USING BANACH ALGEBRAS TO DO ANALYSIS WITH THE UMBRAL CALCULUS

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1. INTRODUCTION

The modern version of the umbral calculus, which was developed by G.C. Rota with a number of collaborators, particularly Steven Roman, has roots which go far back into the nineteenth century. According to them, [13, Section 1, pp.95–98], the umbral calculus was originally an attempt to exploit the fact that one could often substitute appropriate sequences of numbers or polynomials for the sequence $\{x^n\}$ of powers in certain identities. For instance, many polynomial sequences $\{p_n(x)\}$ satisfy the binomial identity

$$p_n(x+y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} p_k(y)\, p_{n-k}(x) . \tag{1.1}$$

By the early twentieth century, it was clear that the umbral calculus was closely tied to the Heaviside operational calculus. Recall that when we let t be the differentiation operator d/dx, then for the power series $f(t) = \Sigma_0^\infty a_n t^n$ the operational calculus formula is

$$f(t)\, h(x) = \Sigma a_n h^{(n)}(x) , \tag{1.2}$$

so that, in particular, $e^{at} h(x) = h(x+a)$.

Throughout the nineteenth century, both the umbral calculus and the operational calculus were powerful heuristic devices for discovering useful formulae, but these formulae needed to then be rigorously demonstrated by other means. While there have been many successful rigorous versions of the operational calculus, for instance the elegant Mikusiński calculus, [9], the umbral calculus resisted being made rigorous until relatively recently. In 1970, in a path-breaking paper [10], Mullin and Rota developed a rigorous theory for polynomials of binomial type, that is, polynomials which satisfy formula (1.1). The theory was extended and simplified by Rota and his collaborators [13], [14] and reached a polished form in Roman's book [12], whose terminology and notation we will normally follow. As is frequently the case when a powerful heuristic theory is made rigorous, [3], the scope of the theory must be narrowed. Thus the Roman–Rota umbral calculus and operational calculus apply only to the space $C[x]$ of complex polynomials in x. For some years I have been engaged in a program of extending these operational and umbral methods to spaces of entire functions. In my published papers [6], [7] and in my previous talks on this subject, I have emphasized describing the most general results which show the power of the umbral calculus on spaces of entire functions. In the present paper I would like instead to emphasize the techniques, so that a reader with problems about entire functions can get an idea of how umbral methods might apply. Thus I will not attempt to prove the sharpest results, as I do in [6], [7], and I will often carry proofs only to the stage where the umbral calculus has reduced a problem about entire functions or special polynomial sequences to a "more familiar" problem in analysis.

From the point of view of a functional analyst, the essence of Roman's formulation of the umbral calculus [12] is the representation of the dual space of $C[x]$ by the algebra $C[[t]]$ of formal power series in such a way that important linear operators on $C[x]$ have adjoints that relate well to the algebraic structure of $C[[t]]$. This is done, [12, pp.6–7], by making $(\{t^n\}, \{x^n/n!\})$ a biorthogonal sequence so that the duality is given by

$$\langle \Sigma a_n t^n \mid \Sigma b_n x^n/n! \rangle = \Sigma a_n b_n . \tag{1.3}$$

It is easy to see, [12, Th. 2.2.5, p.13], that under this duality the operational calculus map on $C[x]$ given by $f(t)$, according to formula (1.2), is adjoint to power series multiplication by $f(t)$ in $C[[t]]$. Other important maps on $C[x]$ in the umbral calculus are adjoints of automorphisms and derivations of $C[[t]]$.

In the Roman–Rota umbral calculus, the sum in formula (1.3) is a finite sum, since $\Sigma\, b_n x^n/n!$ is a polynomial. To extend the umbral calculus, one just uses formula (1.3) whenever the sum converges. Suppose that E is a Banach or Fréchet space of entire functions of x whose series converge in the topology of E (that is, $\{x^n/n!\}$ is a Schauder basis of E). Then the dual space E^* contains precisely those series for which formula (1.3) converges for all $\Sigma\, b_n x^n/n!$ in E. For us the most important spaces of entire functions will be the weighted spaces $c_0(n!/w_n)$ given by a sequence $\{w_n\}$ of positive numbers. The space $c_0(n!/w_n)$ consists of all functions $h(x) = \Sigma_0^\infty a_n x^n$ with $\lim_n |a_n n!/w_n| = 0$, and norm $\|h(x)\| = \sup_n |a_n n!/w_n|$. The more familiar spaces of entire functions which we will consider are countable unions or intersections of such weighted c_0-spaces, so we will largely be studying operators between various $c_0(n!/w_n)$. We do this by looking at the adjoints of the operators on the dual of $c_0(n!/w_n)$. Because of the biorthogonality of $\{t^n\}$ and $\{x^n/n!\}$, we must have $\|t^n\| \cdot \|x^n\| = n!$, so that the dual of $c_0(n!/w_n)$ is the space $l^1(w_n)$ of all series $f(t) = \Sigma\, b_n t^n$ with finite norm $\|f\| = \Sigma\, |b_n|\, w_n$. We will almost always assume that $\{w_n\}$ is an *algebra weight*, that is, $w_0 = 1$ and $w_{n+m} \le w_n w_m$, so that $l^1(w_n)$ is a Banach algebra.

Because of the very different roles of x and t in the umbral calculus, it is important to distinguish spaces of functions of x from spaces of functions or series in t. Thus, to be absolutely precise, we should write $l^1(w_n,t)$ for $l^1(w_n)$ and should write $E^0(x)$ for the space E^0 in the next section. Fortunately the context makes this unwieldy notation unnecessary. When a space contains only entire functions, then x is the intended variable; but when the space contains an analytic function or formal power series which is not entire, then t is implicitly understood as the variable.

In the Roman–Rota umbral calculus, algebraic formulae for polynomials are obtained by studying algebraic properties of certain linear operators on $C[x]$. These algebraic properties are usually proved by considering the adjoint operators on $C[[t]]$. We wish to extend this procedure to analytic questions on spaces of entire functions. These are studied by determining the properties of bounded linear maps, including the crucial property of determining between which weighted c_0-spaces the map is defined and continuous. This is done by considering their adjoints between the dual weighted

l^1–algebras (cf. [7, Lem. (2.6), p.137]). When we do this, two very different cases arise. When the entire functions are of exponential type the duals are algebras of analytic functions; but when the spaces of entire functions contain only functions of exponential order less than one, then the duals are algebras containing formal power series with zero radius of convergence. We illustrate some results for each of these two cases.

2. THE OPERATIONAL CALCULUS

The modern umbral calculus makes heavy use of the operational calculus of formula (1.2). The deepest results, both in the Roman–Rota umbral calculus and in its extension to entire functions, also involve these special polynomial sequences, called Sheffer sequences, to which umbral methods apply, but the operational calculus by itself, though much more elementary, has wider applicability. The operational calculus maps include all linear differential and linear difference operators and many other useful operators, [12, pp.14–15], [10, pp.178–180]. In fact, the operational calculus operators on $C[x]$ are precisely the translation–invariant operators, [12, Cor. 2.2.9, p.17]. In this section we study analytic extensions of the Roman–Rota form of the operational calculus, postponing the definition and study of Sheffer sequences to the next section.

For a power series $f(t) = \Sigma a_n t^n$ and a space of entire functions E we will look at the following two questions:

Question 1. For which $g(x)$ in E is $\mathrm{span}\{f(t)^n g(x)\}_0^\infty$ dense in E ?

Question 2. When does the infinite–order differential equation $f(t)\, y(x) = \Sigma a_n y^{(n)}(x) = 0$ have a non–trivial solution in E ?

We will answer a special case of each of the above questions and discuss other cases with references to [6]. In order to avoid some delicate situations treated in [6], [7], we will concentrate on the spaces E^ρ of entire functions of order no more than ρ , where

$0 \le \rho < 1$, and the spaces E_τ of functions of exponential type no more than τ, where $0 \le \tau < \infty$. E^ρ is the Fréchet space of all $h(x) = O(\exp(|x|^r))$ for all $r > \rho$, with the topology given by the norms $\sup_x |h(x)\, e^{-|x|^r}|$. Similarly, E_τ is all entire functions finite in all the norms $\sup_x |h(x)\, e^{-a|x|}|$ for $a > \tau$. We can now answer one case of Question 1.

THEOREM (2.1). *Suppose that* $f(t)$ *is analytic on a neighbourhood of the origin and has* $f'(0) \neq 0$ *and that* $g(x)$ *belongs to the space* E^ρ *for some* $0 \le \rho < 1$. *If* g *is not a polynomial, then* $M = \operatorname{span}\{f(t)^n g(x)\}_0^\infty$ *is dense in* E^ρ.

Proof. Subtraction of the constant term of $f(t)$ does not change the invariant subspace M, so we assume that $f(0) = 0$. Standard estimates on the coefficients of entire functions, [8, Th. 4.12.1, p.74], show that

$$E^\rho = \bigcap_{1<\alpha<1/\rho} c_0(n!^\alpha)\,; \tag{2.2}$$

and it follows from the closed graph theorem that the topology of E^ρ is also given by the norms of the $c_0(n!^\alpha)$. (It is enough to observe that the coefficient functionals are continuous in all the norms we consider.) Thus, by the Hahn–Banach theorem, it will be enough to show that M is dense in each of the $c_0(n!^\alpha)$.

We fix $1 < \alpha < (1/\rho)$ and let $w_n = 1/n!^{\alpha-1}$, so that the dual of $c_0(n!^\alpha)$ is $l^1(w_n)$. We thus need only show that $M^\perp = \{0\}$ in $l^1(w_n)$. It is easy to see that $l^1(w_n)$ is a Banach algebra of power series containing f, and in fact contains all series with positive radius of convergence. Thus $f(t)$ is a bounded linear operator on $c_0(n!^\alpha)$, [7, Th. (3.2), p.140], so $M^\perp$ is $f(t)$–invariant in $l^1(w_n)$ since M is $f(t)$–invariant in $c_0(n!^\alpha)$. Since f has positive radius of convergence, so does its compositional inverse $\bar{f}(t) = \Sigma_1^\infty b_n t^n$, and it is easy to see, [4, Th. (2.3), p.643], that $t = \Sigma\, b_n f(t)^n$, so that $M^\perp$ is t–invariant, and is therefore a closed ideal of $l^1(w_n)$. Since w_{n+1}/w_n decreases monotonically to 0, it is well known, [4, pp.644–645], that the only proper closed non–zero ideals of $l^1(w_n)$ are the ideals $P_k^{\perp}$, where

$$P_k = \mathrm{span}\{1,x,\ldots,x^k\}\ , \tag{2.3}$$

is the space of polynomials of degree no more than k. But if $M^{\perp} = P_k^{\perp}$, then $g \in M \subseteq M^{\perp\perp} = P_k$, contradicting the assumption that g is not a polynomial. This completes the proof.

For the analogous result for E_τ, [6, Th. 3.2, p.421], we need to assume that f has radius of convergence greater than τ and we need to assume not only that g is not a polynomial but also that it is not a finite linear combination of functions $x^i e^{a_j x}$ with $|a_j| \le \tau$. Answers to Question 1 for other spaces and other $f(t)$ also appear in [6].

We now consider Question 2. Following Bade, Dales, and Laursen [1, Def. 1.9, p.18], we say that the algebra weight $\{w_n\}$ is *regulated* if there is a k for which $\lim_{n\to\infty} (w_{n+k}/w_n) = 0$. For regulated weights we have the following equivalent reformulation of Question 2 in terms of ideals of $l^1(w_n)$. Recall that the *standard ideals* of $l^1(w_n)$ are $\{0\}$, the whole space, and the $P_k^{\perp}$, where P_k is as in formula (2.3).

THEOREM (2.4). *Suppose that* $\{w_n\}$ *is a regulated algebra weight and that* $f(t) = \Sigma_1^\infty a_n t^n$ *belongs to* $l^1(w_n)$. *Then the infinite–order differential equation* $\Sigma a_n y^{(n)}(x) = 0$ *has a non–polynomial solution in* $c_0(n!/w_n)$ *if and only if the closed ideal generated by* $f(t)$ *is not standard in* $l^1(w_n)$.

Proof. Let N be the solution space of $\Sigma a_n y^{(n)}(x) = 0$ in $c_0(n!/w_n)$; that is, N is the null–space of the linear operator $f(t)$ acting on $c_0(n!/w_n)$. Hence $N^{\perp}$ is the weak*–closure of the range of $f(t)^*$, that is, of multiplication by $f(t)$, in the dual space $l^1(w_n)$. In other words $N^{\perp}$ is the weak*–closure of the principal ideal $J = l^1(w_n)\, f(t)$.

First suppose that the closure of J is standard so that it equals some $P_k^{\perp}$. Then

$N = N^{\perp\perp} = J^{\perp} = P_k^{\perp\perp} = P_k$ and the differential equation has only polynomial solutions.

Conversely suppose that the closure of J is not standard. Since the weight $\{w_n\}$ is regulated, it follows from [1, Th. 5.3, p.79] that the weak*–closure $N^{\perp}$ is not standard either. Thus $N \neq P_k$ for any k, and since N is a closed t–invariant subspace, this implies that N cannot be composed only of polynomials. This completes the proof.

In the case of $a_0 \neq 0$, which is omitted from the above theorem, $f(t)$ is an invertible element of $l^1(w_n)$, since $\lim_{n\to\infty} (w_n)^{1/n} = 0$, so that $f(t)$ is an invertible linear operator on $c_0(n!/w_n)$ and the solution space $N = \{0\}$.

Examples of $l^1(w_n)$ with only standard ideals have been known for a long time. One particularly simple condition, [4, Th. (2.10), p.645], is that there is some k for which $\{w_{n+k}/w_n\}$ decreases monotonically to zero. This shows non–polynomial solutions cannot exist in $c_0(n!^{\alpha})$, and thus, by formula (2.2), in E^{ρ} for $\rho < 1$. On the other hand Marc Thomas has constructed $l^1(w_n)$ with non–standard ideals, [16], and the $\{w_n\}$ can be regulated, [17, Th. 3.3.1, p.144], so non–standard solutions of the differential equation occur in these $c_0(n!/w_n)$.

For spaces of functions of exponential type, in addition to the polynomials, one has "standard" solutions of the form $x^k e^{t_0 x}$ of $f(t)\, y(x) = 0$ wherever $f(t)$ has a zero of order greater than k (cf. [6, pp.140–141]). Since the solution space is a closed t–invariant subspace, there are no other solutions for the space E_{τ}, [6, Th. 3.2, p.421]. For other spaces the situation is more complicated.

Suppose that f is a bounded analytic function on the disc and consider the solution space N of the differential equation $f(t)\, y = 0$ on the space $l^2(n!)$, whose dual under formula (1.3) is the Hardy space H^2 of square–summable power series. Let ϕ be the inner factor of f under the Beurling factorization, [15, Th. 17.17, p.344]. Then $N^{\perp}$, the closure of $f H^2$, is just the space ϕH^2, [15, Th. 17.23, p.350]. So N is just $(\phi H^2)^{\perp}$. Thus $f(t)\, h(x) = 0$ if and only if $\langle \phi(t)\, t^n | h(x) \rangle = \langle \phi(t) | h^{(n)}(x) \rangle = 0$ for all $n \geq 0$, with the duality defined by formula (1.3). When f is equal to its inner factor ϕ, then $f H^2 = \phi H^2$ is closed, [15, Th. 17.21(a), p.348]. Since multiplication by f on $H^2 \subseteq C[[t]]$ must be one–to–one, we have that f is bounded below as an operator on H^2, so that $f(t)$ maps $l^2(n!)$ onto itself. That is, the nonhomogeneous differential equation $f(t)\, y(x) = \phi(t)\, y(x) = h(x)$ has a solution in $l^2(n!)$ whenever h belongs to $l^2(n!)$.

3. SHEFFER SEQUENCES AND UMBRAL METHODS

The Sheffer sequences $\{s_n(x)\}$, which include many classical polynomial sequences, [12, Chap. 4, pp.53–130], such as Abel, Bernoulli, factorial, Hermite, Laguerre, and Stirling polynomials, are essentially the polynomial sequences to which the Roman–Rota rigorization of the classical umbral calculus applies. In analysis, Sheffer sequences occur most naturally through generating functions $\Sigma\, (s_n(x)/n!)\, t^n$. To be more precise, let A and B be analytic functions with $A(0) \neq 0$ and $B(0) = 0$ but $B'(0) \neq 0$, so that A has an algebraic inverse and B has a compositional inverse which we denote by $\bar{B}$. Then we can find analytic functions f and g with

$$\text{(3.1)} \qquad \begin{array}{ll} \text{(a)} & A(t) = 1/g(\bar{f}(t)) \quad \text{and} \quad B(t) = \bar{f}(t)\,, \ \text{or, equivalently,} \\[1em] \text{(b)} & g(t) = 1/A(\bar{B}(t)) \quad \text{and} \quad f(t) = \bar{B}(t)\,. \end{array}$$

Following Roman, [12, p.19], we say that the sequence of polynomials $\{s_n(x)\}$ is *Sheffer* for $(g(t), f(t))$ if

$$\text{(3.2)} \qquad \sum_0^\infty \frac{s_n(x)}{n!}\, t^n = A(t)\, e^{xB(t)} = \frac{1}{g(\bar{f}(t))}\, e^{x\bar{f}(t)}\,.$$

To avoid some of the more delicate arguments from [6], [7], which make heavy use of results about radical Banach algebras of power series from [5], we will only consider $f(t)$ and $g(t)$ with positive radius of convergence. The simplest non–trivial example of a Sheffer sequence is the *(lower) factorial polynomials*, [12, Ex. 4.1.2, pp.56–63], [7, Ex. 5.1, pp.152–154] $(x)_n = x(x-1)...(x-n+1)$, whose generating formula is just the binomial formula

$$\sum_{0}^{\infty} \frac{(x)_n}{n!} t^n = (1+t)^x = e^{x \log(1+t)} . \tag{3.3}$$

Thus the factorial polynomials are Sheffer for $(1, e^t-1)$.

While the generating formula (3.2) is the most common way that Sheffer sequences arise in analysis, in the umbral calculus it is much more useful to have a definition, [12, p.17] exploiting the duality of formula (1.3) between $C[x]$ and $C[[t]]$. The equivalence of the two definitions is essentially [12, Th. 2.3.4, p.18] which we now rephrase.

THEOREM (3.4). *The polynomial sequence* $\{s_n(x)\}_0^\infty$ *is Sheffer for* $(g(t), f(t))$ *if and only if* $(\{g(t)\, f(t)^n\}, \{s_n(x)/n!\})$ *is a biorthogonal sequence for the dual spaces* $C[[t]]$ *and* $C[x]$.

The power series $g(t)\, f(t)^n$ has order n , so that $s_n(x)$ has degree n , and therefore $\{s_n(x)/n!\}_0^\infty$ and $\{g(t)\, f(t)\}_0^\infty$ are dual Schauder bases for the natural topologies on $C[x]$ and $C[[t]]$. (In fact $\{s_n(x)/n!\}$ is a Hamel basis and $\{g(t)\, f(t)^n\}$ is a basis for the Zariski topology.) We thus have the following expansions for polynomials $h(x)$ and formal power series $\phi(t)$, [12, Ths. 2.3.2 and 2.3.3, p.18],

$$h(x) = \sum_{0}^{\infty} \frac{\langle g(t)\ f(t)^n \,|\, h(x)\rangle}{n!}\, s_n(x) \tag{3.5}$$

$$\phi(t) = \sum_{0}^{\infty} \frac{\langle \phi(t) \,|\, s_n(x)\rangle}{n!}\, g(t)\, f(t)^n . \tag{3.6}$$

In the Rota–Roman umbral calculus $h(x)$ is a polynomial so that the expansion of formula (3.5) is a finite sum. Similarly, the expansion in formula (3.6) is finite in each coordinate. For a Banach or Fréchet space E of entire functions, the natural question suggested by formula (3.5) is:

Question 3. For which $h(x)$ in E does formula (3.5) give a convergent expression in the topology of E ?

Related questions about uniqueness of expansions, rapidity of convergence, and properties of $h(x)$ in terms of the expansion are treated, along with many cases of Question 3, in [7]. Here we will concentrate on some simpler special cases of Question 3 which are closely related to the classical expansion theorems in [2]. Analogous questions suggested by formula (3.6) are treated for functions of exponential type in [7, pp.150–151] and for functions of order less than one in [7, pp.166–167]. Several examples of both types of expansion theorems are worked out in [7, Sect. 5, pp.152–157].

With the Sheffer sequence $\{s_n(x)\}$ for $(g(t), f(t))$, Roman [12, p.42] associates the *Sheffer operator* $\lambda = \lambda_{g,f}$ for $s_n(x)$ or for $(g(t), f(t))$ defined as the linear operator on $C[x]$ for which

$$\lambda_{g,f}(x^n) = \lambda(x^n) = s_n(x) .$$

In the Roman–Rota umbral calculus it is the Sheffer operator which precisely expresses what is meant by replacing x^n by $s_n(x)$ in appropriate formulae. In our extension of the umbral calculus to entire functions, [7], the main question seems to be:

Question 4. Between which Banach or Fréchet spaces of entire functions does the Sheffer operator $\lambda_{g,f}$ extend to a continuous linear operator?

Since, as we will see below, λ^{-1} is also a Sheffer operator, the next result (cf. [7, Th. (4.7), pp.148f.]) shows that answers to Question 4 give answers to Question 3 as well.

LEMMA (3.7). *Suppose that* D, E, *and* F *are Banach or Frechet spaces of entire functions and that* $\{x^n\}$ *is a Schauder basis of* E . *If the Scheffer operators* $\lambda = \lambda_{g,f}$ *and* λ^{-1} *extend to continuous linear operators* $\lambda: E \to F$ *and* $\lambda^{-1}: D \to E$, *then every* $h(x)$ *in* D *has an expansion* $h(x) = \Sigma_0^\infty c_n s_n(x)$ *converging in the topology of* F . *Moreover one can take* $c_n = \langle g(t)\, f(t)^n \,|\, h(x)\rangle/n!$ *as in formula* (3.5).

Proof. Let $\lambda^{-1}(h(x)) = \phi(x) = \Sigma_{n=0}^{\infty} c_n x^n$ in E. Then $h(x) = \lambda(\phi(x)) = \lambda(\Sigma c_n x^n) = \Sigma c_n \lambda(x^n) = \Sigma_0^{\infty} c_n s_n(x)$ in F. We postpone the proof of the formula for c_n until we develop some umbral methods.

Just as answers to Question 4 give answers to Question 3, information about the properties of λ gives additional information about the expansions. For instance, uniqueness and non-uniqueness results depend on whether λ is one-to-one or not. The nicest case is of course when $D = F$, or, normalizing to $f'(0) = 1$, when $D = E = F$ so that λ is a continuous isomorphism of E. This happens for functions of exponential order less than 1, [7, Th. (6.6), p.161], which gives unique expansions in E for all $h(x)$ in E, [7, Th. (7.2), p.165]. In these cases, λ being an isomorphism also shows that $\Sigma c_n s_n(x)$ and $\Sigma c_n x^n$ have the same order (and type), provided the order is less than 1. For functions of exponential type, similar results hold only for Appell sequences, [7, Th. (4.8), p.150], which is the case $f(t) = t$, [12, pp.17–18].

To determine the properties of λ on $C[x]$ we need to look at λ^* on $C[[t]]$. We first treat the important case where $g(t) = 1$. If $\{p_n(x)\}$ is Sheffer for $(1, f(t))$, then $\{p_n(x)\}$ is called the *associated sequence* for $f(t)$ [12, p.17]. We then write λ_f for the Sheffer operator $\lambda_{g,f}$ and call λ_f the *umbral operator* for $\{p_n(x)\}$ or for $f(t)$, [12, pp.37–38]. Associated sequences are precisely the polynomial sequences which satisfy the binomial identity of formula (1.1), [12, Th. 2.4.6, p.26], [10, Th. 1, p.182], while general Sheffer sequences only satisfy a related identity, [12, Th. 2.3.9, p.21].

We now obtain Roman's formulas for λ^* for umbral and Sheffer operators. If $\{p_n(x)\}$ is the associated sequence for $f(t)$ and λ_f is the umbral operator, then the formulae $\lambda_f(x^n) = p_n(x)$ together with the biorthogonality of $\{f(t)^n\}$ and $\{p_n(x)/n!\}$ yield, [12, form. (3.4.1), p.38],

$$\lambda_f^*(g) = g \circ \bar{f}\,. \tag{3.8}$$

In other words, λ_f^* is the automorphism of $C[[t]]$ given by composition with $\bar{f}$, the compositional inverse of f. Similarly, when $\{s_n(x)\}$ is Sheffer for $(g(t), f(t))$, the biorthogonality conditions give

$$\lambda^*_{g,f}(g(t)\ f(t)^n) = t^n\ .$$

One can factor this map in two ways, giving the following commutative diagram:

$$\begin{tikzcd}
g(t)\ f(t)^n \arrow[r, "1/g(t)"] \arrow[d, "\lambda^*_f"'] \arrow[dr, "\lambda^*_{g,f}"] & f(t)^n \arrow[d, "\lambda^*_f"] \\
g(\bar{f}(t))t^n \arrow[r, "1/g(\bar{f}(t))"'] & t^n
\end{tikzcd}$$

Taking adjoints yields, [12, forms. (3.5.1) and (3.5.2), p.42],

$$\lambda_{g,f} = \frac{1}{g(t)}\,\lambda_f = \lambda_f\,\frac{1}{g(\bar{f}(t))}\ . \tag{3.9}$$

Either by taking inverses in formula (3.9) or by using formula (3.9) to calculate $\lambda^*_{g,f}(t^n) = A(t)\ B(t)^n$, we find that, [12, form. (3.5.3), p.43],

$$\lambda_{g,f}^{-1} = \lambda_{A,B}$$

where $A(t)$ and $B(t)$ are as in formula (3.1) and generating formula (3.2). Many other formulae in the umbral calculus are easily obtained by using biorthogonality to determine the adjoints of the operators involved. For instance, from the formula $f(t)(g(t)\ f(t)^{n-1}) = g(t)\ f(t)^n$, one obtains, [12, Th. 2.3.7, p.20], $f(t)\ s_n(x) = ns_{n-1}(x)$.

We can now verify the formula for c_n claimed in Lemma (3.7). With $\phi(x) = \lambda^{-1}(h(x)) = \Sigma\, c_n\, x^n$ as in the proof of the lemma, we obtain

$$n!\ c_n = \langle t^n \,|\, \phi(x)\rangle = \langle t^n \,|\, \lambda^{-1}(h)\rangle = \langle \lambda^{*-1}(t^n)\,|\,h\rangle = \langle g(t)\ f(t)^n \,|\, h(x)\rangle\ ,$$

as claimed in the lemma.

With formulae (3.8) and (3.9) for umbral and Sheffer operators, we can now answer a simple special case, [7, Th. (3.12)(A), p.145], of Question 3. Recall that E_σ is the Fréchet space of all entire functions of exponential type no more than σ .

THEOREM (3.10). *Let* $\lambda_{g,f}$ *be the Sheffer operator for* $(g(t), f(t))$. *If both* $A(t) = 1/(g(\bar{f}(t))$ *and* $B(t) = \bar{f}(t)$ *are analytic on a neighbourhood of the closed disc* $\bar{D}(\rho)$ *of radius* ρ, *and if the maximum value of* $|\bar{f}(t)|$ *or* $\bar{D}(\rho)$ *is less than or equal to* σ, *then* $\lambda_{g,f}$ *extends to a continuous linear operator from* E_ρ to E_σ.

Proof. Standard estimates of the coefficients of entire functions of exponential type, [8, Th. 4.13.1, p.78], show that

$$E_\rho = \bigcap_{r>\rho} c_0(n!/r^n), \tag{3.11}$$

and similarly for E_σ. Thus we need only show that if $s > \sigma$, then there is an $r > \rho$ with $\lambda_{g,f}: c_0(n!/r^n) \longrightarrow c_0(n!/s^n)$ continuously. It is easy to see that it suffices, [7, Lemma (2.6), p.137], to show that $\lambda^*_{g,f}$ maps the dual space $l^1(s^n)$ to $l^1(r^n)$. To do this we choose $r > \rho$ such that $A(t)$ and $B(t)$ are analytic on a neighbourhood of the closed disc $\bar{D}(r)$ and that the maximum value of $|B(t)| = |\bar{f}(t)|$ on this disc is less than s. We use the factorization of $\lambda^*_{g,f}$ given by formula (3.9), considering each of the factors separately.

Since $A(t) = 1/g(\bar{f}(t))$ is analytic on a neighbourhood of $\bar{D}(\rho) \supseteq \bar{D}(r)$, it belongs to $l^1(r^n)$. Since $l^1(r^n)$ is a Banach algebra, multiplication by $A(t)$ then maps $l^1(r^n)$ into itself. Thus we must show that λ^*_f maps $l^1(s^n)$ into $l^1(r^n)$. The maximal ideal space of $l^1(r^n)$ is the closed disc $\bar{D}(r)$, so that the spectral radius of $\bar{f}(t)$ in $l^1(r^n)$ is, by assumption, less than s. Hence there is an $M > 0$ for which $\|\bar{f}(t)^n\| \le M\, s^n$ for all $n \ge 0$; where the norm is taken in the Banach algebra $l^1(r^n)$. Now suppose that $\phi(t) = \Sigma\, c_n\, t^n$. Then from (3.8) we obtain

$$\|\lambda^*_f(\phi)\| = \|\Sigma c_n\, \lambda^*_f(t^n)\| = \|\Sigma c_n (\bar{f}(t))^n\| \le M\, \Sigma\, |c_n|\, s^n = M\|\phi\|.$$

Thus λ^*_f is a bounded linear operator from $l^1(s^n)$ to $l^1(r^n)$, and the proof is complete.

For specific Sheffer sequences, it is usually easy to apply Theorem (3.10), and the sharper answers to Question 4 given in [7], to obtain specific spaces which answer

Question 3 (see the examples in [7, Sect. 5, pp.152–157]). For instance the factorial polynomials $(x)_n$ given in generating formula (3.3) are the associated polynomials for $f(t) = e^t - 1$, so that $\bar{f}(t) = \log(1+t)$. It is easy to see that

$$\max_{|t| \leq r} |\bar{f}(t)| = -\log(1-r), \quad \text{for } 0 < r < 1$$

and

$$\max_{|t| \leq q} |f(t)| = e^q - 1 .$$

Thus if $r = e^q - 1$ and $s = -\log(1-t) = -\log(2-e^q)$, then for $0 < q < \log 2$, it follows from Theorem (3.10) that $\lambda_f : E_r \to E_s$ and $\lambda_f^{-1} = \lambda_{\bar{f}} : E_q \to E_r$. Hence Lemma (3.7) implies that, if $h(x)$ has exponential type no more than $q < \log 2$, then the expansion of $h(x)$ according to formula (3.5) converges in the topology of E_s. For further related results about factorial polynomials see [7, ex. 5.1, pp.152–154].

Uniqueness and non–uniqueness results are obtained by determining when Sheffer operators are one–to–one. Some of these results can be delicate, but for the domain space E_ρ, we have the following simple answer [7, Th. (3.12), p.145].

Theorem (3.12). Under the hypothesis of Theorem (3.10) we have

(a) $\lambda_{g,f} : E_\rho \to E_\sigma$ is one–to–one if $A(t)$ has no zeros on $\bar{D}(\rho)$ and if $\bar{f}(t)$ is univalent on a neighbourhood of the disc $\bar{D}(\rho)$.

(b) If $A(t)$ has a zero or if $\bar{f}(t)$ is not univalent on the closed disc $\bar{D}(\rho)$, then $\lambda_{g,f}$ is not one–to–one on E_ρ.

We will not give a full proof of the theorem here, but only of the key lemma which gives a sufficient condition for λ_f to be one–to–one on E_ρ. We actually show that λ_f is one–to–one on some $c_0(n!/r^n) \supseteq E_\rho$ (cf. formula (3.11)). The proof, though short, is a nice blend of umbral methods with both functional and classical analysis.

LEMMA (3.13). *If* $\bar{f}(t)$ *is analytic and univalent on a neighbourhood of the closed disc* $\bar{D}(r)$, *then* λ_f *is one–to–one on* $c_0(n!/r^n)$.

Proof. Choose some s greater than the maximum value of $|\bar{f}(t)|$ on the closed disc. As we showed in the proof of Theorem (3.10), λ_f is then a bounded linear operator from $c_0(n!/r^n)$ to $c_0(n!/s^n)$. To show that λ_f is one–to–one, it will be enough to show that the adjoint $\lambda_f^*: l^1(s^n) \longrightarrow l^1(r^n)$ has dense range. Since λ_f^* is an algebra homomorphism, this is equivalent to showing that $\lambda_f^*(t) = \bar{f}(t)$ is a Banach algebra generator of $l^1(r^n)$ (cf. [7, Th. (3.5), pp.141f.]).

Let U be an open disc containing $\bar{D}(r)$ on which $\bar{f}(t)$ is univalent and let $\bar{f}(U) = V$. Then $\lambda_f^*(\phi) = \phi \circ \bar{f}$ is an isomorphism from the algebra $H(V)$ of analytic functions on V onto $H(U)$. Also λ_f^* is a homeomorphism with respect to the topologies of uniform convergence on compacta in $H(U)$ and $H(V)$. Since V is simply connected, it follows from Runge's theorem that the polynomials are dense in $H(V)$. Hence $\bar{f}(t) = \lambda_f^*(t)$ is a topological algebra generator of $H(U)$ (cf. [11, p.467]). Since $H(U)$ is continuously imbedded in $l^1(r^n)$, it then follows that the closed algebra generated by $\bar{f}(t)$ in $l^1(r^n)$ contains $H(U)$ for all open discs $U \supseteq \bar{D}(r)$. That is, the closed algebra generated by $\bar{f}(t)$ in $l^1(r^n)$ contains the dense subalgebra of functions analytic on a neighbourhood on $\bar{D}(r)$. This completes the proof.

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