SUPERAMENABILITY

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Let A be a Banach algebra, M a Banach A–bimodule. A continuous linear operator $D : A \to M$ is a *derivation* if $D(ab) = Da.b + a.Db$ $(a, b \in A)$. The set of all such derivations will be denoted $Z^1(A, M)$. For each $m \in M$, the map $\delta_m : A \to M :$ $a \to m.a - a.m$ is an *inner derivation.* The set of such inner derivations will be denoted $B^1(A, M)$. A is *superamenable* if $H^1(A, M) = Z^1(A, M) / B^1(A, M) = 0$ for all M.

The prefix 'super' is due to Barry Johnson, as is the term 'amenability' applied to A, meaning $H^1(A, M) = 0$ for all dual A–bimodules M. The same notion has been around for some time in algebra, where the term 'separable' is used.

In the purely algebraic situation, ignoring topological considerations altogether, any separable algebra over $\mathbb{C}$ is necessarily finite dimensional and semisimple (and so a direct sum of full matrix rings). In the Banach case, this is suspected to still hold true, but it need not hold in the general topological situation – the algebra of distributions on a compact Lie group is superamenable.

In terms of the tensor product $A\hat{\otimes}A$ and the natural map $\pi : a\otimes b \to ab$ of $A\hat{\otimes}A$ to A, superamenability of A is easily characterized by (i) A has an identity 1, and (ii) there is an element $u \in A\hat{\otimes}A$ such that $u.a = a.u$ for all $a \in A$, and $\pi u = 1$.

An important consequence of superamenability is the fact that for any short exact sequence of left (or right) A–modules, $0 \to X \to Y \to Z \to 0$, with the given module homomorphisms $f : X \to Y$, $g : Y \to Z$ satisfying $\mathrm{Im}\, f = \mathrm{Ker}\, g$ complemented in Y, then $\mathrm{Im}\ f$ is a module direct summand of Y (that is, the sequence splits). It follows that any closed complemented ideal of a superamenable algebra is generated by a central idempotent. In particular, a commutative superamenable algebra is isomorphic to $\mathbb{C}^k$ for some k.

Making condition (ii) above more explicit, with $u = \sum a_i \otimes b_i$,

$$\sum aa_i \otimes b_i = \sum a_i \otimes b_i a \quad (a \in A), \ \underline{\text{and}} \ \sum a_i b_i = 1.$$

By the universality of tensor products, for any A–bimodules X, Y, and $T \in B(X, Y)$,

$$\sum aa_i T(b_i x) = \sum a_i T(b_i ax) \qquad (a \in A,\ x \in X).$$

Noting that $B(X, Y)$ is a left $A \hat{\otimes} A^{op}$–module under the operation $(a \otimes b)T(x) = a.T(b.x)$, we have

$$(u.T)(a.x) = \sum a_i T(b_i ax) = \sum aa_i T(b_i x) = a.(u.T)(x),$$

so that $u.T$ is a left A–module homomorphism. In particular, if $X = Y = A$, $x = 1$, $(u.T)(a) = a.(u.T)(1)$, and choosing, as we may, $\{\| a_i \|\} \in \ell^1$ and $b_i \to 0$, we have

$$\| (u.T)(a) - a \| = \| \sum a_i [T(b_i a) - b_i a] \| \leq \left(\sum \| a_i \| \right) \sup_j \| [T(b_j) - b_j] \| . \| a \|.$$

Thus

$$\| u.T - Id_A \| \leq const.\ \sup_j \{\| T(b_j) - b_j \| \}.$$

Now if T is compact, then so is $u.T$, so if A has the compact approximation property, letting T run over a net of compact operators converging to the identity uniformly on compact sets shows that Id_A is compact, whence A is finite dimensional.

This 1972 result of Joseph Taylor has recently been sharpened by Barry Johnson, who showed that if A is superamenable then any irreducible representation of A on a space with the compact approximation property is finite dimensional.

The above result settles the case for C*–algebras since for such algebras superamenable $\Rightarrow$ amenable $\Rightarrow$ nuclear $\Rightarrow$ approximation property. In fact a direct proof of a stronger result can be given. If A is a unital C*–algebra and $H^1(A, B) = 0$ for all C*–algebras $B \supseteq A$, then A is finite dimensional. (In the absence of the C*–condition on B this hypothesis is equivalent to superamenability).

As a consequence, if A is the W*–algebra of all bounded sequences $\{T_n\}$, $T_n \in B(\mathbb{C}^n)$, then A is not superamenable. (It is known that A fails to have the approximation property, so A is not even amenable.) *Is there a simple construction of a bimodule M with $H^1(A, M) \neq 0$? What if $\{\mathbb{C}^n\}$ here is replaced by a sequence of finite dimensional Banach spaces $\{X_n\}$ with dim $X_n \to \infty$? Surely the resulting algebra is never superamenable ? Is it ever amenable ?*

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