LOCALLY INNER DERIVATIONS

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1. INTRODUCTION

Let A be an infinite-dimensional, unital complex Banach algebra. An outstanding conjecture on 'super-amenability' or 'contractibility' (see e.g. P.C. Curtis and R. J. Loy [2], A.J. Helemskii [4]) is equivalent to the existence, for every such algebra A, of a non-inner, continuous derivation from A into some Banach A-bimodule.

In this paper we give a method for the construction of certain derivations into various A-bimodules (both Banach and Fréchet) of A-valued functions. A natural name for the derivations to be constructed is *locally inner*. The aim is then to construct a derivation that is locally inner, but not inner. These problems lead, rather naturally, to certain questions concerning the homological nature of some particular modules of A-valued functions.

There is, of course, a well-developed mathematical language in which to discuss questions concerning relations between local and global properties, namely the theory of presheaves and sheaves. The questions raised in the present paper are closely linked with the first (Čech) cohomology groups with coefficients in certain sheaves. The paper can however be read without a knowledge of sheaf cohomology. It is, of course, possible that to answer some of the problems might require more of the technical machinery.

2. SHEAVES OF A-VALUED FUNCTIONS

We shall introduce, in a concrete way, some ideas about presheaves and sheaves. The more general 'abstract' notion of a presheaf is not, in fact, very different.

Let A be a given unital Banach algebra and let X be a topological space. For every open $U \subseteq X$, let $\mathcal{C}(U, A)$ be the algebra of all continuous A-valued functions on U. We also consider $\mathcal{C}(U, A)$ as an A-bimodule, the module actions being multiplication on the left and right by elements of A (regarded as constant functions). By a *presheaf $\mathcal{A}$, of modules of continuous A-valued functions, on X* we mean an assignment $U \mapsto \mathcal{A}(U)$, to every open subset U of X, of a submodule $\mathcal{A}(U)$ of $\mathcal{A}(U, A)$ such that:

PS1 : $\mathcal{A}(\emptyset) = 0$;

PS2 : for every pair of open subsets U, V of X with $V \subseteq U$ and every $f \in \mathcal{A}(U)$, the restriction $f \mid V \in \mathcal{A}(V)$.

The presheaf $\mathcal{A}$ is called a *sheaf* (of modules of continuous A-valued functions on X) provided that it satisfies the extra condition:

PS3: for every open $U \subseteq X$, every open covering $\{U_\lambda : \lambda \in \Lambda\}$ of U and every collection $\{f_\lambda : \lambda \in \Lambda\}$ where $f_\lambda \in \mathcal{A}(U_\lambda)$ and where $f_\lambda \mid U_\lambda \cap U_\mu = f_\mu \mid U_\lambda \cap U_\mu$ (for all $\lambda, \mu \in \Lambda$), there exists $f \in \mathcal{A}(U)$ such that $f \mid U_\lambda = f_\lambda$ $(\lambda \in \Lambda)$.

[**Remark.** In the case of an 'abstract' sheaf there is also another condition, asserting the uniqueness of f in PS3; this extra condition is automatically fulfilled in the case of a presheaf of functions.]

Examples. (i) If we set $\mathcal{A}(U) = \mathcal{C}(U, A)$ for every open $U \subseteq X$, we clearly obtain a sheaf; we call it the *sheaf of continuous A-valued functions* on X.

(ii) If we take $\mathcal{A}(U) = \mathcal{C}_b(U, A)$, the algebra of all *bounded* continuous A-valued functions on U, we have an example of a presheaf that is not a sheaf (except for very particular X). This is because, given $f_\lambda \in \mathcal{C}_b(U_\lambda, A)$ as in PS3 above, the unique $f \in \mathcal{C}(U, A)$, such that $f \mid U_\lambda = f_\lambda$ $(\lambda \in \Lambda)$, will not in general be bounded, unless the covering $\{U_\lambda\}$ is finite.

(iii) If we now take X to be an open subset of $\mathbf{C}^n$ (or, more generally, a complex manifold) then we may set for every open $U \subseteq X$, $\mathcal{A}(U) = \mathcal{O}(U, A)$, the algebra of all *holomorphic* A-valued functions on U obtaining the sheaf $\mathcal{O}_A$ of *holomorphic A-valued functions* on X.

(iv) We may similarly define sheaves of smooth A-valued functions, of C^k A-valued functions, A-valued Lipschitz functions and pre-sheaves of *bounded* A-valued functions of these kinds, in each case for a suitable space X.

(v) Let Ω be a bounded open subset of $\mathbf{C}^n$ and let X be its closure. For every (relatively) open subset U of X we take

$$\mathcal{A}(U) = \{f \in \mathcal{C}(U, A) : f \mid U \cap \Omega \text{ is holomorphic}\} .$$

Then $\mathcal{A}$ is a sheaf on X; notice that $\mathcal{A}(X)$ is then the algebra of all continuous A-valued functions on $\overline{\Omega}$ that are holomorphic on Ω. We may call $\mathcal{A}$ the *sheaf of holomorphic A-valued functions with continuous boundary values on Ω.*

We remark that the examples of presheaves which are not sheaves that we have given are really not very different from sheaves. They have the property of being '*finitely sheaf-like*' as follows:

PS4: the condition of PS3 holds for every open $U \subseteq X$ and every *finite* open covering $\{U_1, \ldots, U_n\}$ of U.

We should warn the reader that the term 'finitely sheaf-like' is not standard.

In the theory of derivations from Banach algebras — and more generally in the homological theory of Banach algebras — we are interested not in general bimodules but in Banach (and sometimes Fréchet) A-bimodules. Let $\mathcal{A}$ be a presheaf of modules of continuous A-valued functions on X. We shall call $\mathcal{A}$ a *Fréchet pre-sheaf* (this is an adaptation of a standard usage) provided that:

FS1: each $\mathcal{A}(U)$ has a topology τ_U, finer than pointwise convergence, in which it is a Fréchet A-bimodule.

Suppose now that $\mathcal{A}$ is a Fréchet presheaf on X and that U, V are open subsets of X, with $V \subseteq U$. Let $\rho_{VU} : \mathcal{A}(U) \to \mathcal{A}(V)$ be the restriction map. Suppose that $f_n \in \mathcal{A}(U)$ $(n \geq 1)$, that $f_n \to f \in \mathcal{A}(U)$ $(n \to \infty)$ in the topology τ_U, while $\rho_{VU}(f_n) \to g \in \mathcal{A}(V)$ for the topology τ_V. By FS1, we then have that $f_n(t) \to f(t)$ $(t \in U)$ and $f_n(t) \to g(t)$ $(t \in V)$; hence $\rho_{VU}(f) = g$. It follows from the closed graph theorem that a Fréchet presheaf necessarily also satisfies:

FS2: for every pair $V \subseteq U$ of open subsets of X, the restriction map $\rho_{VU} : \mathcal{A}(U) \to \mathcal{A}(V)$ is continuous.

Finally we say that the presheaf $\mathcal{A}$ is a *Banach presheaf on* X (*not* a standard term) provided that $\mathcal{A}$ is a Fréchet presheaf and that:

BS: the Fréchet space $\mathcal{A}(X)$ is a Banach space.

Remark. A moment's reflection on the examples given will show that to require *every* $\mathcal{A}(U)$ to be a Banach space would be much too strong a requirement to yield interesting examples. In the examples given, the presheaves of bounded holomorphic functions, bounded continuous functions and the sheaf of holomorphic functions with continuous boundary values (on bounded open $\Omega \subseteq \mathbf{C}^n$) are all examples of Banach presheaves of modules.

Now let $\mathcal{A}$ be a given presheaf of modules of continuous A-valued functions on a topological space X and let $\mathcal{U} \equiv \{U_\lambda\}_{\lambda\in\Lambda}$ be a given open covering of X. We shall introduce the *Čech cochain complex* $C^*(\mathcal{U};\mathcal{A})$ on $\mathcal{U}$ with coefficients in the presheaf $\mathcal{A}$. (Our terminology here is not precisely standard.)

We shall use the standard notation $U_{\lambda\mu} \equiv U_\lambda \cap U_\mu$, $U_{\lambda\mu\nu} \equiv U_\lambda \cap U_\mu \cap U_\nu$ etc. for intersections of sets in the given cover $\mathcal{U}$. For $n = 0, 1, 2, \ldots$, an *n-cochain* on $\mathcal{U}$ assigns an element $f \in \mathcal{A}(U_{\lambda_1\lambda_2\ldots\lambda_{n+1}})$ to every $(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \in \Lambda^{n+1}$ such that $U_{\lambda_1\ldots\lambda_{n+1}} \neq \emptyset$; more formally the space $C^n(\mathcal{U},\mathcal{A})$ of n-cochains is,

$$C^n(\mathcal{U},\mathcal{A}) \equiv \prod_{\substack{(\lambda_1,\ldots,\lambda_{n+1})\in\Lambda^{n+1} \\ U_{\lambda_1\ldots\lambda_{n+1}}\neq\emptyset}} \mathcal{A}(U_{\lambda_1\ldots\lambda_{n+1}})\,.$$

In particular $C^0(\mathcal{U};\mathcal{A}) = \prod_{\lambda\in\Lambda}\mathcal{A}(U_\lambda)$.

It will also be convenient to extend the sequence one place back by setting

$$C^{-1}(\mathcal{U};\mathcal{A}) = \mathcal{A}(X)\,.$$

We now define the coboundary maps

$$\delta_n : C^n(\mathcal{U};\mathcal{A}) \to C^{n+1}(\mathcal{U};\mathcal{A}) \qquad (n \geq -1)\,.$$

For $n = -1$: we define $\delta_{-1} : \mathcal{A}(X) \to \prod_{\lambda\in\Lambda}(\mathcal{U}_\lambda)$ by $\delta_{-1}(f) = (f \mid U_\lambda)_{\lambda\in\Lambda}$ $(f \in \mathcal{A}(X))$.

$n = 0$: let $\mathbf{F} \equiv (f_\lambda)_\lambda \in \Lambda \in C^0(\mathcal{U}; \mathcal{A})$. Then for each $(\lambda, \mu) \in \Lambda^2$ with $U_{\lambda\mu} \neq \emptyset$ we define $\delta_0(\mathbf{F})_{\lambda\mu} = f_\lambda \mid U_{\lambda\mu} - f_\mu \mid U_{\lambda\mu}$.

Observe that, very obviously, for every $f \in \mathcal{A}(X) \equiv C^{-1}(\mathcal{U}; \mathcal{A})$ and $(\lambda, \mu) \in \Lambda^2$,

$$(\delta_0 \circ \delta_{-1})(f)_{\lambda\mu} = f \mid U - \lambda\mu - f \mid U_{\lambda\mu} = 0\,;$$

so that $\delta_0\delta_{-1} = 0$ and the complex condition is verified at the beginning. For the next steps we *omit restriction signs*, to avoid cluttering the notation. Thus, the definition of δ_0 would be written just as $\delta_0(f)_{\lambda\mu} = f_\lambda - f_\mu$, with 'restriction to $U_{\lambda\mu}$' being understood. In general (for $n \geq 0$) let $\mathbf{F} \equiv (f_{\lambda_1\ldots\lambda_{n+1}})$ be an n-cochain; then for every choice of $(\lambda_1, \ldots, \lambda_{n+2}) \in \Lambda^{n+2}$ with $U_{\lambda_1\ldots\lambda_{n+2}} \neq \emptyset$ we define

$$\delta_n(f)_{\lambda_1\ldots\lambda_{n+2}} = \sum_{k=1}^{n+2} (-1)^{k-1} f_{\lambda_1\ldots\hat{\lambda}_k\ldots\lambda_{n+2}}\,,$$

with $\hat{\lambda}_k$ indicating omission of the suffix λ_k.

Then it is a simple calculation to verify that

$$0 \to C^{-1}(\mathcal{U}; \mathcal{A}) \xrightarrow{\delta_{-1}} C^0(\mathcal{U}; \mathcal{A}) \xrightarrow{\delta_0} \cdots \xrightarrow{\delta_{n-1}} C^n(\mathcal{U}; \mathcal{A}) \\ \xrightarrow{\delta_n} C^{n+1}(\mathcal{U}; \mathcal{A}) \to \ldots$$

is a cochain complex of complex vector spaces.

Since we are dealing only with presheaves of functions, the complex is exact at $C^{-1}(\mathcal{U}; \mathcal{A})$, i.e. δ_{-1} is injective. Also, if $\mathcal{A}$ is a *sheaf* then the complex is exact at $C^0(\mathcal{U}; \mathcal{A})$; indeed to say that the presheaf $\mathcal{A}$ is a sheaf is *equivalent* to the complex being exact at C^0 for *every* open cover of every open $U \subseteq X$. For $\mathcal{A}$ to be finitely sheaf-like is for this complex to be exact at C^0 for every *finite* open cover of every open $U \subseteq X$.

The complex

$$0 \to C^0(\mathcal{U}; \mathcal{A}) \xrightarrow{\delta_0} C^1(\mathcal{U}; \mathcal{A}) \xrightarrow{\delta_1} \cdots\,,$$

i.e. the previous complex with the $C^{-1}(\mathcal{U}; \mathcal{A})$ term removed, is called the Čech complex $C^*(\mathcal{U}; \mathcal{A})$.

Generally we write, as usual, $Z^n \equiv Z^n(\mathcal{U};\mathcal{A}) = \ker \delta_n$, the space of n-cocycles, and $B^n \equiv B^n(\mathcal{U};\mathcal{A}) = \operatorname{im} \delta_{n-1} \quad (n \geq 1)$, the space of n-coboundaries; put $B^0 = 0$. The nth Čech cohomology group of the covering $\mathcal{U}$, with coefficients in the presheaf $\mathcal{A}$ is, of course,

$$H^n \equiv H^n(\mathcal{U};\mathcal{A}) = Z^n/B^n .$$

It is worth looking explicitly at the $n = 1$ case. Thus a 1-cocycle $\mathbf{F} \equiv (f_{\lambda\mu})$ assigns to each $U_{\lambda\mu} \neq \emptyset$ an element $f_{\lambda\mu} \in \mathcal{A}(U_{\lambda\mu})$; it is easily checked that the cocycle condition $\delta_1(\mathbf{F}) = 0$ is equivalent to requiring that for every $U_{\lambda\mu\nu} \neq \emptyset$,

$$f_{\lambda\mu} + f_{\mu\nu} + f_{\nu\lambda} = 0 \quad \text{on } U_{\lambda\mu\nu} .$$

Such a 1-cocycle is sometimes (especially in complex analysis) called a set of *Cousin data*, with values in $\mathcal{A}$. The set of data is called *soluble* precisely if it is a 1-coboundary, i.e. if and only if there is some 0-cochain $\mathbf{g} \equiv (g_\lambda)_{\lambda\in\Lambda}$ such that, on every non-empty $U_{\lambda\mu}$, $f_{\lambda\mu} = g_\lambda - g_\mu$.

A sheaf (or presheaf) $\mathcal{A}$ on a (paracompact) space X is called *acyclic* if $H^n(\mathcal{U};\mathcal{A}) = 0$ for every open covering $\mathcal{U}$ of X and every $n \geq 1$.

For example, let X be an open subset of $\mathbf{C}^n$. Then, using partitions of unity, it is not too hard to prove that the sheaf of *all* continuous (A-valued) functions, and also the sheaf of smooth functions, are both acyclic. However, the sheaf $\mathcal{O}_A$ of holomorphic A-valued functions is only acyclic for 'nice' open subsets of $\mathbf{C}^n$ ('nice' means, for example, being a domain of holomorphy; in case $n = 1$, *all* open subsets of $\mathbf{C}$ are nice).

Exercise. Let $X = \mathbf{C}^2\backslash\{(0,0)\}$,

$$U_1 = \{(z_1, z_2) \in \mathbf{C}^2 : z_1 \neq 0\} ,$$

$$U_2 = \{(z_1, z_2) \in \mathbf{C}^2 : z_2 \neq 0\} ,$$

Then $\{U_1, U_2\}$ is an open covering of X. Define $f(z_1, z_2) = 1/(z_1 z_2)$ on U_{12}. Show that it is *not* possible to find $f_1 \in \mathcal{O}(U_1)$, $f_2 \in \mathcal{O}(U_2)$ such that $f = f_1 - f_2$ on U_{12}.

[*Note:* you need to know the theorem of Hartogs, that for every f holomorphic on $\mathbf{C}^2\backslash\{(0,0)\}$, the point (0,0) is a removable singularity.]

This example shows, then, that $H^1(X, \mathcal{O}) \neq 0$, where $\mathcal{O} \equiv \mathcal{O}_{\mathbf{C}}$, the sheaf of holomorphic complex-valued functions on X.

The fact that, if D is a domain of holomorphy in $\mathbf{C}^n$ (or, more generally, a 'Stein manifold'), then $H^p(D, \mathcal{O}) = 0 \quad (p \geq 1)$ is part of the classical "Théorème B" of H. Cartan. Using representation theorems of Grothendieck [3] or Bishop [1] it follows that also $H^p(D, \mathcal{O}_E) = 0$ $(p \geq 1)$, where, again, D is a domain of holomorphy, E is a complex Banach space (or even a Fréchet space) and $\mathcal{O}_E$ is the sheaf of holomorphic E-valued functions on D. In the converse direction it is very simple to see that if $H^p(G, \mathcal{O}_E) = 0$, for some open $G \subset \mathbf{C}^n$ and some (non-zero) Banach space E, then also $H^p(G, \mathcal{O}_F) = 0$ where F is any complemented closed subspace of E (and, less obviously, for F an arbitrary closed subspace). In particular, taking F as a 1-dimensional subspace, we would deduce $H^p(G, \mathcal{O}) = 0$. In particular, therefore, with $X = \mathbf{C}^2 \backslash \{(0,0)\}$ and E any non-zero Banach space, then $H^1(X, \mathcal{O}_E) \neq 0$.

3. LOCALLY INNER DERIVATIONS

Let X be a topological space and let $\mathcal{A}$ be a Fréchet presheaf, of bimodules of continuous A-valued functions, on X. (As before, A is a complex, unital Banach algebra). We shall, moreover, require that $\mathcal{A}$ be either a *sheaf* or at least be finitely sheaf-like. Now let $\mathcal{U} = \{U_\lambda\}_\lambda \in \Lambda$ be an open covering of X; in case that $\mathcal{A}$ is merely finitely sheaf-like, we take $\mathcal{U}$ to be a *finite* open covering.

For each $\lambda \in \Lambda$, let $f_\lambda \in \mathcal{A}(U_\lambda)$. Then we define the associated inner derivation $D_\lambda : A \to \mathcal{A}(U_\lambda)$ by setting

$$D_\lambda(a) = a.f_\lambda - f_\lambda.a \qquad (a \in A)\,.$$

We now wish to define a derivation $D : A \to \mathcal{A}(X)$ such that $D(a) \mid U_\lambda = D_\lambda(a)$ $(a \in A,\ \lambda \in \Lambda)$. Evidently this is possible if and only if, whenever $U_{\lambda\mu} \neq \emptyset$ we have, for all $x \in U_{\lambda\mu}$,

$$af_\lambda(x) - f_\lambda(x)a = af_\mu(x) - f_\mu(x)a \qquad (a \in A)\,,$$

that is,

$$a\big(f_\lambda(x) - f_\mu(x)\big) = \big(f_\lambda(x) - f_\mu(x)\big)a \qquad (a \in A)\,.$$

Thus, the inner derivations $\{D_\lambda\}_{\lambda\in\Lambda}$ 'fit together' to give a single derivation $D : A \to \mathcal{A}(X)$ if and only if, for all $\lambda, \mu \in \Lambda$ such that $U_{\lambda\mu} \neq \emptyset$, we have $f_\lambda(x) - f_\mu(x) \in Z$, where $Z \equiv Z_A$ is the centre of A, for all $x \in U_{\lambda\mu}$. A derivation D defined in this way will be called *locally inner*.

We may describe the above process conveniently in terms of Čech cohomology. We are given the presheaf $\mathcal{A}$. Define the 'sub-presheaf' $\mathcal{Z}$ of $\mathcal{A}$ as follows: for every open $U \subseteq X$,

$$\mathcal{Z}(U) = \big\{f \in \mathcal{A}(U) : f(x) \in Z \quad \text{for all } x \in U\big\}\,.$$

It is clear that, if $\mathcal{A}$ is a sheaf or is finitely sheaf-like, then also $\mathcal{Z}$ has the corresponding property. (Of course, $\mathcal{Z}(U)$ is a Z-module not an A-module.)

Now let $\mathcal{U}$ be an open covering of X and let $\mathbf{F} \equiv (f_\lambda)_{\lambda\in\Lambda}$ be a 0-cochain on $\mathcal{U}$, with values in $\mathcal{A}$, and such that the 1-cochain $\delta_0(\mathbf{F}) \equiv (f_\lambda - f_\mu : \lambda, \mu \in \Lambda,\ U_{\lambda\mu} \neq 0)$ has values in the sub-presheaf $\mathcal{Z}$ of $\mathcal{A}$. Then there is a unique derivation $D : A \to \mathcal{A}(X)$ such that $D(a)(x) = af_\lambda(x) - f_\lambda(x)a$ $(a \in A,\ x \in U_\lambda,\ \lambda \in \Lambda)$. The derivation D will be inner if and only if there is some $f \in \mathcal{A}(X)$ such that $f - f_\lambda \in \mathcal{Z}(U_\lambda)$ for every λ; but this is equivalent to requiring that $\delta_0(\mathbf{F})$ be a 1-coboundary as a $\mathcal{Z}$*-valued* cocycle. In other words, it is equivalent to the existence of a $\mathcal{Z}$*-valued* 0-cochain, say $\mathbf{g}$, on $\mathcal{U}$ such that $\delta_0(\mathbf{F}) = \delta_0(\mathbf{g})$.

To construct a locally inner, but non-inner, derivation $D : A \to \mathcal{A}(X)$ it would, therefore, be *sufficient* (though not necessary) to find X and $\mathcal{A}$ such that $H^1(X, \mathcal{A}) = 0$, $H^1(X, \mathcal{Z}) \neq 0$.

In practice, the method will only stand a reasonable chance of working provided that the centre Z of A is *not* complemented as a Banach subspace of A. This is because, in the contrary case, if $\pi : A \to Z$ is a continuous projection of A onto Z and if, as above, $\mathbf{F} \equiv (f_\lambda)$ is an $\mathcal{A}$-valued 0-cochain on $\mathcal{U}$ with $\delta_0(\mathbf{F})$ being $\mathcal{Z}$-valued, then, for most reasonable presheaves, $\pi \circ \mathbf{F} \equiv (\pi \circ f_\lambda)$ will be a $\mathcal{Z}$-valued 0-cochain such that $\delta_0(\mathbf{F}) = \delta_0(\pi \circ \mathbf{F})$.

One would expect that Z would fail to be complemented in A for 'most' A, though not for most 'naturally occurring' A (which tend to be either commutative or, at the other extreme, to have a trivial centre!). I am indebted to E. Albrecht for the following example.

Example. A Banach algebra A with uncomplemented centre Z.

Let Δ be the closed unit disc in $\mathbf{C}$ and write

$$X_0 = C(\Delta)\,,$$

$$X_1 = B^1(\Delta) \equiv \{f \in X_0 : \overline{\partial} f \text{ exists and is continuous on } \Delta\}\ .$$

Let X be the Banach space $X_0 \oplus X_1$. Then we may represent elements of $L(X)$ by (2×2)-matrices, according to this direct-sum decomposition. If we identify $f \in X_0$ with the operation of multiplication by f, then we may define A to be the closed subalgebra of $L(X)$ generated by all operators with one of the matrix representations:

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \qquad (f \in X_0,\ g \in X_1)$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ , \qquad \begin{pmatrix} 0 & \overline{\partial} \\ 0 & 0 \end{pmatrix}\ .$$

Recalling that $\overline{\partial}$ commutes with multiplication by f if and only if f is analytic, we see that the centre Z of A is

$$Z = \left\{ \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} : f \in C(\Delta)\,,\ f \text{ analytic on } \Delta^\circ \right\}\ .$$

But then, if Z were complemented in A, it would follow that the disc algebra $A(\Delta)$ would be complemented in $C(\Delta)$, which is known to be false. (See e.g. [6], following (5.19) and a simple argument.)

4. HOLOMORPHIC A-VALUED FUNCTIONS

In seeking a suitable candidate for a pre-sheaf $\mathcal{A}$, in the attempt to construct a non-inner, but locally inner, derivation, it is clear, after the discussion of the previous section, that there is no use picking an acyclic sheaf. Thus, if we took $\mathcal{A}$ to be the sheaf of *all* continuous A-valued functions, on any paracompact topological space X, then every locally inner derivation $A \to \mathcal{A}(X)$ would be inner. The analogous result would also hold if $\mathcal{A}$ were the sheaf of all smooth A-valued functions on a manifold M.

The first case in which there is no immediately obvious reason for failure *and*, very importantly, where it seems reasonable to expect to be able to determine the outcome, seems to be as follows: let Ω be an open subset of $\mathbf{C}^n$ that is *not* a domain of holomorphy and let $\mathcal{A}$ be the sheaf $\mathcal{O}_A$ of all holomorphic A-valued functions on Ω. For the reasons explained in §3, we only have hope of success in case the centre Z of A does not have a Banach space complement. Unfortunately, even for the sheaf $\mathcal{O}_A$, it turns out that, again, all locally inner derivations are inner. We shall give two proofs: one a direct proof and one based on some of the homological ideas expounded by A.J. Helemskii [4]. In fact the homological proof gives a stronger result; if nothing else, therefore, the 'interesting failure' of this section gives yet another good reason for getting to grips with homological algebra!

THEOREM 1. *Let Ω be an open subset of $\mathbf{C}^n$ (or any complex manifold), let A be a complex unital Banach algebra and let $\mathcal{O}_A$ be the sheaf of holomorphic A-valued functions on Ω. Then every locally inner derivation from A into $\mathcal{O}_A(\Omega)$ is inner.*

Proof. Let $D : A \to \mathcal{O}_A(\Omega)$ be a locally inner derivation. Then D can be represented by an open covering $\mathcal{U} \equiv (U_\lambda)_\lambda \in \Lambda$ of Ω and an $\mathcal{O}_A$-valued 0-cocycle, $\mathbf{F} \equiv (f_\lambda)_{\lambda \in \Lambda}$ on $\mathcal{U}$ such that, on every $U_{\lambda\mu} \neq \emptyset$,

$$f_\lambda(x) - f_\mu(x) \in Z \qquad (x \in U_{\lambda\mu}) .$$

We have that $D(a)(x) = af_\lambda(x) - f_\lambda(x)a$ for all $a \in A$, $x \in U_\lambda$, $\lambda \in \Lambda$.

Now let $\pi : A \to A/Z$ be the quotient mapping onto the Banach space A/Z. Since $(\pi \circ f_\lambda)(x) = (\pi \circ f_\mu)(x)$ $(x \in U_{\lambda\mu})$ there is a well-defined holomorphic mapping $\phi : \Omega \to A/Z$ such that, for every $\lambda \in \Lambda$, $\phi \mid U_\lambda = \pi \circ f_\lambda$. But, in consequence of Bishop's representation theorem [1], as pointed out explicitly by J. Leiterer [5], every holomorphic mapping into a quotient Banach space has a holomorphic lift. In particular, therefore, there is a holomorphic function $\psi : \Omega \to A$ such that $\pi \circ \psi = \phi$. Thus, for every $\lambda \in \Lambda$, $\pi \circ (\psi \mid U_\lambda) = \pi \circ f_\lambda$, i.e. $\psi(x) - f_\lambda(x) \in Z \quad (x \in U_\lambda)$. It follows that $D(a) = a.\psi - \psi.a \quad (a \in A)$, so that D is an inner derivation.

The second method of proof gives the stronger result.

THEOREM 2. *With the notation of Theorem 1, every continuous derivation from A into $\mathcal{O}_A(\Omega)$ is inner.*

Proof. By the representation theorem of Grothendieck [3],

$$\mathcal{O}_A(\Omega) \simeq A\hat{\otimes}\mathcal{O}(\Omega)\,,$$

a projective tensor product of Fréchet spaces. Thus $\mathcal{O}_A(\Omega)$ is a free, in particular *projective*, A-bimodule. In particular the Hochschild–Kamowitz cohomology group $H^1(A, \mathcal{O}_A(\Omega)) = 0$, which is the required result.

Remark. The basic complex analysis needed to prove the Grothendieck representation is essentially the same as that needed to prove the lifting result of Leiterer used in the first proof.

5. SOME PROBLEMS

With a distinct taste of sour grapes one might remark that, even had the attempt of §4 succeeded, it would only have supplied a derivation into a Fréchet module, rather than a Banach module, which is what we are really after.

The next thing to try is clearly the following: let Ω be a bounded open subset of $\mathbf{C}^n$ (even the unit disc in $\mathbf{C}$) and let $\mathcal{A}$ be either:

(i) the sheaf of holomorphic A-valued functions with continuous boundary values on Ω; we shall denote this sheaf by $\mathcal{O}_A^c$; or

(ii) the finitely sheaf-like presheaf, say $\mathcal{H}_A^\infty$, of bounded holomorphic A-valued functions on Ω.

Problem. *Either* extend one of the proofs in §4 to deal with the above examples, *or* give a locally inner derivation that is not inner (for suitable A and Ω).

Final Remark. The main aim of this paper has been to suggest a possible pattern for construction of a non-inner derivation. Any hope of success would almost surely involve varying many points of detail! But even a proof that, *in every case*, all locally inner derivations are inner would be interesting.

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