THE WEDDERBURN DECOMPOSITION FOR QUOTIENT ALGEBRAS ARISING FROM SETS OF NON-SYNTHESIS

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1. INTRODUCTION

Let B be a complex commutative unital Banach algebra and let R be the radical for B. We say B has a *Wedderburn splitting* if there exists a subalgebra C of B such that $B = C \oplus R$. If a closed subalgebra C can be found, we say B has a *strong splitting*. The question of the existence of Wedderburn splittings has been investigated in several papers. See for example [3] and [8]. There are algebras B for which no splitting exists [3, Theorem 5.2] and also algebras having an algebraic splitting but no strong splitting [2, Theorem 6.1].

Let G be a non-discrete locally compact Abelian group. Our purpose in this note is to explore the question of a Wedderburn splitting for the non-semisimple quotient algebras of $A(G)$ which arise from compact sets of non-synthesis in G. Let E be such a set of non-synthesis and $J(E)$ be the minimal ideal whose hull is E. In 1961, Katznelson and Rudin [9] moved that $B = A(G)/\overline{J}(E)$ never has an algebraic Wedderburn splitting $B = C \oplus R$ (where $R = \text{rad}\,(B)$ in the case that G is totally disconnected and B is generated by its idempotents. In 1987, Bachelis and Saeki proved without any additional hypotheses, that $A(G)/\overline{J}(E)$ never has a strong splitting. We shall prove here that if $A(G)/\overline{J}(E)$ has an algebraic splitting, then it has a strong one, and, hence, none at all. Actually, the proof shows that $A(G)/H$ never has a splitting, whenever H is a closed ideal satisfying $\overline{J}(E) \subseteq H \subseteq K(E)$, $H \neq K(E)$.

2. PRELIMINARIES

Let A be a complex commutative semi simple Banach algebra with unit 1 and structure space Φ_A. We suppose in the present discussion that A is a Silov algebra. This means that A, considered as a subalgebra of $C(\Phi_A)$, is a normal algebra of functions ([6, Section 39]). If E is a closed subset of Φ_A, we denote by $J(E)$ the ideal of functions $f \in A$ which vanish in neighbourboods of E, and let $K(E) = \{f \in A \mid f(E) = 0\}$. As is well known, $J(E)$ is the smallest ideal whose hull is E, while $\overline{J}(E)$ and $K(E)$ are, respectively, the smallest and largest closed ideals with this property. If $\phi \in \Phi_A$, we write $J(\phi)$ for $J(\{\phi\})$ and $M(\phi)$ for the maximal ideal $K(\{\phi\})$.

The algebra A is called *strongly regular* if $\overline{J}(\phi) = M(\phi)$ for each $\phi \in \Phi_A$. We say *A has bounded relative units* if for each $\phi \in \Phi_A$, there exists a constant $K = K_\phi$ such that for each $g \in J(\phi)$, an element $h \in J(\phi)$ can found so that $gh = g$ and $\|h\| \leq K$. It will be important for us that the algebra $A(\mathbf{T})$ of absolutely convergent Fourier series and many related algebras have these two properties.

The following theorem generalizing part of Theorem 4.3 of [2] is announced without proof in a footnote at the end of that paper. For convenience we prove it here.

2.1 THEOREM. *Let A be a unital Silov algebra which is strongly regular and has bounded relative units. Let $\nu : A \to B$ be an algebraic homomorphism into a Banach algebra $B = \overline{\nu}(A)$. Then ν has a splitting $\nu = \mu + \lambda$, where $\mu : A \to B$ is a continuous homomorphism and $\lambda(A) \subseteq \operatorname{rad} B$.*

Proof. By Theorem 3.7 of [2] there exists a finite subset $F = \{\phi_1, \dots, \phi_n\}$ of Φ_A and a constant M such that

$$\|\nu(f)\| \leq M\|f\|\,\|h\|$$

for each $f,\ h \in J(F)$ such that $fh = f$. Let $\mathcal{K}$ be the subalgebra of A consisting of those functions f which are constant in some neighbourhood of each of the points of F. Now select functions $e_i \in A$ $(1 \leq i \leq n)$ such that $e_i e_j = 0,\ \ i \neq j$, and such that each e_i is identically one in a neighbourhood of the point $\phi_i \in F$.

Then for any fixed $f \in \mathcal{K}(F)$, the function $g = f - \sum_{i=1}^n f(\phi_i)e_i$ belongs to $J(F)$. Choose $h_i \in J(\phi_i)$ so that $gh_i = g$ and $\|h_i\| \leq K = \sup\{K_\phi \mid 1 \leq i \leq n\}$. Then if

$h = h_1 \dots h_n,\ \ h \in J(F)$ and $gh = g$. Hence we have

$$\begin{aligned}
\|\nu(f)\| &= \|\nu(g)\| + \left\| \sum_{i=1}^{n} f(\phi_i)\nu(e_i) \right\| \\
&\le M\|g\|\,\|h\| + \|f\| \sum_{i=1}^{n} \|\nu(e_i)\| \\
&\le MK^n \left(\|f\| + \left\| \sum_{i=1}^{n} f(\phi_i)e_i \right\| \right) + \|f\| \sum_{i=1}^{n} \|\nu(e_i)\| \\
&\le M'\|f\|,
\end{aligned}$$

where M' is a constant independent of $f \in K(F)$.

Define $\mu(f) = \nu(f)\ \ (f \in \mathcal{K}(F))$. Since A is strongly regular, $\mathcal{K}(F)$ is a dense subalgebra of A on which μ is bounded. We denote again by μ its unique continuous extension to all of A. Clearly μ is a homomorphism. Define $\lambda(f) = \nu(f) - \mu(f)\ \ (f \in A)$, so $\nu = \mu + \lambda$.

Finally we show λ maps A into $R = \operatorname{rad} B$. For $\Theta \in \Phi_B$, define $\Theta_\nu = \Theta \circ \nu$ and $\Theta_\mu = \Theta \circ \mu$. Then θ_ν and θ_μ belong to Φ_A and coincide on $K(F)$. Thus $\theta_\nu = \theta_\mu$, so $\theta\big(\lambda(f)\big) = 0\ \ (f \in A)$. Since $\theta \in \Phi_B$ is arbitrary, $\lambda(f) \in R$. Consequently $\nu(A) \subseteq \mu(A) + R$.

Under the hypotheses of the last theorem we cannot prove that $\mu(A)$ is closed or that $\mu(A) \cap R = (0)$. The next theorem gives a special situation in which these two conclusions hold.

2.2 THEOREM. *Let B be a commutative Banach algebra with unit 1 and radical R. Let $A = B/R$ have its quotient norm and suppose that A is a Silov algebra which is strongly regular and has bounded relative units. Let $B = C \oplus R$ be the algebraic direct sum of its radical and a subalgebra C. Then there exists a closed subalgebra D of B such that $B = D \oplus R$.*

Proof. Let $\mathcal{G} : B \to A$ be the Gelfand map. Then the restriction $\mathcal{G} \mid C$ is a norm decreasing isomorphism from C onto A. Let $\nu : A \to C \subset B$ be its inverse. By Theorem 2.1, $\nu = \mu + \lambda$, where μ is a continuous homomorphism and $\lambda(A) \subseteq R$.

Since $\nu(A) \cap R = C \cap R = (0)$, $\mu(a) = 0$ implies $\nu(a) = \lambda(a) \in R$, so $a = 0$. Thus μ is a continuous isomorphism. Moreover, $\mu(A) \cap R = (0)$, since if $\mu(a) = r = \nu(a) - \lambda(a)$, then $\nu(a) \in R$, so $a = 0$ and $r = \mu(a) = 0$.

Finally we note that $\mu(A)$ is closed. For this we note that since $a = \mathcal{G}\big(\mu(a)\big)$, we have $\|a\|_A \leq \|\mu(a)\| \quad (a \in A)$. Hence if $b_0 = \lim_{\mu \to \infty} \mu(a_n)$, then

$$\left\|a_n - a_m\right\|_A \leq \|\mu(a_n) - \mu(a_m)\| \to 0$$

as $m, n \to \infty$. Let $a_n \to a_0 \in A$. Then $\mu(a_n) \Rightarrow \mu(a_0) = b_0$. Since $B = \nu(A) \oplus R \subseteq \mu(A) \oplus R$, we must have $B = \mu(A) \oplus R$, as desired.

Question. Is the strong Wedderburn splitting provided by the last theorem necessarily unique?

3. ELEMENTS IN SUBALGEBRAS COMPLEMENTARY TO THE RADICAL

Let $B = C \oplus R$ be a Wedderburn splitting of a commutative unital Banach algebra B. We are concerned with which elements of B must necessarily lie in C. The easiest result of this sort is the fact that, even if C is not closed, it must contain every idempotent in B. For if $e = c + r = e^2$, it follows that $c^2 = c$, and that $(e-c)^2 = (e-c)^4 = \cdots = (e-c)^{2^n} \quad (n \in \mathbf{N})$. Thus $r = e - c$ cannot lie in R unless $r = 0$. An important result of this type is due to Bachelis and Saeki [1]. They prove that if the splitting is a strong one, i.e. C is closed, then C contains every doubly power bounded element. That is every element $b \in B$ for which $\sup\{\|\ell^n\| \mid n \in \mathbf{Z}\} < \infty$. In the next theorem we identify certain larger classes of elements which must lie in C. These elements were also considered in [4], where it was shown they are necessarily mapped to zero under any bounded derivation from B into a Banach B-module. See also [5].

3.1 THEOREM. *Let B be a commutative unital Banach algebra and let $R = \mathrm{rad}\,(B)$. Suppose that B has a strong Wedderburn splitting $B = C \oplus R$. Let $b \in B$. Suppose* *either that*

$$\text{(i)} \qquad \|\exp(nb)\|\,\|\exp(-nb)\| = o(n) \quad \textit{as } n \to \infty$$

or that b is invertible and

$$\text{(ii)} \qquad \|b^n\|\,\|b^{-n}\| = o(n) \quad \textit{as } n \to \infty\,.$$

Then $b \in C$.

Proof. The proof is essentially the same as that of Bachelis and Saeki [1], taken together with an observation from [5].

Let P be the projection of B onto C with kernel R. Then P is a continuous homomorphism. Let b satisfy (ii) and write $b = C + r$, where $c \in C$ and $r \in R$. Since $1 \in C$, $P(b^{-1}) = c^{-1}$, and $1 + c^{-1}r = c^{-1}b$. But $c^{-1}r \in R$, so we have $Sp(c^{-1}b) = \{1\}$. Moreover

$$\begin{aligned}\|(c^{-1}b)^n\| &\le \|[P(b^{-1}]^n\|\,\|b^n\| \\ &\le \|P\|\,\|(b^{-1})^n\|\,\|b^n\| = o(n)\end{aligned}$$

as $n \to \infty$. By Hille's generalization of a theorem of Gelfand (see [7, 4.10.1]) it follows that $c^{-1}b = 1$, so $b = c \in C$. Now suppose b satisfies (i). Then by what we have just proved, $e^b \in C$. Let $b = c + r$. Then

$$e^b = e^c e^r = e^c + r_1\,,$$

where

$$r_1 = \sum_{n=1}^{\infty} \frac{r^n}{n!} = r\left(1 + \sum_{n=1}^{\infty} \frac{r^n}{(n+1)!}\right).$$

Since $e^c \in C$, $r_1 = 0$, and hence $r = 0$, as the second factor is invertible. Thus $b \in C$. This last part of the proof is taken from [5].

3.2 COROLLARY. [5, Corollary 5.3]. *Let B be a commutative unital Banach algebra containing a family of elements satisfying either (i) or (ii) which has dense span. Then B has no strong Wedderburn splitting.*

4. QUOTIENT ALGEBRAS OF $A(G)$

Let G be a non-discrete locally compact abelian group and let $A(G)$ be the Fourier algebra of G. It is well known that $A(G)$ is a Silov algebra which is unital if and only if G is compact. Let E be a compact subset of G and let $J(E)$ and $K(E)$ have their meanings as in Section 2. If E is not of synthesis, i.e. $\overline{J}(E) \subsetneqq K(E)$, then it is known that there exist infinitely many distinct closed ideals H such that $\overline{J}(E) \subseteq H \subseteq K(E)$ [10]. For such H, $A(G)/H$, with its quotient norm, has structure space E and radical $K(E)/H$. The algebras $A(G)/H$ are a convenient source of non-semisimple commutative Banach algebras. We can now complete the result of Katznelson and Rudin.

4.1 THEOREM. *Let G be a non-discrete locally compact abelian group and let $E \subseteq G$ be a compact set not of synthesis. Let H be a closed ideal in $A(G)$ satisfying $\overline{J}(E) \subseteq H \subseteq K(E)$, $H \neq K(E)$. Then $A(G)/H$ has no algebraic Wedderburn splitting.*

Proof Let $B = A(G)/H$. The Gelfand map $\mathcal{G}$ of B carries B into the restriction algebra $A(E) = A(G)/K(E)$. It is well known that $A(E)$ is a strongly regular Silov algebra which has bounded relative units [10]. If B has an algebraic Wedderburn splitting, then by Theorem 2.2 it has a strong splitting. We now repeat an argument of Bachelis and Saeki [1] to show this is impossible. They show that there is a family of doubly power bounded elements whose span is dense in B. (They consider the case that $H = \overline{J(E)}$, but this is not essential.) Let f be any function in $A(G)$ which is identically one in a neighbourhood of the set E. Then $[f + H]$ is the unit in B, and if γ belongs to the character group Γ of G, $[\gamma f + H]$ is invertible in B. Also

$$[\gamma f + H]^n = [\gamma^n f + H] \qquad (n \in \mathbf{Z}) ,$$

and $[\gamma f + H]$ is doubly power bounded, since

$$\|[\gamma f + H]^n\| \leq \|\gamma^n f\|_{A(G)} = \|f\|_{A(G)} \qquad (n \in \mathbf{Z}) .$$

By [6, Section 40.17], if $g \in A(G)$ there exist sequences $(\alpha_k) \subseteq \mathbf{C}$ and $(\gamma_k) \in \Gamma$ such that $\sum_{n=1}^{\infty} |\alpha_k| < \infty$ and $g = \sum_{n=1}^{\infty} \alpha_k \gamma_k$ on some neighbourhood of E. Then

$$[g + H] = \sum_{n=1}^{\infty} \alpha_k [\gamma_k f + H] ,$$

and an application of Corollary 3.2 completes the proof.

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