambda = \lambda.$$

Besides the unphysical metric $g_{\mu\nu}$ and the conformal factor Ω the unknowns in these equations are given by

the function	$s = \frac{1}{4}\nabla_\mu \nabla^\mu \Omega,$
the trace free part of the Ricci tensor of $g_{\mu\nu}$	$s_{\mu\nu} = \frac{1}{2}(R_{\mu\nu} - \frac{1}{4}\, R\, g_{\mu\nu})$
the rescaled Weyl tensor	$d^\mu{}_{\nu\lambda\rho} = \Omega^{-1} C^\mu{}_{\nu\lambda\rho}.$

The curvature tensor, which appears on the left hand side of equation (4) is thought here as being given by the metric and its derivatives or, corresponding to the formalism used to obtain "nice" propagation equations, by the connection coefficients and their derivatives. The right hand side of equation (4) represents the decomposition of the curvature tensor into its irreducible parts. Writing $R=24\Lambda$, where R denotes the Ricci scalar, we have $L_{\mu\nu}=s_{\mu\nu}+\Lambda g_{\mu\nu}$. The specification of the real-valued function Λ is considered as a gauge freedom. When the Ricci scalar has been chosen arbitrarily, the conformal factor is determined by the field equations and data on a suitable initial surface. It can be shown that the algebraic equation (9) represents a constraint equation in the sense that it will be satisfied everywhere, if it is satisfied on the initial surface.

The regular conformal field equations (4)–(9) have been derived from the source free field equations (1) under the assumption, that the conformal factor be positive and they in turn imply these equations. Since the sign of the conformal factor does not play any role any longer in the regular conformal field equations they can be thought of as a generalization of Einstein's equation into regimes where the conformal factor vanishes or becomes negative.

Before I discuss properties of the conformal equations, which depend more specifically on the signature and the dimension, it may be worthwhile to point out, that the derivation of the regular conformal field equations does not depend on the signature and may be performed in a similar way for dimensions $n \geq 5$, in which case the variables which have to be introduced and the regular conformal equations which are obtained depend on n. In the positive definite case in dimension four it is posible to derive a regular elliptic system of second order. The three dimensional case is somewhat special, because the Weyl tensor vanishes there and solution of the source free Einstein equations are then just spaces of constant curvature. However the equations implied by Einstein's vacuum field equations (four dimensions, Lorentz signature) in the presence of a static Killing field on a slice of constant Killing time may be written as three dimensional Einstein equations with sources and the asymptotic behaviour of their solutions has been analysed with profit by using conformal methods [3]. These investigations have been extended later to the case of the stationary Einstein equations [4,14].

After the overdetermined system of regular conformal field equations has been obtained, one has to find ways to show the existence of solutions. In the case of a four dimensional Lorentz space, which will be of interest to us in the following, one may try to do this by solving initial value problems for the regular conformal field equations. Decisive for the possibility to do this is the

Reduction theorem [8,13]:

Initial value problems for the regular conformal field equations can be reduced to initial value problems for symmetric hyperbolic systems of propagation equations.

This short statement of a somewhat lengthy operation means the following. Going to a spin frame formalism and using the torsion free condition to obtain equations for the frame field we may express all unknown tensor fields by their components with respect to a suitably chosen spin frame. Taking certain linear combinations of the resulting equations, we can extract a symmetric hyperbolic system of propagation equations for all unknowns. These evolution equations "propagate the constraints", i.e. assuming that we are given a solution of these propagation equations, which satisfies on some Cauchy surface the constraint equations implied by the complete system (4)–(9) on this surface, it can be shown by deriving a certain symmetric hyperbolic system of subsidiary equations, that the solution satisfies the constraints and consequently the complete system of regular conformal field equations everywhere.

In these calculations the two component spin frame formalism turned out to be extremely useful, because it allows to take care of the symmetry properties of the tensor fields involved in a simple way and it shows immediately how to extract the symmetric hyperbolic systems of propagation and subsidiary equations. Moreover, the use of the first order system and of the spinor formalism proves particularly useful in the analysis of the structure of the solutions near the conformal boundary. However, it may be pointed out here, that neither the spinor nor the frame formalism is decisive for the possibility to obtain hyperbolic propagation equations from the system of regular conformal field equations. There are various possibilities to derive, by taking derivatives of equations (4)–(9), propagation equations of wave equation type. An example for this is given in [6]. It is clear from the reduction theorem above, that any such system will propagate the constraints. If an argument for this is wanted which is independent of the result above, one has to go through some complicated and lengthy calculations.

In the reduction process to obtain hyperbolic equations, one has to fix the gauge freedom. This can be done by choosing in a suitable way certain "gauge source functions" which appear in the equations and which may be specified arbitrarily. Once these have been chosen, the gauge dependant quantities, which in the spin frame formalism mentioned above are given by the coordinates, the spin frame, and, as remarked before, by the conformal factor, are determined by the field equations and some initial data for these quantities. In a local situation the particular choice of the gauge source functions is irrelevant. If one wants to show long time existence of solutions and, if possible, keep the same gauge everywhere, a more careful choice of the gauge source functions is needed. To derive the global and semi-global existence results mentioned later, the coordinate gauge source function has been fixed by constructing a harmonic map of the prospective solution space-time onto a suitable comparison solution and the other gauge source functions have then be related to corresponding gauge source functions for a global gauge on the comparison solution.

The Conformal Boundary

On some four dimensional manifold M let be given a smooth Lorentz metric $g_{\mu\nu}$ and a smooth real-valued function Ω such that the set $\mathscr{I} = \{\Omega=0\}$ is not empty. Assume furthermore that $d\Omega\neq 0$ on $\mathscr{I}$ such that $\mathscr{I}$ forms a smooth hypersurface of M and that we are given a time orientation. On the set $\tilde{M} = \{\Omega>0\}$ we can then define by the relation (2) a smooth Lorentz metric $\tilde{g}_{\mu\nu}$. It is then easy to show, that any null geodesic on $\tilde{M}$ with respect to the metric $\tilde{g}_{\mu\nu}$ which has future (say) endpoint on $\mathscr{I}$ is future complete. In this sense the hypersurface $\mathscr{I}$ represents an infinity for the Lorentz space $(\tilde{M},\tilde{g}_{\mu\nu})$. Conversely, given some Lorentz space $(\tilde{M},\tilde{g}_{\mu\nu})$, we can ask, whether we can find an extension M of the manifold $\tilde{M}$ and a metric $g_{\mu\nu}$ together with a conformal factor Ω on M which are related to $(\tilde{M},\tilde{g}_{\mu\nu})$ as described above. If this is the case, it can be shown that the surface $\mathscr{I}$ and the metric $g_{\mu\nu}$ on $\tilde{M}\cup\mathscr{I}$ are determined essentially uniquely by the conformal class of $\tilde{g}_{\mu\nu}$ up to conformal rescalings of $g_{\mu\nu}$ with everywhere positive conformal factors and certain completeness requirements on $\mathscr{I}$, which make sure that one has not left out part of it or has made unwanted identifications. Therefore the surface $\mathscr{I}$ is called "the conformal boundary at infinity" associated with $(\tilde{M},\tilde{g}_{\mu\nu})$.

For the three conformally flat, geodesically complete solutions of the source free Einstein equations, Minkowski-, De-Sitter-,and Anti-De-Sitter -space, the conformal boundaries are easily constructed and may be described also group theoretically. It turns out, that in the case of

De-Sitter-space the conformal boundary is space-like and consists of two components $\mathscr{I}^-$, $\mathscr{I}^+$ which represent past and future null and time-like infinity respectively, while in the case of the Anti-De-Sitter-space the conformal boundary is time-like and consists of one component which represents space-like and null infinity. In the case of Minkowski-space the situation is more complicated and I shall describe it in more detail to illustrate the statements which will be made later.

Consider on the manifold $M_0 = \mathbb{R} \times S^3$ the metric $g_0 = dt^2 - d\omega^2$ and the "conformal factor" $\Omega_0 = \cos t + \cos \chi$, where $d\omega^2$ is the standard line element on the 3-dimensional unit sphere, given in standard spherical coordinates φ, θ, χ with $0 \le \varphi < 2\pi$, $0 \le \theta \le \pi$, $0 \le \chi \le \pi$ by $d\omega^2 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$. A maximal connected subset of M_0 where Ω_0 is positive, is given for example by $\tilde{M} = \{0 \le \chi < \pi,\ \chi - \pi < t < \pi - \chi\}$ and the restriction of $\tilde{g} = \Omega_0^{-2}\, g_0$ to this submanifold makes $(\tilde{M},\tilde{g})$ isomorphic to Minkowski space. The transition from one metric to a rescaled metric will in general be accompanied by a transition to another coordinate systems which is adapted to the new metrical situation. I shall not give this coordinate transformation in the case of our example.

Denoting the point $\{\chi = 0\}$ of S^3 by N and $\{\chi = \pi\}$ by S, the set $\tilde{M}$ can also be described in the following way. The null geodesics of g_0 which start from $i^- = (-\pi, N) \in M_0$ into the future, sweep out a smooth null hypersurface $\mathscr{I}^-$, on which $\Omega_0 = 0$, $d\Omega_0 \neq 0$, until they reconverge again at the point $i^0 = (0,S)$. After passing i^0 they generate another smoth null hypersurface $\mathscr{I}^+$, on which again $\Omega_0 = 0$, $d\Omega_0 \neq 0$, until the reconverge at the point $i^+ = (\pi, N)$. The "physical" space $\tilde{M}$ is the intersection of the time-like future of i^- with the time-like past of i^+. The surface $\mathscr{I}^-$ ($\mathscr{I}^+$) is past (future) conformal infinity for the physical space-time $(\tilde{M},\tilde{g})$. While the physical null geodesics have past and future endpoints on $\mathscr{I}^-$ and $\mathscr{I}^+$ respectively, physical geodesics which are time-like have past (future) endpoints at "past (future) time-like infinity" i^- (resp. i^+). The point i^0, "space-like infinity", is the endpoint (in both directions) of space-like geodesics. At the points i^0, $i^\pm$ the function Ω_0 vanishes and has a non-degenerate critical point. Any Cauchy hypersurface of the physical space-time "touches" i^0 and is compactified to a topological 3-dimensional sphere by the inclusion of i^0.

Later we will consider hypersurfaces with boundary of M, which intersect the hypersurface $\mathscr{I}^+$ in a space like 2-surface in such a way that they meet any null generator of $\mathscr{I}^+$ once. Such hypersurfaces will be called "hyperboloidal", because an example is provided by a space-like unit hyperbola. Though one may get from the standard Minkowski picture the impression, that at infinity this hyperbola becomes tangent to the light cone through the origin of the Minkowskian coordinate system in which the hyperbola is given, this is not the case. The hyperbola may be extended smoothly as a space-like hypersurface through $\mathscr{I}^+$, which intersects $\mathscr{I}^+$ as well as the light cone through the origin transversely.

In general the decision whether a given space time allows the construction of a smooth conformal boundary at infinity may be quite difficult. However, for theoretical investigations the notion of a conformal boundary is very convenient, since it allows the discussion of asymptotic properties of space-times and of the physical fields on them in terms of local differential geometry.

Penrose conjectured, that the far-fields of isolated gravitating systems behave in such a way, that the fields allow the construction of a smooth conformal boundary. Evidence for this was provided at the time by a number of explicitely known solutions of Einstein's field equation which are interpreted as representing such fields and posses a smooth conformal boundary, by an extensive study of formal expansion type solutions of the field equations, by the fact, that the vacuum Bianchi identities show a certain conformal invariance (which has been used to derive equation (8)), and by the possibility to define various physical notions in an elegant and natural way. Later certain doubts were raised, mainly motivated by the results of some approximation calculations, as to whether the notion of the conformal boundary reflects the asymptotic behaviour of gravitational fields appropriately.

There are in fact two different kinds of conditions involved. On the one hand one has the geometrically defined fall-off condition of the fields which is contained implicitely in the requirement that the conformal boundary be smooth and on the other hand the gravitational field is governed by field equations which together with suitable initial data determine the fall-off of the field uniquely. In view of the transformation behaviour (3) it is a priori not clear, whether these conditions of quite a different nature should work together harmoniously and this question certainly cannot be answered satisfactorily by the study of formal expansions or by (uncontrolled) approximation procedures. To come here to a decision one would have to follow the evolution of the field from appropriate Cauchy data into the far future and observe the decay of the field in the direction of outgoing null geodesics. This amounts to proving global existence theorems for solutions to Einstein's field equations and on top of that of deriving decay estimates which are so sharp and so precisely related to the properties of the data, that a decision can be made for which data the solutions will admit a smooth conformal boundary. The regular conformal field equations have been derived for this purpose.

Existence Results

Since quite general existence results are available for solutions of quasi-linear symmetric hyperbolic systems we may use the reduction theorem to derive the existence of solutions to the conformal field equations which extend into regions where the conformal factor becomes zero or even negative. The corresponding physical solutions to Einstein's field equations will then extend at least in parts up to infinity.

This program has been carried out with a certain completeness for the coupled Einstein-Maxwell-Yang-Mills equations in the case where the cosmological constant has the same sign as in the case of the De-Sitter solution [9,10,13]. For convenience I shall in the following only consider smooth (i.e. C^{∞}) data which will give smooth solutions. *There is available a general semi-global existence theorem for past time-like and null geodesically complete solutions, which have a smooth conformal boundary in the past.* It is a consequence of the sign of the cosmological constant, that this boundary is space-like. The data for these solutions are specified on the conformal boundary and the freedom to prescribe data is essentially the same as in the standard Cauchy problem (there is a certain interesting difference though). *There is a general stability result which says that for any solution with compact Cauchy surface, which has a smooth and complete conformal boundary in the infinite past as well in as in the infinite future, all solutions sufficiently "close" (measured in terms of Cauchy data) have the same property. As a consequence of the last statement and the fact that De-Sitter-space provides a solution with all the properties stated above we obtain a global existence result which says, that data close enough to De-Sitter-data develop into a solution* *which is geodesically complete and has a smooth conformal boundary in the future and in the past.* By using the conformal field equations we obtain thus global existence results and precise and complete information on the asymptotic behaviour of the solution at one stroke. It appears that in the case where the sign of the cosmological constant is such that conformal infinity is space-like the idea of the conformal boundary is quite natural.

The situation for Einstein's field equations with vanishing cosmological constant, to which will be devoted the rest of this article, is not so clear yet. There are a number of partial results available, but I shall concentrate only on those which I consider as most interesting.

By analogy with the global existence result for small data for De-Sitter type solutions mentioned above, one may expect, that aymptotically flat Cauchy data on $\mathbb{R}^3$ sufficiently close to Minkowski data develop into a solution which has an asymptotic structure similar to Minkowski-space, i.e. which will possess a smooth conformal boundary consisting of two components $\mathscr{I}^-$, $\mathscr{I}^+$, which are complete in the sense that in a suitable conformal scaling the null geodesics generating these hypersurfaces (it is a consequence of the Einstein equations with vanishing cosmological constant and conformally invariant sources that these hypersurfaces are necessarily null) are complete and each physical null geodesic has past endpoint on $\mathscr{I}^-$ and future endpoint on $\mathscr{I}^+$. Furthermore we may require that it will possess points i^-, i^+, representing past resp. future time-like infinity, which are regular in the sense that in a suitable smooth conformal extension of the physical solution $\mathscr{I}^-$ ($\mathscr{I}^+$) represents the future (past) null cone of i^- (i^+).

The question whether and under which conditions on the data such space times exist does not only pose an interesting mathematical problem but is also of considerable physical interest. A nontrivial solution of the vacuum field equations with the properties required above would have an indubitable interpretation as representing pure gravitational radiation coming in from infinity, interacting in a nonlinear though smooth way with itself, and escaping to infinity again. Such spaces will in the following be called "purely radiative space-times".

The diligent reader will have noticed, that I did not mention space-like infinity, which is represented by the point i^0 in the case of Minkowski space. The reason for this is the fact, that we know that for a nontrivial purely radiative space–time a smooth conformal extension through space-like infinity cannot exist, because for data with nonvanishing mass the curvature fields will diverge at space–like infinity in any conformal extension in which space–like infinity will be represented by a point. Nevertheless it will be convenient in the following and to a certain extend justified, to think of space–like infinity as being represented by a point i^o in a conformal extension which yields a metric of low smoothness at i^o and to refer to the divergence of the curvature fields as to "the singularity at i^o".

The singularity at i^o is just one expression for the fact that the global causal structure of a nontrivial purely radiative space–time would be quite different from the global causal structure of Minkowski–space (see [17] for a very illustrative discussion of this point). Because of this singularity we cannot expect to obtain the desired existence results by an immediate application of the known existence results for symmetric hyperbolic systems. Therefore I prefered to divide the problem into two parts:

– The "i^o–problem" asks whether there exist nontrivial asymptotically flat Cauchy data on $\mathbb{R}^3$ for which it can be shown that the corresponding solution allows the construction of "pieces (diffeomorphic to $\mathbb{R} \times S^2$) of $\mathscr{I}^+$ and $\mathscr{I}^-$ near i^o" (thinking in terms of the conformal picture). One would also like a characterization of the subset of these data in the class of all asymptotically flat data.

– If the existence of these pieces can be established, one can construct in the solutions space–like hypersurfaces which intersect $\mathscr{I}^+$ (say) in a space–like 2–sphere. Such hypersurfaces will be called "hyperboloidal". Smooth initial data for the conformal field equations on a three dimensional manifold with boundary which satisfy the constraint equations implied by the conformal field equations on space like–hypersurfaces and which are such that the conformal factor is positive in the interior, vanishes on the boundary, but has nonvanishing differential there, will be called "hyperboloidal initial data". The "hyperbolidal initial value problem" asks, whether such data develop into a solution which has a smooth conformal boundary in the future and possibly a regular point i^+ representing time like infinity for this solution.

Since the second problem deals with a regular situation I studied it first. The following existence result has been obtained.

Theorem [9,13]:

Smooth hyperboloidal initial data develop into a solution of the Einstein–Maxwell–Yang–Mills equations, wich has a "piece of a smooth conformal boundary" in the future of the initial surface. Hyperboloidal initial data "sufficiently close" to Minkowskian hyperboloidal initial data develop into a solution which has a future complete conformal boundary and admits a regular point i^+ *representing the infinite time-like future of this solution.*

It is a quite remarkable consequence of the field equations, that under the assumption that the data be close to Minkowskian data the null generators of $\mathscr{I}^+$ must meet at exactly one point i^+. The first part of the existence statement above is true also for hyperboloidal data on initial surfaces of a more complicated topology than that of the unit ball in $\mathbb{R}^3$, as they are given for example by those hyperboloidal hypersurfaces in the maximal analytically extended Schwarzschild solution, which intersect both $\mathscr{I}^+$'s. However, the solution will then necessarily develop a kind of conformal singularity as a consequence of the properties of the field equations alluded to above. In the vacuum case the solutions are time-like and null geodesically future complete [11], in the more general case this is probably still true but the argument used in [11] has to be modified to obtain the result.

Solutions of the type considered in the theorem will be called "radiative solutions" in the following.

I consider the result above as a preliminary step towards a global existence result, which should follow as soon as the i^0–problem has found a positive answer. Nevertheless is the question of the existence of hyperboloidal data of considerable interest, not only because the generality with which such data can be provided should shed some light on the question of the naturalness of the idea of the conformal boundary, but also because hyperboloidal data are of interest to model certain physical situations. The construction of such data requires the discussion of elliptic equations on complete Riemannian manifolds which behave at infinity similar to spaces of constant negative curvature and in particular a detailed study of the behaviour of their solutions at infinity. The constraint equations implied by the regular conformal field equations on a hyperboloidal hypersurface are of course also regular, but it is not known yet, whether it is possible to find a procedure to construct solutions of the conformal constraint equations by solving finite and regular elliptic boundary value problems in the nonphysical space. The fact that there exist exact radiative solutions [5] and the examples discussed in the following show, that the class of hyperboloidal data does certainly not only consist of Minkowskian data.

By-passing the i^{o}-problem, Cutler and Wald [7] recently made use of the result above by devising smooth asymptotically flat initial data on $\mathbb{R}^3$ for the Einstein-Maxwell equations for which the support of the Maxwell-field is compact and the data outside this set coincide with Schwarzschild data for positive mass. In this situation one has perfect control on the behaviour of the time development of the field near space like infinity where the solution coincides with the Schwarzschild solution. There exist smooth hyperboloidal hypersurfaces and Cutler and Wald show that the hyperboloidal initial data implied on these surfaces can be made to approximate Minkowskian hyperboloidal initial data arbitrarily close. Combined with the existence theorem above this shows for the first time the existence of non-trivial solutions to the Einstein-Maxwell equations with complete $\mathscr{I}^-$, $\mathscr{I}^+$ and regular points i^-,i^+.

Though this construction gives rise to interesting questions and suggests a number of possible generalisations, it supplies no new information on the decisive i^{o}-problem.

In the following I will report on some observation which I made while looking at i^{o} from different angles.

The Radiativity Condition

The point i^{o} is a nondegenerate critical point of the conformal factor. To understand better the behaviour of the equations and their solutions in such a situation without being disturbed by the complicated singularity at i^o, we may analyse in detail the radiative solutions at the point i^+. Assume we are given such a solution which is real analytic. Then we can extend this solution conformally and analytically through i^+ into the nonphysical regime to obtain an extended solution of the conformal field equation. Let S be a space-like analytic hypersurface in this extension, which contains the point i^+. Near this point we can then reconstruct our solution from the data implied on S by solving a backward Cauchy problem for the conformal field equations. For simplicity assume that the (nonphysical) second fundamental form on S vanishes and that the interior metric h_{ab} (where indices a,b,c,.. take values 1,2,3) on S is given to us. Then we find that the conformal factor must satisfy the requirements

(10) $$2\Omega\, \Delta\, \Omega = 3D^a\, \Omega\, D_a\, \Omega - \frac{1}{2}\, \Omega^2\, R \qquad \text{on S near } i^+$$

(11) $$\Omega = 0,\ D_a\, \Omega = 0,\ D_a\, D_b\, \Omega = 2h_{ab} \qquad \text{at } i^+,$$

where D_a denotes the covariant derivative operator, R the Ricci-scalar, and Δ the Laplacian defined by h_{ab}. The analytic solution of these conditions is determined uniquely near i^+ and is given essentially by a fundamental solution of Δ with pole at i^+. We find, that by h_{ab} and Ω all other initial data on S for the conformal field equations are determined uniquely by the requirement that they solve the constraint equations. In the following we will call a set of such initial data, if they are smooth even at the point i^+, a "radiative initial data set". If one calculates the data from h_{ab} and Ω one has to divide by Ω several times. This implies conditions on the metric h_{ab} which must be satisfied to ensure that all initial data are smooth at i^+. Denoting by S_{ab} the trace free part of the Ricci-tensor of h_{ab}, we have the

Theorem [12]:

Assume that the metric h *and the conformal factor* Ω *satisfying* (10), (11) *are analytic in a neighbourhood* V *of* $i^+ \in$ S. *Then the radiative initial data set implied by* h *and* Ω *on* S *is analytic in* V *if and only if* h *is such that the "radiativity condition"*

$$C_{a_n \ldots a_1}\{D_{a_n} \ldots D_{a_1} D_c (D_a D_b \Omega - \tfrac{1}{3} \Delta\Omega\, h_{ab} - \Omega\, S_{ab})\} = 0 \qquad \textit{at } i^+, \qquad n = 0,1,2,\ldots ,$$

is satisfied, where the operator $C_{a_n \ldots a_1}$ *means: "take the symmetric trace free part of the argument with respect to the indices* $a_1, \ldots, a_n$*".*

The radiativity condition, which is in fact a condition on the conformal class of h_{ab}, is of course related to the decay of the physical solution in the infinite time-like future. However, it may be considered also from a different point of view. Since the analytic and conformal extension of our original solution defines a solution of the regular conformal field equations, one finds, that the part of the unphysical space which is space-like related to i^+ and where Ω is negative, also provides a solution of Einstein's field equations, for which i^+ now represents space-like infinity i^o. The radiativity condition may therefore be understood as an additional fall-off condition on asymptotically flat Cauchy data on S such that the solution allows an analytic conformal extension through the point i^o.

The meaning of the radiativity condition is not understood yet in a satisfactory way, but to the question, how one could construct three dimensional conformal structures which satisfy this condition has found a certain answer.

A large class of solutions which possess a smooth and complete conformal boundary is provided by static or stationary space-times. The structure of space-like infinity of static space-times has been analysed extensively. If a space-time with physical metric $\tilde{g}$ admits a time-like hypersurface orthogonal Killing vector field, i.e. if $\tilde{g}$ is static, one can choose coordinates x^o, x^a, a = 1,2,3, such that the metric takes the form

(12) $$\tilde{g} = - e^{2U}(dx^o)^2 + e^{-2U}\,\hat{h}_{ab}\,dx^a\,dx^b$$

with x^o-independent functions U, $\hat{h}_{ab}$. Einstein's vacuum field equations for such a metric reduce to the quasilinear system of equations

(13) $$\hat{R}_{ab} = 2\hat{D}_a U\,\hat{D}_b U,$$

(14) $$\hat{\Delta} U = 0$$

for the function U and the Riemannian metric $\hat{h}$. The covariant operator

$\hat{D}$, the Laplacian $\hat{\Delta}$ and the Ricci-tensor $\hat{R}_{ab}$ are defined here with respect to $\hat{h}$. The equations (13),(14) may be read as three dimensional Einstein equations with source field U. In the following will be considered solutions of these equations, which exist with $U \neq 0$ on some three dimensional manifold $\hat{S}$ (representing a slice $\{x^o = \text{const}\}$) which is diffeomorphic to $\mathbb{R}^3$ minus a closed non-empty ball. The solution will be required to be asymptotically flat and we shall be concerned with its behaviour near spatial infinity (with respect to $(\hat{h},\hat{S})$).

Beig and Simon [3] showed for asymptotically flat solutions of (13),(14) with nonvanishing mass M_o that the rescaled metric

$$(15) \qquad h_{ab} = \omega^2 \hat{h}_{ab} \quad \text{with} \quad \omega = M_o^{-2} U^2, \quad M_o \neq 0,$$

if expressed in suitable coordinates, extends to a real analytic metric on a manifold $S = \hat{S} \cup \{i\}$, which is obtained by attaching a point i, representing space-like infinity for $\hat{h}$, to $\hat{S}$ such that S is diffeomorphic to an open ball in $\mathbb{R}^3$. Using this, it is easy to show, that the corresponding space time with metric given by (12) allows the construction of a conformal boundary and that the analytic extension of the rescaled space time through this boundary gives again the solution (12) but with M_o replaced by $-M_o$. A closer analysis of the extension to i of the metric given by (15) and of the conformal field equations it has to satisfy, gives the

Theorem [12]:

Any asymptotically flat solution of the static field equations (12),(13), defines a conformal class which satisfies at the point i, *which reprents space-like infinity, the radiativity condition.*

Thus we find, somewhat surprisingly, that the fall-off behaviour which is expressed by the radiativity condition is satisfied not only by solutions which extend conformally through i^o in an analytic way (and consequently have vanishing mass), but also by certain solutions which have a nonvanishing mass and consequently a singularity at i^o. Whether this is of any significance at all for the possibility that a solution may admit a smooth conformal boundary or is just an irrelevant coincidence still has to be seen.

The result above has interesting consequences. There is known explicitely a number of static solution and it should not be to difficult to ensure the existence of a large number of them by specifying a fall off behaviour for their multipoles which would ensure the convergence of their multipole expansions (see also [18], where the existence of stationary asymptotically flat solutions is shown). Any such solution will allow to provide a radiative initial data set. Its time development by the conformal field equations determines in the time-like future of the point i a radiative solution of Einstein's vacuum field equations (which shows the existence of hyperboloidal data again).

The fact, that so far no explicite purely radiative space-time has been found has sometimes been used as an argument that the concept of the conformal boundary is inappropriate. It is well known (see [1,2]) that the existence of more than one Killing is not compatible with the desired asymptotic behaviour. It turns out that it is easy to give conditions on the first few multipole coefficients of a static space-time, to ensure that the corresponding radiative space-time admits at most one Killing field. We may therefore expect that among the analytic extensions of the radiative solutions associated with static solutions are good candidates for purely radiative space-times.

It appears that still much is to be learned about the i^0-problem. Its investigation should prove worth any effort, not only because it should lead to a further clarification the asymptotic behaviour of gravitational fields and the related notions of mass, radiation, etc. but also because its understanding should shed a sharp light onto the decisive properties of the field equations which are responsible for the behaviour of their solutions near singularities.

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