ON NORMS ON ALGEBRAS

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1. INTRODUCTION

In order to be quite explicit, let me collect in this section some fundamental definitions.

Throughout, an *algebra* is a linear, associative algebra over a field $\mathbb{F}$, which is always either $\mathbb{R}$ or $\mathbb{C}$. An algebra is *unital* if it has an identity; the identity is usually denoted by e. If A is an algebra without identity, then $A^{\#}$ is the algebra formed by adjoining an identity to A; if A is unital, then $A^{\#} = A$.

1.1. DEFINITION. Let A be an algebra. An *algebra seminorm* on A is a map $\|.\|:A \to \mathbb{R}$ such that:

(i) $\|a\| \geq 0 \quad (a \in A)$;

(ii) $\|a + b\| \leq \|a\| + \|b\| \quad (a,b \in A)$;

(iii) $\|\alpha a\| = |\alpha| \, \|a\| \quad (\alpha \in \mathbb{F}, a \in A)$;

(iv) $\|ab\| \leq \|a\| \, \|b\| \quad (a,b \in A)$.

The map $\|.\|$ is an *algebra norm* if, further:

(v) $\|a\| > 0 \quad (a \in A \setminus \{0\})$.

An algebra A with a norm $\|.\|$ is a *normed algebra*; the normed algebra is a *Banach algebra* if the norm is complete. An algebra A is *normable* if there is an algebra norm on A.

The *spectrum* $\sigma(a)$ of an element a of an algebra A is

$$\sigma(a) = \{z \in \mathbb{C} : ze - a \text{ is not invertible in } A^{\#}\},$$

and the *spectral radius* of a is

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\},$$

where $\nu(a) = 0$ if $\sigma(a) = \emptyset$.

An element a is *quasi-nilpotent* if $\nu(a) = 0$, and we write $\mathcal{Q}(A)$ for the set of quasi-nilpotent elements of A.

Let A be an algebra, and let $a, b \in A$. Then the *quasi-product* of a and b is $a \circ b$, where

$$a \circ b = a + b - ab.$$

An element a is *quasi-invertible* if there exists $b \in A$ with $a \circ b = b \circ a = 0$; we write $\text{q-Inv}\,A$ for the set of quasi-invertible elements of A.

The *radical* of an algebra A, $\operatorname{rad} A$, is the intersection of the maximal modular left ideals of A, so that $\operatorname{rad} A$ is a (two-sided) ideal in A. By definition, A is *semisimple* if $\operatorname{rad} A = \{0\}$.

We have the following standard characterization of $\operatorname{rad} A$.

1.2. PROPOSITION. *Let* A *be an algebra. Then*

$$\operatorname{rad} A = \{a \in A : Aa \subset \text{q-Inv}\,A\}.$$

The most fundamental results about general Banach algebras are the following.

1.3. THEOREM. *Let* A *be a Banach algebra, and let* $a \in A$.

(i) *If* $\nu(a) < 1$, *then* $a \in \text{q-Inv}\,A$.

(ii) $\sigma(a)$ *is a non-empty, compact subset of* $\mathbb{C}$.

(iii) $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

2. UNIQUENESS OF COMPLETE NORM

The first topic that I would like to explore in these lectures is the 'uniqueness of norm problem' for Banach algebras.

2.1. DEFINITION. Let $(A,\|.\|)$ be a Banach algebra. Then A has a *unique complete norm* if, for each algebra norm $|||.|||$ on A such that $(A,|||.|||)$ is a Banach algebra, $|||.|||$ is equivalent to $\|.\|$.

The uniqueness of norm problem is to characterize in some way the class of Banach algebras which have a unique complete norm.

The first result in this area is the classic theorem of Gelfand of 1941. Let me recall a proof that does not use Gelfand theory. First recall ([5]) that, if A is a commutative Banach algebra, if $a_1,\dots,a_n \in A$, and if $\epsilon > 0$, then there is an algebra norm $\|.\|$ on A , equivalent to the given norm, such that

$$\|a_j\| < \nu(a_j) + \epsilon \qquad (j = 1,\dots,n) . \tag{1}$$

It follows that, if A is a Banach algebra and $a,b \in A$ with $ab = ba$, then

$$\nu(a + b) \leq \nu(a) + \nu(b) . \tag{2}$$

2.2. THEOREM. (Gelfand) *Each commutative, semisimple Banach algebra has a unique complete norm.*

Proof. Let $(A,\|.\|)$ be a commutative, semisimple Banach algebra, and let $|||.|||$ be a complete algebra norm on A . Take $(a_n) \subset A$ such that $a_n \longrightarrow 0$ in $(A,\|.\|)$ and $a_n \longrightarrow a$ in $(A,|||.|||)$. For each $b \in A$,

$$\begin{aligned} \nu(ba) &\leq \nu(ba_n) + \nu(b(a-a_n)) \qquad \text{by (2)} \\ &\leq \|b\|\,\|a_n\| + |||b|||\,|||a-a_n||| \\ &\longrightarrow 0 \quad \text{as} \quad n \to \infty . \end{aligned}$$

Thus $ba \in \mathcal{Q}(A)$, $Aa \subset q - \operatorname{Inv} A$, and $a \in \operatorname{rad} A$.

But A is semisimple, so that $a = 0$. By the closed graph theorem, it follows that $|||\cdot|||$ is equivalent to $\|\cdot\|$.

The extension of Gelfand's theorem to general (not necessarily commutative) semisimple Banach algebras was an open question for a long time. The question was brought into prominence by Rickart, who obtained some partial results, and discussed it in Chapter II, §3.5, of his classic treatise ([25]). Eventually the problem was resolved by Johnson ([20]); you will know of the great influence of the ideas of this paper on automatic continuity theory. I had always assumed that Johnson's proof was definitive until the advent of Aupetit's paper ([2]), which contains a different proof. The key fact that Aupetit proved is equation (3), below.

Let E and F be Banach spaces, and let $T:E \to F$ be a linear map. Then the *separating space*, $\mathfrak{S}(T)$, of T is defined to be

$$\mathfrak{S}(T) = \{y \in F: \text{ there exists } (x_n) \subset E \text{ such that } x_n \longrightarrow 0 \text{ and } Tx_n \longrightarrow y\} .$$

Of course, T is continuous if and only if $\mathfrak{S}(T) = \{0\}$.

2.3. THEOREM. (Aupetit) *Let* A *and* B *be Banach algebras, and let* $T:A \to B$ *be a linear map such that* $\nu_B(Ta) \le \nu_A(a) \quad (a \in A)$. *Suppose that* $b \in \mathfrak{S}(T)$. *Then*

$$\nu_B(Ta) \le \nu_B(Ta - b) \quad (a \in A) . \tag{3}$$

The proof that Aupetit gave of this theorem uses some facts about subharmonic functions, and in particular about subharmonic regularizations, that are not quite standard. I now wish to present a proof of the non-commutative version of Gelfand's theorem that is related to Aupetit's proof, but which replaces the background in the theory of subharmonic functions with the modest requirement that one knows the maximum modulus theorem. The proof is due to T.J. Ransford ([24]), and I am grateful to him for permission to give it in these lectures.

The starting point is Hadamard's three circles theorem, which follows immediately from the maximum modulus theorem for analytic functions.

2.4. LEMMA. *Take* R_1, R_2, R_3 *with* $0 < R_1 < R_2 < R_3$, *and suppose that* f *is analytic on a neighbourhood of the annulus* $\{z \in \mathbb{C}: R_1 \le |z| \le R_3\}$. *Set*

$$M_j = \sup\{|f(z)|: |z| = R_j\} \quad (j = 1,2,3) .$$

Then

$$M_2^{\log(R_3/R_1)} \le M_1^{\log(R_3/R_2)} M_3^{\log(R_2/R_1)} . \tag{4}$$

The algebra of polynomials in one indeterminate over an algebra A is denoted by A[X] .

2.5. LEMMA. *Let* A *be a Banach algebra, let* $p \in A[X]$, *and take* $R > 1$. *Then*

$$(\nu(p(1)))^2 \le \sup_{|z|=R} \nu(p(z)) \,.\, \sup_{|z|=1/R} \nu(p(z)) . \tag{5}$$

Proof. Take $q \in A[X]$ and take λ a continuous linear functional on A with $\|\lambda\| = 1$ and $\lambda(q(1)) = \|q(1)\|$, and apply 2.4 with $R_1 = 1/R$, $R_2 = 1$, $R_3 = R$, and $f = \lambda \circ q$. Then, by (4),

$$|f(1)|^2 \le \sup_{|z|=R} |f(z)| \,.\, \sup_{|z|=1/R} |f(z)| ,$$

and so

$$\|q(1)\|^2 \le \sup_{|z|=R} \|q(z)\| \,.\, \sup_{|z|=1/R} \|q(z)\| . \tag{6}$$

Apply (6) with $q = p^{2^n}$, where $n \in \mathbb{N}$. By 1.3(iii) , $\|p^{2^n}(z)\|^{1/2^n} \to \nu(p(z))$ pointwise, and the sequence $(\|p^{2^n}(z)\|^{1/2^n})$ is decreasing, and so (5) follows.

2.6. THEOREM. *Let* A *and* B *be Banach algebras, and let* $T:A \to B$ *be a linear map such that* $\nu_B(Ta) \le \nu_A(Ta)$ $(a \in A)$. *Suppose that* $b \in \mathfrak{S}(T)$. *Then*

$$(\nu_B(Ta))^2 \le \nu_A(a)\nu_B(Ta - b) \qquad (a \in A)\,. \tag{7}$$

Proof. Take $a_n \longrightarrow 0$ in A with $Ta_n \longrightarrow b$ in B, and take $a \in A$ and $\epsilon > 0$. As before, we may suppose that the norms on A and B are such that

$$\|a\| \le \nu_A(a) + \epsilon\,, \quad \|Ta - b\| \le \nu_B(Ta - b) + \epsilon\,.$$

Take $R > 1$, and apply 2.5 with $p(X) = (Ta - Ta_n) + (Ta_n)X \in B[X]$. Then $p(1) = Ta$, and so, by (5),

$$(\nu_B(Ta))^2 \le \sup_{|z|=R} \nu_B(p(z))\,.\, \sup_{|z|=1/R} \nu_B(p(z))\,. \tag{8}$$

By hypothesis, $\nu_B(p(z)) \le \nu_A(a - a_n + za_n)$, and so $\nu_B(p(z)) \le \|a - a_n\| + |z|\,\|a_n\|$. Also $\nu_B(p(z)) \le \|Ta - Ta_n\| + |z|\,\|Ta_n\|$, and so, by (8), we have

$$(\nu_B(Ta))^2 \le (\|a - a_n\| + R\|a_n\|)(\|Ta - Ta_n\| + \tfrac{1}{R}\|Ta_n\|)$$

for all $n \in \mathbb{N}$. Let $n \longrightarrow \infty$ to obtain

$$(\nu_B(Ta))^2 \le \|a\|(\|Ta - b\| + \tfrac{1}{R}\|b\|)\,,$$

and then let $R \longrightarrow \infty$ to obtain

$$\begin{aligned} \nu_B(Ta))^2 &\le \|a\|\ \|Ta - b\|\,, \\ &\le (\nu_A(a) + \epsilon)(\nu_A(Ta - b) + \epsilon)\,. \end{aligned}$$

But this holds for all $\epsilon > 0$, and so the result follows.

Remark. Formula (7) suffices for our purposes, but it is not exactly the same as (3). To obtain (3) itself, we may proceed as follows.

Fix $k \in \mathbb{N}$, and apply Lemma 2.4 with $R_1 = 1/R, R_2 = 1$, and $R_3 = R^k$. Instead of (5), we have

$$(\nu(p(1))^{k+1} \leq \sup_{|z|=R^k} \nu(p(z)) \cdot \left[\sup_{|z|=1/R} \nu(p(z))\right]^k ,$$

and hence

$$(\nu_B(Ta))^{k+1} \leq \nu_A(a)(\nu_B(Ta - b))^k .$$

But now, taking k^{th} roots and letting $k \longrightarrow \infty$, we obtain (3).

We can now prove the general form of Gelfand's theorem.

2.7. THEOREM. *Let* A *be a Banach algebra, let* B *be a semisimple Banach algebra, and let* $\theta:A \to B$ *be an epimorphism. Then* θ *is automatically continuous. In particular, each semisimple Banach algebra has a unique complete norm.*

Proof. Since $\sigma_B(\theta a) \subset \sigma_A(a) \cup \{0\}$ $(a \in A)$, Theorem 2.6 applies.

Take $b_0 \in \mathfrak{S}(\theta)$ and $b \in B$, say $b = \theta a$ and $b_0 = \theta a_0$, where $a, a_0 \in A$. Since $\mathfrak{S}(\theta)$ is an ideal in B, $bb_0 \in \mathfrak{S}(\theta)$, and so, by (7),

$$(\nu_B(bb_0))^2 \leq \nu_A(aa_0)\nu_B(\theta(aa_0) - bb_0) = 0 ,$$

whence $bb_0 \in \mathcal{Q}(B)$. Thus $Bb_0 \subset \mathcal{Q}(B)$, $b \in \text{rad } B$, and $b = 0$, giving the result.

Let us explore a little further the class of Banach algebras that have a unique complete norm. In fact, it is still the case that we have no reasonable characterization of this class, even among commutative Banach algebras, after 50 years.

Certainly one can have commutative Banach algebras A with $\dim \operatorname{rad} A = 1$ which do not have a unique complete norm. The following classical example is due to Feldman (see [3]).

2.8. EXAMPLE. The sequence space ℓ^2 is a Banach algebra with respect to coordinatewise multiplication and the norm

$$\|\alpha\|_2 = \left[\sum_{n=1}^{\infty} |\alpha_n|^2\right]^{\frac{1}{2}} \qquad (\alpha = (\alpha_n) \in \ell^2).$$

Set $A = \ell^2 \oplus \mathbb{C}$, the linear space direct sum, and define multiplication on A by setting

$$(\alpha,z)(\beta,w) = (\alpha\beta, 0).$$

Then A is a commutative algebra and $\operatorname{rad} A = \{0\} \oplus \mathbb{C}$.

Clearly A is a Banach algebra with respect to the norm $\|\cdot\|_1$ given by $\|(\alpha,z)\|_1 = \|\alpha\|_2 + |z|$. Now let λ be any linear functional on ℓ^2 such that $\lambda | \ell^1$ is the functional $(\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n$, and set

$$\|(\alpha,z)\|_2 = \max\{\|\alpha\|_2, |\lambda(\alpha) - z|\} \quad ((\alpha,z) \in A).$$

Then $\|\cdot\|_2$ is a complete norm on A. For $\alpha,\beta \in \ell^2$, we have $\alpha\beta \in \ell^1$, and $|\lambda(\alpha\beta)| \le \sum_{n=1}^{\infty} |\alpha_n\beta_n| \le \|\alpha\|_2\|\beta\|_2$. It follows that $\|\cdot\|_2$ is an algebra norm on A.

Clearly the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent.

For a discussion of the uniqueness of norm problem in Banach algebras with finite-dimensional radical, see [23].

Nevertheless, the class of commutative Banach algebras with a unique complete norm is much wider than the class of semisimple algebras. For example, it contains the following standard examples (see [8]):

(i) the convolution algebras $L^1(\mathbb{R}^+,\omega)$, where ω is a weight function on $\mathbb{R}^+$;

(ii) the radical convolution algebras $L^1_*[0,1]$ and $C_*[0,1]$;

(iii) each Banach algebra of power series.

To prove these results, one argues as follows. Let A and B be commutative Banach algebras, and let $\theta:A \to B$ be an epimorphism. Set $\mathfrak{S} = \mathfrak{S}(\theta)$. Then for each sequence $(b_n) \subset B$, the sequence

$$(\overline{b_1 \ldots b_n \mathfrak{S}})$$

of closed ideals in B is eventually constant. If B is one of the above examples, this can only happen in the case where $\mathfrak{S} = \{0\}$, and so such a θ is then automatically continuous.

A reasonable guess is that every commutative Banach algebra which is an integral domain has a unique complete norm. In this direction, the following has been shown by Cusack ([7]).

2.9. THEOREM. *Assume that there is an integral domain which is a Banach algebra with respect to each of two norms, which are not equivalent to each other. Then there is a commutative, topologically simple Banach algebra.*

(A Banach algebra A is topologically simple if $\dim A > 1$ and if the only closed ideals in A are $\{0\}$ and A. No commutative, topologically simple Banach algebra is known.)

If the existence of topologically simple Banach algebras cannot be ruled out, maybe we should approach the uniqueness of norm problem by attacking these mythical beasts directly. Let me propose the following problem.

2.10. QUESTION. Let A be a commutative algebra which is a topologically simple algebra with respect to each of two norms. Are the norms necessarily equivalent?

3. MINIMUM TOPOLOGIES

I next explore a concept that is related to uniqueness of norm for Banach algebras.

3.1. DEFINITION. Let $(A, \|\cdot\|)$ be a Banach algebra. Then A has a *minimum topology* if, for each algebra norm $|||\cdot|||$ on A, there is a constant K such that $\|a\| \le K|||a|||$ $(a \in A)$.

Clearly, if A has a minimum topology, then A has a unique complete norm.

The fundamental result about these algebras is, of course, Kaplansky's theorem of 1949 ([21]). The uniform norm on a space Ω is denoted by $|\cdot|_\Omega$, and $C_0(\Omega,\mathbb{C})$ is the uniform algebra of all complex-valued, continuous functions on a locally compact space Ω which vanish at infinity.

3.2. THEOREM. *Let Ω be a locally compact space, and let $|||\cdot|||$ be an algebra norm on $C_0(\Omega,\mathbb{C})$. Then*

$$|f|_\Omega \le |||f||| \quad (f \in C_0(\Omega,\mathbb{C})).$$

Thus $C_0(\Omega,\mathbb{C})$ has a minimum norm topology.

I would like to draw your attention to a very old result of Eidelheit, which may even go back to Mazur before 1939. The formulation of the proof is taken from Bonsall ([4]). In the next result, $\|\cdot\|$ denotes the operator norm in $\mathcal{B}(E)$, the algebra of all bounded operators on a Banach space E.

3.3. THEOREM. *Let E be a Banach space, and let A be a subalgebra of $\mathcal{B}(E)$ containing the finite-rank operators. Suppose that $|||\cdot|||$ is an algebra norm on A. Then there exists a constant K such that $\|T\| \le K|||T|||$ $(T \in A)$.*

Proof. To obtain a contradiction, suppose that there is no such constant K. Then there exists $(S_n) \subset A$ with $|||S_n||| = 1$ $(n \in \mathbb{N})$ and such that $\|S_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. By the uniform boundedness theorem, there exists $x_0 \in E\backslash\{0\}$ with $(\|S_n x_0\| : n \in \mathbb{N})$ unbounded, and there exists a continuous linear functional λ on E with $(|\lambda(S_n x_0)| : n \in \mathbb{N})$ unbounded. Set $z_n = \lambda(S_n x_0)$ $(n \in \mathbb{N})$.

Define $Tx = \lambda(x)x_0$ $(x \in E)$. Then T is a rank-one operator, and so $T \in A$. Now, for $x \in E$,

$$\begin{aligned}(TS_nT)(x) &= (TS_n)(\lambda(x)x_0) = \lambda(x)TS_nx_0 \\ &= \lambda(x)\lambda(S_nx_0)x_0 = z_n\lambda(x)x_0 \\ &= z_nTx.\end{aligned}$$

So $TS_nT = z_nT$ and $|z_n|\,|||T||| \leq |||T|||^2\,|||S_n|||$ for $n \in \mathbb{N}$. But $|||T||| > 0$, and so $|z_n| \leq |||T|||$ $(n \in \mathbb{N})$, a contradiction.

Thus the result holds.

3.4. COROLLARY. *Let* E *be a Banach space, and let* A *be a closed subalgebra of* $\mathcal{B}(E)$ *containing the finite-rank operators. Then* A *has a minimum topology.*

That an arbitrary C^*-algebra has a minimum topology is a result of Cleveland from 1963 ([6]). Cleveland's proof uses the *main boundedness theorem* of Bade and Curtis; I give a different proof here.

3.5. LEMMA. *Let* $(A, \|\cdot\|)$ *be a* C^**-algebra, let* $|\!|\!|\cdot|\!|\!|$ *be an algebra norm on* A, *and let* B *be the completion of* $(A, |\!|\!|\cdot|\!|\!|)$. *Then*

$$\|a\|^2 \leq |\!|\!|a|\!|\!|\,|\!|\!|a^*|\!|\!| \qquad (a \in A) \tag{9}$$

and $A \cap \mathcal{Q}(B) = \mathcal{Q}(A)$.

Proof. Take $a \in A$. Then aa^* is self-adjoint, and so $\nu_A(aa^*) = \|aa^*\|$. Set $\Omega = \sigma(aa^*)\setminus\{0\}$. Then $f \mapsto |f(aa^*)|$, $C_0(\Omega,\mathbb{C}) \to \mathbb{R}$, is an algebra norm. By 3.2, $\nu_A(aa^*) \le \pmb{|} aa^* \pmb{|}$, and so

$$\|a\|^2 = \|aa^*\| \le \pmb{|} aa^* \pmb{|} \le \pmb{|} a \pmb{|}\, \pmb{|} a^* \pmb{|}\ ,$$

giving (9).

Certainly $\mathcal{Q}(A) \subset A \cap \mathcal{Q}(B)$. Now take $a \in A \cap \mathcal{Q}(B)$. By (9), we have $\|a^n\|^{2/n} \le \pmb{|} a^n \pmb{|}^{1/n} \pmb{|} (a^*)^n \pmb{|}^{1/n}$, and so $\nu_A(a)^2 \le \nu_B(a)\nu_B(a^*) = 0$. Thus $a \in \mathcal{Q}(A)$.

3.6. THEOREM. *Let* $(A, \|\cdot\|)$ *be a* C^**-algebra, and let* $|||\cdot|||$ *be an algebra norm on* A. *Then there exists* $K > 0$ *such that* $\|a\| \le K|||a|||\quad (a \in A)$.

Proof. Let B be the completion of $(A, |||\cdot|||)$, and let $\mathfrak{S}$ be the separating space of the natural embedding of A in B, so that $\mathfrak{S}$ is a closed ideal in B. Let $\pi : B \to B/\mathfrak{S}$ be the quotient map. Then $\pi\ (A, \|\cdot\|) \to B/\mathfrak{S}$ is a continuous map : take K to be the norm of this map, and set $\pmb{|} a \pmb{|} = \|\pi(a)\|\quad (a \in A)$, so that $\pmb{|} \cdot \pmb{|}$ is an algebra seminorm on A and $\pmb{|} a \pmb{|} \le K\|a\|$ and $\pmb{|} a \pmb{|} \le |||a|||$ for all $a \in A$.

By (3) or (7), $A \cap \mathfrak{S} \subset \mathcal{Q}(B)$, and so, by 3.5, $A \cap \mathfrak{S} \subset \mathcal{Q}(A)$. Since $A \cap \mathfrak{S}$ is an ideal in A, $A \cap \mathfrak{S} \subset \operatorname{rad} A$. But A is semisimple, and so $A \cap \mathfrak{S} = \{0\}$. This shows that $\pmb{|} \cdot \pmb{|}$ is actually a norm on A, and so, by (9), $\|a\|^2 \le \pmb{|} a \pmb{|}\, \pmb{|} a^* \pmb{|}\quad (a \in A)$. Thus, for $a \in A$,

$$\|a\|^2 \le K \pmb{|} a \pmb{|}\, \|a^*\| = K \pmb{|} a \pmb{|}\, \|a\|\ ,$$

and so $\|a\| \le K \pmb{|} a \pmb{|} \le K|||a|||$.

4. ORDERED FIELDS

Before introducing the next topic, it is necessary that I recall some standard background from the theory of ordered fields.

Let G be a group (always taken to be abelian and written additively, with identity 0), and let $\leq$ be a partial order on G. Then $(G,\leq)$ is an *ordered group* if $\leq$ is a total order and if $x + z \leq y + z$ whenever $x,y,z \in G$ and $x \leq y$. For $x \in G$, set $|x| = x \vee (-x)$.

Now let K be a real field (so that K is a field with prime field $\mathbb{R}$), and let $\leq$ be an order on K. Then $(K,\leq)$ is an *ordered field* if $(K,+,\leq)$ is an ordered group, if $\alpha a \geq 0$ in K whenever $\alpha \geq 0$ in $\mathbb{R}$ and $a \geq 0$ in K, and if $ab \geq 0$ in K whenever $a,b \geq 0$ in K. We set $K^+ = \{a \in K : a \geq 0\}$.

The real field K has identity 1, and we identify $\mathbb{R}$ with $\mathbb{R}1$. Let $a \in K$. Then a is an *infinitesimal* of K if $n|a| < 1$ $(n \in \mathbb{N})$, a is *finite* if there exists $n \in \mathbb{N}$ with $|a| \leq n$, and a is *infinitely large* if $|a| \geq n$ $(n \in \mathbb{N})$. We write K^* and $K^{\#}$ for the infinitesimals and for the finite elements of K, respectively. Then $K^{\#}$ is a unital subalgebra of K, and K^* is its unique maximal ideal.

The question that I wish to explore here is the following.

4.1. QUESTION. Under what conditions on K is the algebra K^* normable?

Let us give an example of such a field K. Let Ω be a compact space, and let $C(\Omega)$ denote the *real*-valued, continuous functions on Ω. For $f,g \in C(\Omega)$, set

$$f \leq g \quad \text{if} \quad f(x) \leq g(x) \qquad (x \in \Omega).$$

Then $(C(\Omega),\leq)$ is a partially ordered set. For $x \in \Omega$, set $M_x = \{f : f(x) = 0\}$ and $J_x = \{f : f = 0 \text{ on a neighbourhood of } x\}$, so that M_x is a maximal ideal of $C(\Omega)$, J_x is an ideal, and $J_x \subset \bar{J}_x = M_x$. Let P be a prime ideal in $C(\Omega)$. Then there exists a unique point $x \in \Omega$ with $J_x \subset P \subset M_x$; the corresponding maximal ideal is termed M_P. Now take $f \in P$ and $g \in C(\Omega)$ with $0 \leq |g| \leq |f|$. Set $h(x) = g^2(x)/f(x)$ when $f(x) \neq 0$ and $h(x) = 0$ when $f(x) = 0$. Then $h \in C(\Omega)$ and $g^2 = fh$ in $C(\Omega)$. So $g^2 \in P$, and $g \in P$ because P is prime. This shows that P is as an *absolutely convex* ideal in $C(\Omega)$.

4.2. DEFINITION. Let Ω be a compact space, and let P be a prime ideal in $C(\Omega)$. Then

$$A_P = C(\Omega)/P\ ,$$

and $\pi_P : C(\Omega) \to A_P$ is the quotient map. Set $a \geq 0$ in A_P if $a = \pi_P(f)$, where $f \geq 0$ in $C(\Omega)$.

It follows from the fact that P is an absolutely convex ideal in $C(\Omega)$ that $\leq$ is well-defined on A_P. The key fact about this order on A_P is that it is a total order. For let $a \in A_P$, say $a = \pi_P(f)$, where $f \in C(\Omega)$. Then $f = f^+ + f^-$, where $f^+, f^- \in C(\Omega)$, $f^+ \geq 0$, $f^- \leq 0$, and $f^+f^- = 0$. Either $f^+ \in P$ or $f^- \in P$, and so either $a \leq 0$ or $a \geq 0$ in A_P.

The quotient field of A_P will be denoted K_P. It is easy to see that the order $\leq$ on A_P induces an order $\leq$ on K_P, extending the order on A_P, such that $(K_P, \leq)$ is an ordered field. Clearly $K_P{}^* = M_P/P$ and $K_P{}^\# = C(\Omega)/P$, and so a special case of the first question is the following.

4.3. QUESTION. Under what conditions on P is the algebra M_P/P normable?

Our interest in Question 4.3 arose because of its connection with the question of the existence of discontinuous homomorphisms from $C(\Omega,\mathbb{C})$. (See [9], [12]).

4.4. THEOREM. *Let Ω be a compact space. Then the following conditions are equivalent:*

(a) *there is a discontinuous homomorphism from $C(\Omega,\mathbb{C})$ into some Banach algebra;*

(b) *there is an algebra norm on $C(\Omega,\mathbb{C})$ which is not equivalent to the uniform norm;*

(c) *there is a non-maximal, prime ideal P in $C(\Omega)$ such that $K_P{}^* = M_P/P$ is normable.*

However, Questions 4.1 and 4.3 also suggest the following question: for which ordered fields K is there a compact space Ω and a non-maximal, prime ideal P in $C(\Omega)$ such that $K = K_P$ (in the sense that K is isotonically isomorphic to K_P)? I think that this is an interesting question. We shall return to it later.

Let me now describe a yet more special example. For background on ultrafilters and Stone-Čech compactifications, see [18].

Let κ be an infinite cardinal, and let $\mathcal{U}$ be a free ultrafilter on κ. For $f,g \in \mathbb{R}^\kappa$, set

$$f \sim g \quad \text{if} \quad \{\sigma < \kappa : f(\sigma) = g(\sigma)\} \in \mathcal{U}\,.$$

Then $\sim$ is an equivalence relation on $\mathbb{R}^\kappa$, and the set of equivalence classes is denoted by $\mathbb{R}^\kappa/\mathcal{U}$: this is an *ultrapower* of $\mathbb{R}$. The equivalence class containing f is $[f]$. We define

$$[f] + [g] = [f + g]\,,\ \alpha[f] = [\alpha f]\,,\ [f][g] = [fg]\,,$$

and set

$$[f] < [g] \ \text{if} \ \{\sigma < \kappa : f(\sigma) < g(\sigma)\} \in \mathcal{U}$$

for $f,g \in \mathbb{R}^\kappa/\mathcal{U}$ and $\alpha \in \mathbb{R}$. It is standard (e.g., [9]) that $(\mathbb{R}^\kappa/\mathcal{U}, \leq)$ is an ordered field.

We recognise this ultrapower as a special case of the field K_P as follows. Let $\beta\kappa$ be the Stone-Čech compactification of the (discrete) space κ. Then $\beta\kappa$ is the collection of ultrafilters on κ, and a point $\mathfrak{p} \in \beta\kappa \setminus \kappa$ corresponds to the free ultrafilter $\mathcal{U} = \{U \cap \kappa : U$ is a neighbourhood of $\mathfrak{p}$ in $\beta\kappa\}$. In this situation, $\ell^\infty(\kappa)$, the space of all bounded, real-valued sequences on κ, is identified with $C(\beta\kappa)$, and the ideal $J_\mathfrak{p}$ is a prime ideal in $C(\beta\kappa)$. Then we have

$$\left[\frac{\mathbb{R}^\kappa}{\mathcal{U}}\right]^* = \frac{M_\mathfrak{p}}{J_\mathfrak{p}}\,, \quad \left[\frac{\mathbb{R}^\kappa}{\mathcal{U}}\right]^\# = \frac{C(\beta\kappa)}{J_\mathfrak{p}}\,.$$

Thus a third variant of our question is the following.

4.4. **QUESTION.** Under what conditions on κ and $\mathcal{U}$ is the algebra $(\mathbb{R}^{\kappa}/\mathcal{U})^*$ normable?

5. NECESSARY CONDITIONS AND SUFFICIENT CONDITIONS

In this section I give necessary conditions and sufficient conditions on a field K for the algebra K^* to be normable. First I recall the definition of the value group of a field.

Let $(G, \leq)$ be an ordered group, and let $x,y \in G$. Then $x \precsim y$ if there exists $n \in \mathbb{N}$ such that $|y| \leq n|x|$, and $x \sim y$ if both $x \precsim y$ and $y \precsim x$. Clearly $\sim$ is an equivalence relation on G. The quotient $\Gamma = (G\backslash\{0\})/\sim$ is the *value set* of G, the elements of Γ are the *archimedean classes* of G, and the quotient map $v:(G\backslash\{0\}) \to \Gamma$ is the *archimedean valuation.* We note that $v(x+y) \geq v(x) \wedge v(y)$ $(x,y \in G)$, and $v(-x) = v(x)$ $(x \in G)$; for convenience we set $v(0) = \infty$.

For $x,y \in G$, set $v(x) \leq v(y)$ if $x \precsim y$. Then $\leq$ is well-defined on Γ, and $(\Gamma, \leq)$ is a totally ordered set.

Now let $(K, \leq)$ be an ordered field, with value set Γ_K. For $a,b \in K\backslash\{0\}$, set

$$v(a) + v(b) = v(ab) .$$

Then $+$ is well-defined on Γ_K, and it is easily checked that $(\Gamma_K, \leq)$ is an ordered group; it is the *value group* of K. We have $K^* = \{a \in K : v(a) > 0\}$ and $K^{\#} = \{a \in K: v(a) \geq 0\}$.

Throughout we denote the cardinality of a set S by $|S|$. We shall refer to ω, the first infinite ordinal, and to ω_1, the first uncountable ordinal; we have $|\omega| = \aleph_0$ and $|\omega_1| = \aleph_1$. The cardinal of the continuum is $c = 2^{\aleph_0}$. The Continuum Hypothesis (CH) is the assertion that $c = \aleph_1$, and the Generalized Continuum Hypothesis (GCH) implies also that $2^{\aleph_1} = \aleph_2$. If we appeal to CH or GCH in a proof, this will be specifically stated.

The following theorem is essentially contained in [12].

5.1. THEOREM. *Let* K *be an ordered field, and suppose that* K^* *is normable. Then* $|\Gamma_K| \leq \mathfrak{c}$.

Proof. Let $\|.\|$ be an algebra norm on K^* . Take $a,b \in K^*\setminus\{0\}$ with $v(a) > v(b)$. Then there exists $c \in K^*\setminus\{0\}$ with $a = bc$. We have

$$\|a^n\| \leq \|b^n\| \; \|c^n\| \qquad (n \in \mathbb{N}) .$$

Since $c \in K^* = \operatorname{rad} K^{\#}$, $\|c^n\|^{1/n} \longrightarrow 0$ as $n \longrightarrow \infty$, and so $\|c^n\| < 1$ eventually. Thus $\|a^n\| < \|b^n\|$ eventually.

Consider the map $\varphi : a \mapsto (\|a^n\|)$, $K^* \to \mathbb{R}^{\mathbb{N}}$. The above paragraph shows that, if $\varphi(a) = \varphi(b)$, then $v(a) = v(b)$. Thus φ induces an injection from Γ_K into $\mathbb{R}^{\mathbb{N}}$, and so $|\Gamma_K| \leq |\mathbb{R}^{\mathbb{N}}| = \mathfrak{c}$.

The following theorem, and other unacknowledged results in the remainder of this report, are taken from a paper of myself and W.H. Woodin ([10]).

5.2. THEOREM. *Let* K *be an ordered field such that* $|\Gamma_K| = \kappa$, *where* κ *is an infinite cardinal. Then :*

(i) K *does not contain any strictly increasing sequence of length* κ^+ ;

(ii) $|K| \leq 2^{\kappa}$.

Proof. (i) Set $\lambda = \kappa^+$ (so that λ is the successor cardinal to κ).

To obtain a contradiction, suppose that $(a_\sigma : \sigma < \lambda)$ is a strictly increasing sequence of length λ in $(K, \leq)$.

We begin with two choices. First, for each $s \in \Gamma_K$, choose $x_s > 0$ in K with $v(x_s) = s$. Second, for each $\alpha > 0$ in $\mathbb{R}$, choose $q(\alpha) \in \mathbb{R}$ with $q(\alpha) \in (\frac{3}{4}\alpha, \frac{3}{2}\alpha)$; we have $|\alpha - q(\alpha)| < \frac{1}{3}q(\alpha)$.

We shall now define a map $\Phi : \lambda \to \Gamma_K \times \mathbb{Q}$. For each $\sigma < \lambda$, consider the sequence $(v(a_\tau - a_\sigma) : \sigma < \tau < \lambda)$. This is an increasing sequence of length λ in Γ_K , and so the sequence is eventually constant because $|\Gamma_K| < \lambda$; this constant value is s_σ, say. For each τ such that $v(a_\tau - a_\sigma) = s_\sigma$, there exists $\alpha_{\tau,\sigma} \in \mathbb{R}$ such that

$$(a_\tau - a_\sigma)/x_{s_\sigma} \in \alpha_{\tau,\sigma} 1 + K^* .$$

The sequence $(\alpha_{\tau,\sigma})$ is an increasing sequence of length λ in $\mathbb{R}$. Since λ is a regular cardinal ([22, 10.37]), and since $\lambda \geq \aleph_1$, the sequence $(\alpha_{\tau,\sigma})$ is eventually constant; this constant value is α_σ, say. Now define

$$\Phi : \sigma \mapsto (s_\sigma, q(\alpha_\sigma)) , \quad \lambda \to \Gamma_K \times \mathbb{Q} .$$

The cardinality of $\Gamma_K \times \mathbb{Q}$ is κ, and so there is a cofinal subset of λ such that Φ takes a constant value, say (s,β), on this subset. Thus we may successively choose $\sigma_1, \sigma_2, \sigma_3 < \lambda$ such that $\sigma_3 > \sigma_2 > \sigma_1$ and such that

$$v(a_{\sigma_2} - a_{\sigma_1}) = v(a_{\sigma_3} - a_{\sigma_1}) = v(a_{\sigma_3} - a_{\sigma_2}) = s ,$$
$$(a_{\sigma_2} - a_{\sigma_1})/x_s \in \alpha_{\sigma_1} 1 + K^* , \ (a_{\sigma_3} - a_{\sigma_1})/x_s \in \alpha_{\sigma_1} 1 + K^* ,$$
$$(a_{\sigma_3} - a_{\sigma_2})/x_s \in \alpha_{\sigma_2} 1 + K^* , \ q(\alpha_{\sigma_1}) = q(\alpha_{\sigma_2}) = \beta .$$

Since $\alpha_{\sigma_1} \in (\frac{2}{3}\beta, \frac{4}{3}\beta)$, it follows that

$$|a_{\sigma_2} - a_{\sigma_1} - \beta x_s| < \tfrac{1}{3}\beta x_s \tag{10}$$

and that

$$|a_{\sigma_3} - a_{\sigma_1} - \beta x_s| < \tfrac{1}{3}\beta x_s . \tag{11}$$

Since $a_{\sigma_2} \in (\frac{2}{3}\beta, \frac{4}{3}\beta)$, it follows that

$$|a_{\sigma_3} - a_{\sigma_2} - \beta x_s| < \tfrac{1}{3}\beta x_3 . \tag{12}$$

By (10) and (12), $a_{\sigma_3} - a_{\sigma_2} > \frac{4}{3}\beta x_s$, and, by (11) $a_{\sigma_3} - a_{\sigma_2} < \frac{4}{3}\beta x_s$. This is the required contradiction.

(ii) This is now an immediate consequence of the standard Erdös-Rado theorem of infinite combinatorics (e.g.[22,VIII, (B.1)]). To obtain a contradiction, suppose that $|K| \geq (2^\kappa)^+$, and let $\preceq$ be a well-ordering of K. Define a map φ from the set of pairs of elements of K into $\{0,1\}$ by setting

$$\varphi(\{a,b\}) = \begin{cases} 0 \text{ if } < \text{ and } \prec \text{ coincide on } \{a,b\}, \\ 1 \text{ otherwise.} \end{cases}$$

Then the Erdös-Rado theorem asserts that K contains a homogeneous set S of cardinality κ^+ : this means that φ is constant when restricted to pairs of elements of S. Thus K contains a set S with $|S| = \kappa^+$ such that either S or $-S$ is well-ordered by the original ordering $\leq$, and so K contains a strictly increasing sequence of length κ^+. This is a contradiction of (i).

5.3. THEOREM. *Let* K *be an ordered field such that* K* *is normable. Then* :

(i) $|K| \leq 2^{\mathbf{c}}$;

(ii) (GCH) $|K| \leq \aleph_2$.

Proof. (i) By 5.1, $|\Gamma_K| \leq \mathbf{c}$, and so, by 5.2(ii), $|K| \leq 2^{\mathbf{c}}$.

(ii) With GCH, $\mathbf{c} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$.

Thus we have obtained a necessary condition on K for K* to be normable. Before turning to a sufficient condition, I wish to raise another open question.

Let $(S, \leq)$ be a totally ordered set. The *weight of* S, $w(S)$, is the minimum cardinal of an order-dense subset of S.

5.4. QUESTION. Let K be an ordered field such that K* is normable. Is $w(K) = \mathbf{c}$?

It does not follow from the fact that $|\Gamma_K| = \mathbf{c}$ that $w(K) = \mathbf{c}$ (see [10]). If $w(K) = \mathbf{c}$, then certainly this implies that $|K| \leq 2^{\mathbf{c}}$.

The sufficient condition for the algebra K* to be normable is a consquence of a very deep theorem of Esterle ([14, 5.3(i)]). I will make further remarks about this theorem after 6.15 below.

5.5. THEOREM. (Esterle) *Let* K *be an ordered field such that* $|K| = \aleph_1$. *Then* K^* *is normable.*

It should be said that, as stated, the above theorem may be a little misleading: necessarily each real field has cardinality at least c, and so the hypotheses of the above theorem are vacuous unless CH holds.

Let K be an ordered field, and assume that GCH holds. Then we know whether or not K^* is normable in every case except where $|K| = \aleph_2$ and $|\Gamma_K| = \aleph_1$; if $|K| > \aleph_2$ or if $|\Gamma_K| > \aleph_1$, then K is not normable, if $|K| < \aleph_2$, then K is normable, and the case where $|\Gamma_K| < \aleph_1$ and $|K| = \aleph_2$ does not arise.

However, it is not entirely obvious that the case where $|K| = \aleph_2$ and $|\Gamma_K| = \aleph_1$ arises. In particular we ask whether or not it arises in the case where K has the form K_P for some non-maximal, prime ideal P in some $C(\Omega)$.

6. AN EXAMPLE.

In this section I shall give a construction of a field K of formal power series with $|K| = \aleph_2$ and $|\Gamma_K| = \aleph_1$ (with GCH). First I require some standard notions about ordered sets.

6.1. DEFINITION. Let $(S, \leq)$ be a totally ordered set. Then S is:

(i) an α_1*-set* if each non-empty subset of S has a countable coinitial and cofinal subset;

(ii) a β_1*-set* if $S = \cup S_\nu$, where $\{S_\nu\}$ is an increasing chain of α_1-subsets;

(iii) an η_1*-set* if, for each pair $\{S_1, S_2\}$ of countable subsets of S such that $s_1 < s_2$ whenever $s_1 \in S_1$ and $s_2 \in S_2$, there exists $s \in S$ with $s_1 < s < s_2$ for all $s_1 \in S_1$ and $s_2 \in S_2$;

(iv) a *semi-η_1-set* if, for each strictly increasing sequence (s_n) and each strictly decreasing sequence (t_n) in S such that $s_n < t_n$ $(n \in \mathbb{N})$, there exists $s \in S$ with $s_n < s < t_n$ $(n \in \mathbb{N})$.

6.2. EXAMPLE. Let S be the set of all sequences $\alpha = (\alpha_\tau : \tau < \omega_1)$ of length ω_1 such that $\alpha_\tau \in \{0,1\}$ $(\tau < \omega_1)$, and let $\mathbf{Q}$ be the set of elements α of S such that $\{\tau < \omega_1 : \alpha_\tau = 1\}$ has a largest element.

If α and β are distinct elements of S, then there exists a minimum ordinal σ such that $\alpha_\sigma \neq \beta_\sigma$: set $\alpha \prec \beta$ if $\alpha_\sigma = 0$ and $\beta_\sigma = 1$, and set $\alpha \preceq \beta$ if $\alpha \prec \beta$ or if $\alpha = \beta$. Then $\preceq$ is a total order on S, called the *lexicographic order.*

The set $\mathbf{Q}$ was introduced by Sierpiński. Note that, for $\alpha = (\alpha_\tau) \in \mathbf{Q}$, $\alpha_\tau = 0$ for all but countably many values of τ, and so one may think of $\mathbf{Q}$ as the analogue of $\mathbb{Q}$ "one cardinal higher". The following result is proved in [18, Chapter 13].

6.3. PROPOSITION. *The set* $(\mathbf{Q}, \preceq)$ *is a totally ordered* β_1-η_1-*set of cardinality* $\mathbf{c}$.

6.4. DEFINITION. Let S be a totally ordered set. Then

$$\mathfrak{F}(\mathbb{R},S) = \{f \in \mathbb{R}^S : \text{supp } f \text{ is well ordered}\},$$

$$\mathfrak{F}_{(1)}(\mathbb{R},S) = \{f \in \mathfrak{F}(\mathbb{R},S) : \text{supp } f \text{ is countable}\}.$$

Here, for $f \in \mathbb{R}^S$, $\text{supp } f = \{s \in S : f(s) \neq 0\}$. Since $\text{supp}(f-g) \subset \text{supp } f \cup \text{supp } g$ $(f,g \in \mathbb{R}^S)$, $\mathfrak{F}(\mathbb{R},S)$ and $\mathfrak{F}_{(1)}(\mathbb{R},S)$ are subgroups of $\mathbb{R}^S$; indeed, they are real linear spaces, and hence divisible groups. For $f \in \mathfrak{F}(\mathbb{R},S)$ with $f \neq 0$, set $v(f) = \inf \text{supp } f$, and set $f > 0$ if $f(v(f)) > 0$ in $\mathbb{R}$. Then $(\mathfrak{F}(\mathbb{R},S), \leq)$ is a totally ordered group, and $f \sim g$ in $\mathfrak{F}(\mathbb{R},S)$ if and only if $v(f) = v(g)$. Thus we can identify S as the value set of $\mathfrak{F}(\mathbb{R},S)$ and of $\mathfrak{F}_{(1)}(\mathbb{R},S)$, and v with the archimedean valuation on each of these groups.

6.5. DEFINITION. Let $(G,\leq)$ be a totally ordered group. Then G is :

(i) an α_1*-group* if $(G,\leq)$ is an α_1-set ;

(ii) a β_1*-group* if $G = \cup G_\nu$, where $\{G_\nu\}$ is an increasing chain of α_1-subgroups;

(iii) an η_1*-group* if $(G,\leq)$ is an η_1-set.

Part (i) of the following theorem is [1, 2.2] and part (ii) is [13, 2.3b].

6.6. PROPOSITION. *Let* S *be a totally ordered set.*

(i) *If* S *is an* η_1-set , *then* $\mathfrak{F}(\mathbb{R},S)$ *and* $\mathfrak{F}_{(1)}(\mathbb{R},S)$ *are both* η_1*-groups.*

(ii) *If* S *is a* β_1-set, *then* $\mathfrak{F}_{(1)}(\mathbb{R},S)$ *is a* β_1*-group.*

6.7. DEFINITION. Set $G = \mathfrak{F}_{(1)}(\mathbb{R},\mathbb{Q})$.

6.8. PROPOSITION. *The set* G *is a totally ordered, divisible,* β_1-η_1*-group* , *and* $|G| = c$.

6.9. DEFINITION. Let G be a totally ordered group. For $f,g \in \mathfrak{F}(\mathbb{R},G)$, set

$$(f * g)(s) = \sum\{f(t_1)g(t_2) : t_1,t_2 \in G\ ,\ t_1 + t_2 = s\} \qquad (s \in G)\ . \tag{13}$$

One checks that the sum in (13) is a finite sum, and then it is easy to verify that $*$ is a commutative, associative product, that $\mathfrak{F}(\mathbb{R},G)$ is a real algebra, and that $\mathfrak{F}_{(1)}(\mathbb{R},G)$ is a subalgebra of $\mathfrak{F}(\mathbb{R},G)$.

The algebra $\mathfrak{F}(\mathbb{R},G)$ is the *formal power series algebra over* G; it was introduced by Hahn in 1907 ([19]). See [17], for example.

We write X^s for the characteristic function of $\{s\}$ for $s \in G$. Then $X^s \in \mathfrak{F}(\mathbb{R},G)$ and $X^s * X^t = X^{s+t}$ $(s,t \in G)$. One can think of a typical element of $\mathfrak{F}(\mathbb{R},S)$ as having the form $\Sigma\{\alpha_s X^s : s \in G\}$, where $\{\alpha_s\} \subset \mathbb{R}$; the formula for the product is consistent with this symbolism.

For example, $\mathfrak{F}(\mathbb{R},\mathbb{Z})$ is the set of Laurent series of the form $\sum_{n=n_0}^{\infty} \alpha_n X^n$, where $n_0 \in \mathbb{Z}$ and $(\alpha_n) \subset \mathbb{R}$. This algebra is also denoted by $\mathbb{R}((X))$; it is the quotient field of $\mathbb{R}[[X]]$, the algebra of formal power series in one indeterminate over $\mathbb{R}$.

Part (i) of the following theorem is a slight extension of a classical result of Hahn ([17, page 137]), and part (ii) is [1, 3.2]. A real field K is *real-closed* if its complexification $K(\sqrt{-1})$ is algebraically closed.

6.10. PROPOSITION. *Let* G *be a totally ordered group.*

(i) $\mathfrak{F}(\mathbb{R},G)$ *and* $\mathfrak{F}_{(1)}(\mathbb{R},G)$ *are totally ordered real fields, with value groups* G.

(ii) *If* G *is a divisible group, then* $\mathfrak{F}(\mathbb{R},G)$ *and* $\mathfrak{F}_{(1)}(\mathbb{R},G)$ *are real-closed fields.*

6.11. DEFINITION. Let $(K,\leq)$ be an ordered field. Then K is :

(i) an α_1*-field* if $(K,\leq)$ is an α_1-set;

(ii) a β_1*-field* if $K = \cup K_\nu$, where $\{K_\nu\}$ is an increasing chain of α_1-real-subfields;

(iii) a *(semi)-*η_1*-field* if $(K,\leq)$ is a (semi)-η_1-set.

The following result is a small variation of results in [18]. For details, see [8,§ 3.5], which is available on request.

6.12. PROPOSITION. *Let* Ω *be a compact space, and let* P *be a non-maximal, prime ideal in* $C(\Omega)$. *Then* :

(i) K_P *is a real-closed field;*

(ii) K_P *is a semi-*η_1*-field;*

(iii) K_P *is an* η_1*-field if and only if* $K_P^+\setminus\{0\}$ *does not contain a strictly decreasing, coinitial sequence.*

In particular, each ultrapower $\mathbb{R}^{\kappa}/\mathcal{U}$ is a real-closed, η_1-field. Thus, since we hope eventually to have examples of fields K of the form K_P with $|K| = \aleph_2$ and $|\Gamma_K| = \aleph_1$, we should restrict ourselves to real-closed, η_1-fields.

6.13. THEOREM.

(i) $\mathfrak{F}_{(1)}(\mathbb{R},G)$ *is a totally ordered, real-closed, β_1-η_1-field of cardinality* c .

(ii) $\mathfrak{F}(\mathbb{R},G)$ *is a totally ordered, real-closed, η_1-field of cardinality* $2^{\aleph_1}$.

Proof. All of this theorem follows easily from earlier results, save perhaps for the claim that $|K| = 2^{\aleph_1}$, where $K = \mathfrak{F}(\mathbb{R},G)$: we just prove that $|K| \geq 2^{\aleph_1}$.

For $\sigma < \omega_1$, δ_σ is the sequence in $\mathbf{Q}$ with 1 in the σ^{th} position and 0 elsewhere. Then $(\delta_\sigma : \sigma < \omega_1)$ is a strictly decreasing sequence in $\mathbf{Q}$. Now set

$$\mu_\sigma = \chi_{\delta_1} - \chi_{\delta_{\sigma+1}} \qquad (\sigma < \omega_1) .$$

Then $S = \{\mu_\sigma : \sigma < \omega_1\}$ is a strictly increasing sequence of length ω_1 in G . Each $f \in \mathbb{R}^S$ belongs to K , and so $|K| \geq |\mathbb{R}^S| = (2^{\aleph_0})^{\aleph_1} = 2^{\aleph_1}$.

We have now obtained the example requested at the end of §5.

6.14. EXAMPLE. Set $K = \mathfrak{F}(\mathbb{R},G) = \mathfrak{F}(\mathbb{R}, \mathfrak{F}_{(1)}(\mathbb{R},\mathbf{Q}))$. Then K is a totally ordered, real-closed, η_1-field with $|K| = 2^{\aleph_1}$ and $|\Gamma_K| = 2^{\aleph_0}$.

Of course, with GCH , $|K| = \aleph_2$ and $|\Gamma_K| = \aleph_1$.

Let me conclude this section by discussing the field $\mathfrak{F}_{(1)}(\mathbb{R},G)$. A strengthened form of Theorem 5.5 holds; this important theorem was given by Esterle in [15], and a detailed proof is given in [8].

6.15. THEOREM.

(i) *The set of infinitesimals in the field* $\mathfrak{F}_{(1)}(\mathbb{R},G)$ *is normable.*

(ii) *Let* K *be any real-closed, totally ordered,* β_1*-field. Then there is an isotonic isomorphism from* K *into* $\mathfrak{F}_{(1)}(\mathbb{R},G)$, *and so* K^* *is normable.*

To deduce Theorem 5.5 from this theorem, we proceed as follows. Let K be an ordered field with $|K| = \aleph_1$. Then K has a real-closure : there is a real-closed, ordered field L and an isotonic embedding of K into L such that each element $a \in L$ is a root of some $p \in K[X]$. Thus $|L| = \aleph_1$. This implies fairly easily that L is a β_1-field. By 6.15(ii) , L^* is normable. The importance of Esterle's theorem 6.15 is that it holds in the theory ZFC, whereas 5.5 is vacuous unless CH holds.

The following well-known result is rather straightforward (it is essentially [9, 1.13]).

Let $\mathfrak{p} \in \beta\mathbb{N}\backslash\mathbb{N}$. Then we set

$$A_{\mathfrak{p}} = C(\beta\mathbb{N})/J_{\mathfrak{p}} ,$$

and we write $K_{\mathfrak{p}}$ for the quotient field of $A_{\mathfrak{p}}$. Thus $K_{\mathfrak{p}}$ has the form $\mathbb{R}^{\omega}/\mathcal{U}$, where $\mathcal{U}$ is a free ultrafilter on ω .

6.16. PROPOSITION. *Suppose that* $A_{\mathfrak{p}}$ *is normable for some* $\mathfrak{p} \in \beta\mathbb{N}\backslash\mathbb{N}$. *Then there is a discontinuous homomorphism from* $C(\Omega)$ *into some Banach algebra for each infinite, compact space* Ω .

Thus, combining 6.15 and 6.16, we obtain the following result.

6.17. THEOREM. *Assume that there is a free ultrafilter* $\mathcal{U}$ *on* ω *such that* $\mathbb{R}^{\omega}/\mathcal{U}$ *is a* β_1*-field. Then there is a discontinuous homomorphism from* $C(\Omega)$ *into some Banach algebra for each infinite, compact space* Ω .

With CH, $|\mathbb{R}^\omega/\mathcal{U}| = \aleph_1$ for each ultrafilter $\mathcal{U}$ on ω, and so $\mathbb{R}^\omega/\mathcal{U}$ is a β_1-field. Thus, with CH, there is discontinuous homomorphism from $C(\Omega)$ for each infinite, compact space Ω. But can we have $\mathbb{R}^\omega/\mathcal{U}$ as a β_1-field without CH being true ? In this direction, we have a result of Dow ([11]): if CH be false, then there exists a free ultrafilter $\mathcal{U}$ on ω such that $\mathbb{R}^\omega/\mathcal{U}$ is not a β_1-field. Also, under the hypothesis MA + $\neg$CH, there is no free ultrafilter $\mathcal{U}$ on ω such that $\mathbb{R}^\omega/\mathcal{U}$ is a β_1-field ([9, Corollary 6.28]). (Here MA is Martin's Axiom : see [9, Chapter 5].)

It is certainly not the case that it is a theorem of ZFC that there is a discontinuous homomorphism from $C(\Omega)$ for each infinite compact space Ω. Indeed it is a theorem of Woodin from 1978 that there is a model of ZFC + MA (in which CH is necessarily false) such that every homomorphism from $C(\Omega)$ into a Banach algebra is automatically continuous for each compact space Ω. This theorem is the main result of the book [9]; the notion of a model of ZFC, and the interpretation of the existence of models in terms of the independence of certain results, is fully explained in that book. Thus we have known for 10 years that the existence of discontinuous homomorphisms from the algebras $C(\Omega)$ is independent of the theory ZFC.

Nevertheless, it has been an important open question for some time whether or not CH is a necessary hypothesis for the existence of discontinuous homomorphisms from the algebras $C(\Omega)$. I am grateful to Hugh Woodin for his permission to announce the following recent theorem of his at this meeting.

6.18. THEOREM. (Woodin) *There exists a model of* ZFC *in which:*

(i) CH *is false*;

(ii) $\mathbb{R}^\omega/\mathcal{U}$ *is a* β_1*-field for some free ultrafilter* $\mathcal{U}$ *on* ω;

(iii) *there is an isotonic isomorphism from* $\mathbb{R}^\omega/\mathcal{U}$ *into* $\mathfrak{F}_{(1)}(\mathbb{R},G)$.

By combining Woodin's theorem 6.18 with Esterle's theorem 6.15, we obtain the following theorem.

6.19. THEOREM. *There is a model of* ZFC *in which* CH *is false and in which there is a discontinuous homomorphism from* $C(\Omega)$ *into a Banach algebra for each infinite, compact space* Ω .

Thus the question of existence of discontinuous homomorphisms from the algebras $C(\Omega)$ is also independent of the theory ZFC + ¬CH . Whether or not there is a model of ZFC + MA + ¬CH in which these discontinuous homomorphisms exist is an interesting, and probably very deep, open problem; you will see that to find such homomorphisms would require quite new methods.

7. EXPONENTIATION ON ORDERED FIELDS

Let P be a non-maximal, prime ideal in an algebra $C(\Omega)$. I said at the end of §5 that the question of the normability of the algebra $K_P{}^* = M_P/P$ was resolved in every case save (with GCH) where $|K_P| = \aleph_2$ and $|\Gamma_{K_P}| = \aleph_1$. In §6, I exhibited a totally ordered, real-closed, η_1-field K such that $|K| = \aleph_2$ and $|\Gamma_K| = \aleph_1$. However, we do not know that this field has the form K_P for some P . In this section, I shall show that in fact the field K is *not* of the form K_P : to do this, of course, we shall describe a property that all fields K_P possess, but which K does not have.

7.1. DEFINITION. Let K be an ordered field. A *strong interval* of K^+ is a subset I of K^+ such that

(i) if $a \in I$ and $b \in K$ with $0 \le b \le a$, then $b \in I$;

(ii) if $a \in I$, then $2a \in I$;

(iii) $1 \in I$.

It follows that, if I is a strong interval in K^+ , then $K^{\#+} \subset I$, and that, if $a, b \in I$, then $a + b \in I$.

7.2. DEFINITION. Let K be an ordered field, and let I be a strong interval of K^+. Then an *exponentiation on* I is a map $\exp: I \to K$ such that:

(i) $\exp(a+b) = (\exp a)(\exp b) \quad (a,b \in I)$;

(ii) $\exp 0 = 1$, $\exp 1 = e1$;

(iii) $\exp a < \exp b$ whenever $a, b \in I$ with $a < b$;

(iv) for each $c \in K$ with $c \geq 1$, there exists $a \in I$ with $\exp a = c$.

It follows from (i) - (iii) that $\exp(\alpha 1) = e^{\alpha}1 \quad (\alpha \in \mathbb{R})$, that $\exp(K^{\#}) \subset K^{\#+}\backslash K^*$, and that $\exp(I\backslash K^{\#}) \subset K^{\#+}\backslash K^{\#}$. The main effect of condition (iv) is to ensure that $\exp(I\backslash K^{\#}) = K^+\backslash K^{\#}$ and that $I \neq K^{\#+}$.

A function F on $\mathbb{R}^+$ is *locally Lipschitz* if, for each $k \in \mathbb{N}$, there exists $\alpha > 0$ such that

$$\left\{\frac{|F(s)-F(t)|}{|s-t|^{\alpha}} : s,t \in [0,k], s \neq t\right\}$$

is bounded.

Now let Ω be a compact space, and let P be a non-maximal, prime ideal in $C(\Omega)$. If $f,g \in C(\Omega)^+$ with $f-g \in P$ and if F is locally Lipschitz, then $F\circ f - F\circ g \in P$. For take $n \in \mathbb{N}$ and $M \in \mathbb{R}^+$ with $|F(s) - F(t)| \leq M|s-t|^{1/n}$ for $s,t \in f(\Omega) \cup g(\Omega)$. Then

$$|(F\circ f)(x) - (F\circ g)(x)|^n \leq M^n|f(x) - g(x)| \qquad (x \in \Omega),$$

and so $|F\circ f - F\circ g|^n \leq M^n|f - g|$ in $C(\Omega)$. Since P is absolutely convex, $F\circ f - F\circ g \in P$. Thus, if F is locally Lipschitz, we can define F on ${A_P}^+$ as follows: for $a \in {A_P}^+$, take $f \in C(\Omega)^+$ with $\pi_P(f) = a$, and set $F(a) = \pi_P(F\circ f)$. Then $F(a)$ is well defined.

For example, take $F(t) = e^t \quad (t \in \mathbb{R}^+)$. Then F is locally Lipschitz, and it is easily checked that the map $a \mapsto F(a) = e^a$, ${A_P}^+ \to A_P$, satisfies (i) - (iii) of Definition 7.2. We now wish to extend the domain of F so that (iv) also holds. We fix a particular function G, namely

$$G(t) = \begin{cases} \exp\left[\frac{-1}{t}\right] & (t \in \mathbb{R}^+ \setminus \{0\}) , \\ 0 & (t = 0) . \end{cases}$$

Then G is locally Lipschitz, and so $G(a)$ is defined for $a \in {A_P}^+$. We set

$$J_G = \{a \in {A_P}^+ : G(a) = 0\} ,$$

$$I_G = \{a \in {K_P}^+ \setminus (0) : a^{-1} \notin J_G\} \cup \{0\}$$

$$= \{b^{-1} : b \in {K_P}^+ \setminus J_G\} \cup \{0\} .$$

(The prime ideal P is a *z-ideal* if $f \in P$ whenever $f \in C(\Omega)$ and $f^{-1}(0) = g^{-1}(0)$ for some $g \in P$. If P is a z-ideal, then $J_G = \{0\}$ and $I_G = {K_P}^+$, and this is the case to bear in mind. However, there are prime ideals in $C(\Omega)$ which are not z-ideals.)

It is easy to check that I_G is a strong interval of K^+ .

7.3. DEFINITION. Let G and I_G be as above. Set

$$\exp a = \begin{cases} e^a & (a \in {A_P}^+) , \\ \left[G(a^{-1})\right]^{-1} & (a \in I_G \setminus {A_P}^+) . \end{cases}$$

I claim that $\exp : a \mapsto \exp a$ is an exponentiation on I_G .

Let me check a special case of 7.2(i). Take $a,b \in I_G \setminus {A_P}^+$. We must verify that

$$G((a + b)^{-1}) = G(a^{-1})G(b^{-1}) . \tag{14}$$

Take $f,g \in C(\Omega)^+$ with $\pi_P(f) = a^{-1}$ and $\pi_P(g) = b^{-1}$, set $X = f^{-1}(0) \cup g^{-1}(0)$, and set

$$h(x) = \begin{cases} \dfrac{f(x)g(x)}{f(x) + g(x)} & (x \in \Omega \setminus X) , \\ \\ 0 & (x \in X) . \end{cases}$$

Then $h \in C(\Omega)^+$. Set $c = \pi_P(h)$. Since $(f + g)h = fg$ in $C(\Omega)$, $(a^{-1} + b^{-1})c = a^{-1}b^{-1}$ in A_P , and so $(a + b)c = 1$ and $c = (a + b)^{-1}$. Since

$$\exp\left[\frac{-1}{h(x)}\right] = \exp\left[\frac{-1}{f(x)}\right] \cdot \exp\left[\frac{-1}{g(x)}\right] \qquad (x \in \Omega) ,$$

$G \circ h = (G \circ f)(G \circ g)$, and so (14) follows.

Other cases of 7.2(i) , and 7.2(ii) and (iii) are trivial or are checked similarly.

We finally verify that 7.2(iv) holds. Take $c \in K_P$ with $c \geq 1$. If $c \in A_P$, then there exists $a \in A_P$ with $e^a = c$, and so we may suppose that $c \in {K_P}^+ \setminus A_P$. Set $b = c^{-1} \in {K_P}^{*+} \setminus \{0\}$: we require $a \in {A_P}^+ \setminus J_G$ with $G(a) = b$. Take $g \in C(\Omega)^+$ with $\pi_P(g) = b$; we may suppose that $|g|_\Omega < 1$. Let $H : [0,1) \to \mathbb{R}^+$ be such that $(G \circ H)(t) = t \quad (t \in [0,1))$. Then $H \circ g \in C(\Omega)$: set $a = \pi_P(H \circ g)$. (We cannot say that $a = H(b)$ in A_P because H is not a locally Liptschitz function.) Then $G(a) = \pi_P(G \circ H \circ g) = \pi_P(g) = b$, as required.

We have obtained the following result.

7.4. THEOREM. *Let* Ω *be a compact space, and let* P *be a non-maximal, prime ideal in* $C(\Omega)$. *Then there is an exponentiation on a strong interval of* ${K_P}^+$. *If* P *is a z-ideal, there is an exponentiation on* ${K_P}^+$ *itself.*

On the other hand we have the following theorem. The proof appeals to the Continuum Hypothesis, but in fact the result holds as a theorem of ZFC ([10]).

7.5. THEOREM. *It is not the case that there is an exponentiation on a strong interval of* $\mathfrak{F}(\mathbb{R},G)$.

Proof. (CH) Set $K = \mathfrak{F}(\mathbb{R},G)$. To obtain a contradiction, suppose that I is a strong interval of K^+ and that $\exp : I \to K$ is an exponentiation on I . Then there exists $f_0 \in I\backslash K^{\#}$. Set $y_0 = -v(f_0)$ and $\alpha^0 = v(y_0)$, so that $y_0 \in G^+\backslash\{0\}$ and $\alpha^0 \in \mathbf{Q}$. Take $\tau < \omega_1$ such that $\alpha^0_\tau = 0$.

We define a map $\alpha \mapsto x_\alpha$, $\mathbf{Q} \longrightarrow G^+$. For each $\rho < \omega_1$, ϵ_ρ is the sequence with

$$\epsilon_\rho(\sigma) = \begin{cases} 1 & (\sigma \leq \tau + \rho) \ , \\ 0 & (\tau + \rho < \sigma < \omega_1) \ . \end{cases}$$

Then, for each $\rho < \omega_1$, $\epsilon_\rho \in \mathbf{Q}$ and $\alpha^0 \prec \epsilon_\rho$, and $\{\epsilon_\rho : \rho < \omega_1\}$ is a well-ordered subset of $\mathbf{Q}$.

For $\alpha = (\alpha_\tau) \in \mathbf{Q}$, define $x_\alpha : \mathbf{Q} \to \mathbb{R}$ by setting

$$x_\alpha(\epsilon_\rho) = \alpha_\rho \text{ for } \rho < \omega_1 \ , \ x_\alpha(\beta) = 0 \quad \text{otherwise.}$$

Then $\operatorname{supp} x_\alpha$ is a countable, well ordered subset of $\mathbf{Q}$, and so $x_\alpha \in G^+$. The map $\alpha \mapsto x_\alpha$, $(\mathbf{Q},\preceq) \to (G,\leq)$, is isotonic. Since $y > 0$ in G and $v(x_\alpha) \succeq \alpha^0 = v(y_0)$, we have $x_\alpha < y_0$ in G for each $\alpha \in \mathbf{Q}$.

Now define $\iota : \alpha \mapsto x_\alpha - y_0$, $\mathbf{Q} \to G$. Then ι is an isotonic map with $\iota(\mathbf{Q}) \subset G^-\backslash\{0\}$. We identify $\mathbf{Q}$ with $\iota(\mathbf{Q})$, and then we can regard $\mathfrak{F}(\mathbb{R},\mathbf{Q})$ as a subgroup of K . For each $f \in \mathfrak{F}(\mathbb{R},\mathbf{Q})^+\backslash\{0\}$, we have $v(f) > v(f_0)$, and so $f < f_0$ in K . By the definition of a strong interval, $f \in I$, and so $\mathfrak{F}(\mathbb{R},\mathbf{Q})^+ \subset I$ and $\mathfrak{F}(\mathbb{R},\mathbf{Q}) \cap K^{\#} = \{0\}$.

Finally we define a map

$$\psi : f \mapsto -v(\exp f) \ , \ \mathfrak{F}(\mathbb{R},\mathbf{Q})^+ \to G \ .$$

I claim that ψ is an injection. For take $f,g \in \mathfrak{F}(\mathbb{R},\mathbf{Q})^+$ with $f < g$, say $g = f + h$, where $h \in I\backslash K^{\#}$. Since $\exp g = (\exp f)(\exp h)$, we have $\psi(g) = \psi(f) + \psi(h)$. Since $\exp h \in \exp(I\backslash K^{\#}) \subset K^+\backslash K^{\#}$, $v(\exp h) < 0$ and $\psi(h) > 0$ in G . Thus $\psi(f) < \psi(g)$, and so ψ is an injection.

However $\mathbf{Q}$ contains a well-ordered subset of cardinality $\aleph_1$, and so $|\mathfrak{F}(\mathbb{R},\mathbf{Q})| \geq 2^{\aleph_1}$, whereas $|G| = 2^{\aleph_0}$ by 6.8. With CH (but not as a theorem of ZFC!) , $2^{\aleph_1} > 2^{\aleph_0}$, and so we have reached a contradiction.

Thus, from 7.4 and 7.5 , we obtain the following result, proved in ZFC + CH , but true as a theorem of ZFC .

7.6. **THEOREM.** *The real-closed,* η_1*-field* $\mathfrak{F}(\mathbb{R},G)$ *is not of the form* K_P *for any prime ideal* P *in an algebra* $C(\Omega)$.

8. A SECOND EXAMPLE

So far we have not found a field K of the form K_P such that $|K| = \aleph_2$ and $|\Gamma_K| = \aleph_1$. In this final section, I shall give an example of such a field which is an ultrapower, and so, in particular, is a field of the form K_P. First we have a result which enables us to calculate some cardinalities.

8.1. **THEOREM.** *Let* κ *be a cardinal, let* $\mathcal{U}$ *be a free ultrafilter on* κ , *and set* $K = \mathbb{R}^\kappa/\mathcal{U}$. *Then*

$$|\Gamma_K| = w(K) = |\mathbb{Q}^\kappa/\mathcal{U}| .$$

Proof. Certainly $w(K) \geq |\Gamma_K|$.

We show that $w(K) \leq |\mathbb{Q}^\kappa/\mathcal{U}|$ by showing that $\mathbb{Q}^\kappa/\mathcal{U}$ is order-dense in K . Take [f] , [g] $\in$ K with [f] < [g] , and set $S = \{\sigma < \kappa : f(\sigma) < g(\sigma)\}$. Then $S \in \mathcal{U}$. For each $\sigma \in S$, choose $h(\sigma) \in \mathbb{Q} \cap (f(\sigma), g(\sigma))$. Then $h \in \mathbb{Q}^\kappa/\mathcal{U}$ and [f] < [h] < [g] in K . Thus $\mathbb{Q}^\kappa/U$ is order-dense in K .

Finally we show that $|\mathbb{Q}^\kappa/\mathcal{U}| \leq |\Gamma_K|$ by giving an injection $\psi : \mathbb{Q}^\kappa/\mathcal{U} \longrightarrow \Gamma_K$. Let $\iota : \mathbb{Q} \to \mathbb{N}$ be a fixed injection, and let f_0 be a fixed infinitely large element of K . For $f \in \mathbb{R}^\kappa$, set

$$\Theta f : \sigma \mapsto f_0(\sigma)^{\iota(f(\sigma))} , \quad \kappa \longrightarrow \mathbb{R} ,$$

and set $\psi([f]) = v_K([\Theta f])$, where v_K is the archimedean valuation on K. Then $\psi([f])$ is well-defined on Γ_K. Take $[f], [g] \in K$ with $[f] \neq [g]$, and set $S = \{\sigma < \kappa : \iota(f(\sigma)) < \iota(g(\sigma))\}$ and $T = \{\sigma < \kappa : \iota(f/\sigma)) > \iota(g(\sigma))\}$. Then $S \cup T \in \mathcal{U}$ because ι is an injection, and so either $S \in \mathcal{U}$ or $T \in \mathcal{U}$, say $S \in \mathcal{U}$. For each $\sigma \in S$,

$$(\Theta f)(\sigma) = f_0(\sigma)^{\iota(f(\sigma)-g(\sigma))}(\Theta g)(\sigma) \geq f_0(\sigma)(\Theta g)(\sigma)$$

because $\iota(f(\sigma) - g(\sigma)) \geq 1$, and so $[\Theta f] > n[\Theta g]$ for each $n \in \mathbb{N}$. Thus $\psi(f) < \psi(g)$, and ψ is an injection, as required.

8.2. DEFINITION. Let κ be a cardinal, and let $\mathcal{U}$ be an ultrafilter on K. Then $\mathcal{U}$ is *uniform* if $|S| = \kappa$ for each $S \in \mathcal{U}$.

Thus, if $\mathcal{U}$ is a uniform ultrafilter on ω_1, the complement of every countable subset of ω_1 is in $\mathcal{U}$. The following proposition is a special case of a standard result.

8.3. PROPOSITION. *Let $\mathcal{U}$ be a uniform ultrafilter on ω_1. Then $|\mathbb{R}^{\omega_1}/\mathcal{U}| \geq \aleph_2$.*

Proof. To obtain a contradiction, suppose that $\{[f_\xi] : \xi < \omega_1\}$ is an enumeration of $\mathbb{R}^{\omega_1}/\mathcal{U}$. For each $\xi < \omega_1$, $\{f_\eta(\xi) : \eta < \xi\}$ is a countable subset of $\mathbb{R}$, and so there exists $f(\xi) \in \mathbb{R}$ with $f(\xi) \neq f_\eta(\xi)$ for each $\eta < \xi$.

For each $\eta < \omega_1$, we have

$$\{\xi < \omega_1 : \xi < \eta\} \subset \{\xi < \omega_1 : f(\xi) \neq f_\eta(\xi)\},$$

and the set on the left has a countable complement. Hence the set on the right belongs to $\mathcal{U}$, and so $[f] \neq [f_\eta]$. Thus $[f] \notin \{[f_\eta] : \eta < \omega_1\}$, the required contradiction.

8.4. THEOREM. (GCH) *Let $\mathcal{U}$ be a uniform ultrafilter on ω_1, and set $K = \mathbb{R}^{\omega_1}/\mathcal{U}$. Then*

$$|K| = \aleph_2, \quad \textit{and} \quad |\Gamma_K| = |\mathbb{Q}^{\omega_1}/\mathcal{U}|.$$

Proof. This follows from 8.1 and 8.3, noting that $|K| \le |\mathbb{R}^{\omega_1}| = 2^{\aleph_1} = \aleph_2$, with GCH.

Thus we would achieve an ultrapower of the form we are considering if we could show that there is a uniform ultrafilter $\mathcal{U}$ on ω_1 such that $|\mathbb{Q}^{\omega_1}/\mathcal{U}| = \aleph_1$.

It has been proved by Hugh Woodin that such an ultrafilter exists (using deep results that appear in the paper [16], and which in turn are based on earlier results of Woodin) but only under a certain "large cardinal" axiom. A large cardinal axiom is a statement that a cardinal with certain properties exists; for example, analysts are familiar with the axiom that measurable cardinals exist (see [18, Chapter 12]). The large cardinal required for Woodin's theorem is a "huge" cardinal, although he allowed himself the remark that "a super-compact cardinal would probably suffice". These large cardinal axioms are known to be independent of the theory ZFC + GCH. Thus we finally obtain the following result. The theorem is in fact a "relative consistency" result, as before (see [9]).

8.5. THEOREM. *Assume that the theory* "ZFC + GCH + *'there is a huge cardinal'*" *is consistent. Then the theory* "ZFC + GCH + *'there is a uniform ultrafilter $\mathcal{U}$ on ω_1 such that*

$$|K| = \aleph_2 \quad \textit{and} \quad |\Gamma_K| = \aleph_1,$$

where $K = \mathbb{R}^{\omega_1}/\mathcal{U}$'" is also consistent.

On the other hand, we also know that we cannot prove in ZFC + GCH that there is a uniform ultrafilter $\mathcal{U}$ on ω_1 with the properties stated in the theorem: the consistency of such a theory implies the consistency of a theory with some large cardinal axiom.

Also, in this talk I have not given a construction in ZFC + GCH of a non-maximal, prime ideal P in an algebra $C(\Omega)$ such that $|K_P| = \aleph_2$, and $|\Gamma_{K_P}| = \aleph_1$. At the time of writing I do not have such a construction, but one will probably emerge soon.

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