SOME REMARKS ABOUT IDEAS AND RESULTS OF TOPOLOGICAL HOMOLOGY

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The area which recently began to be referred to as "topological homology" by Michai Putinar, [40], and some other mathematicians, was initiated in 1962 by Kamowitz [1]. It was he who introduced the first important concept in the area by carrying into Banach algebra the purely algebraic notion of Hochschild cohomology groups, [0].

Let A be a Banach algebra, X a Banach A-bimodule. We call an n-linear continuous operator $f : A \times \ldots \times A \to X$ an *n-cochain.* These cochains form a Banach space denoted by $C^n(A,X)$, $n > 0$; we take $C^0(A,X)$ as X. Now let us consider the so-called *standard cohomological complex*

$$0 \longrightarrow C^0(A,X) \xrightarrow{\delta^0} \ldots \longrightarrow C^n(A,X) \xrightarrow{\delta^n} C^{n+1}(A,X) \longrightarrow \ldots \qquad (\mathcal{C}(A,X))$$

where δ^n is given by

$$(\delta^n f)(a_1,\ldots,a_{n+1}) = a_1 f(a_2,\ldots,a_{n+1}) + \sum_{k=1}^{n} (-1)^k f(a_1,\ldots,a_k a_{k+1},\ldots,a_n) + (-1)^{n+1} f(a_1,\ldots,a_n) a_{n+1}$$

(it is indeed a complex since $\delta^{n+1}\delta^n = 0$; $n \geq 0$) .

DEFINITION (Kamowitz, [1] Guichardet, [2]). The n-th cohomology of $\mathcal{C}(A,X)$ is called the *n-dimensional cohomology group of* A *with coefficients in* X (and is denoted by $\mathcal{H}^n(A,X)$).

In the spirit of Hochschild, Kamowitz applied the groups $\mathcal{H}^2(A,X)$ to the investigation of questions concerning the Wedderburn structure of some extensions of Banach algebras. Let us call an extension

$$0 \longrightarrow I \longrightarrow \mathcal{A} \longrightarrow A \longrightarrow 0 \qquad (\mathcal{E})$$

singular if 1) $I^2 = 0$ (and, as a consequence, I becomes an A-bimodule), and 2) I has, as a subspace, a Banach complement in $\mathcal{A}$. In this case we have:

THEOREM (Kamowitz) [1]. *Every singular extension* $\mathcal{E}$ *with given* A *and* I *splits* (*in the strong sense*; *see* [55]) *if and only if* $\mathcal{H}^2(A,I) = 0$.

It is worth noting in this connection that condition 2) in the definition of singular extensions is not too heavy. In some cases it is automatically satisfied. Here is an example.

THEOREM (Karyaev, Yakovlev, [53], proved in 1988). *Let* $\mathcal{E}$ *be annihilator* (*that is,* $\mathcal{A}I = I\mathcal{A} = 0$) *and* A *have a b.a.u.* (*=bounded approximate unit*). *Then* $\mathcal{E}$ *is singular.*

Nevertheless, there exist examples of Banach algebras $\mathcal{A}$ with annihilator radical R (and commutative $\mathcal{A}$ among them) such that R, as a subspace, has no Banach complement in $\mathcal{A}$. Moreover, for every Banach space E and an uncomplemented subspace F there exists an example with R = F (Yakovlev, [54], proved in 1988).

There are a lot of other applications and representations of the groups $\mathcal{H}^n(A,X)$; these include problems concerning derivations, automorphisms, perturbations, fixed point theorems, invariant means, topology of the spectrum,... (Johnson, Kadison, Ringrose, Christensen, Taylor, Raeburn,...; see, for example, [42] for the references). But applications are not, as a rule, a topic of these talks.

The first result of a computation of "Banach" cohomology is also due to Kamowitz [1] and is as follows. A bimodule X over A is called *symmetric* if $a \cdot x = x \cdot a$ for all $a \in A$, $x \in X$.

THEOREM. *Let* A *be* $C(\Omega)$ *and let* X *be a symmetric Banach A-bimodule. Then* $\mathcal{H}^n(A,X) = 0$ *for* $n = 1$ *and* 2.

It is still an open question whether this equality holds for all $n \geq 3$ or at least for some of them. Further, it is well known that one cannot brush aside the hypothesis of symmetry. Actually, there is the following theorem of rather general character [4], [16].

THEOREM. *Let* A *be a commutative Banach algebra.*

1) *If* $\mathcal{H}^1(A,X) = 0$ *for all Banach* A*-bimodules* X , *then* $A = \mathbb{C}^n$ *for some integer* $n \geq 0$.

2) *If* $\mathcal{H}^2(A,X) = 0$ *for all Banach* A*-bimodules* X , *then the spectrum* (*that is, maximal ideal space*) *of* A *is finite.*

COROLLLARY. *Every infinite dimensional Banach function* (*that means commutative and semisimple*) *algebra has at least one non-splitting singular extension.*

The proof of part 2) of this theorem is rather complicated. The result itself is strongly connected with the existence of subspaces without Banach space complement and with other "pathological" properties of Banach space geometry.

Now let us present one of numerous examples of connections between "homological" and "functional-analytic" properties of well known Banach algebras.

THEOREM (Selivanov, proved in 1988). *Let* $\mathcal{N}(E)$ *be an algebra of nuclear operators in a Banach space* E . *Then*

1) $\mathcal{H}^2(\mathcal{N}(E), \mathcal{N}(E)) = 0$ *for arbitrary* E ,

2) $\mathcal{H}^3(\mathcal{N}(E), \mathcal{N}(E)) = 0$ *if and only if* E *has the approximation property.*

We shall now discuss a general approach to computing cohomology groups which saves us from being tied to the definition of these groups in terms of the standard complex. This approach is in essence one of "full" homology (homological algebra) of Cartan, Eilenberg, MacLane, which was carried over from pure algebra to Banach algebras by the present speaker ([4], 1970) and to locally convex algebras by Taylor ([9], 1972). The main concepts are essentially the same. Let us introduce them for Banach algebras and modules.

In what follows **A-mod** is the notation for the category of Banach left modules over a Banach algebra A . Let us recall that every (Banach) right A-module can be naturally identified with a left A^{op}-module, and every A-bimodule with a left A^{env} - module, where A^{op} is the opposite algebra of A , and $A^{env} = A_+ \otimes A_+^{op}$ is the *enveloping algebra of* A (A_+ denotes the unitization of A) . Therefore, when speaking about left modules, we actually cover all possible types of (bi)modules.

As in the case of an arbitrary category, **A-mod** is equipped with two families of morphism functors; for every fixed $X \in$ **A-mod** we have a covariant functor ${}_A\mathbf{h}(X,?)$: **A-mod** → **Ban** , and for every fixed $Y \in$ **A-mod** we have a contravariant functor ${}_A\mathbf{h}(?,Y)$: **A-mod** → **Ban** (here, and later, **Ban** = **0-mod** denotes the category of Banach spaces and continuous operators). Analogous functors for the category **A-mod-A** of Banach A-bimodules (and of their morphisms) are denoted by ${}_A\mathbf{h}_A(X,?)$: **A-mod-A** → **Ban** and ${}_A\mathbf{h}_A(?,Y)$: **A-mod-A** → **Ban** .

The initial notion of "full" Banach homology is that of projectivity (and the dual notion of injectivity as well). But these notions, before being defined, need the preparatory

DEFINITION. Let $\mathcal{X}$ be a complex of left Banach modules. It is called *admissible* if it splits as a complex of Banach spaces (in other words, it has a contracting homotopy consisting of continuous linear operators).

Thus, we see that admissibility is something more than exactness.

DEFINITION. We call $P \in$ **A-mod** *projective* if the complex ${}_A\mathbf{h}(P,\mathcal{Y})$ is exact for every admissible complex $\mathcal{Y}$ in **A-mod** . "Dually", we call $J \in$ **A-mod** *injective* if the complex ${}_A\mathbf{h}(\mathcal{X},J)$ is exact for every admissible complex $\mathcal{X}$ in **A-mod** .

The simplest example of a projective A-module is the "natural" left

A-module A_+ ; its dual, A_+^* with $a\cdot f(b) := f(ba)$, is injective. If A has no unit, the factor module $\mathbb{C} = A_+/A$ is not a projective A-module.

For every Banach space E the left Banach A-module $A_+ \hat{\otimes} E$ with action $a\cdot(b \otimes x) := ab \otimes x$ is certainly projective. We call modules of this form *free.* The following fact gives an intrinsic definition of projectivity.

THEOREM. $P \in$ **A-mod** *is projective if and only if it is, up to a topological isomorphism, a direct module summand of some free module.*

A similar theorem is valid for right modules and for bimodules, but for the former free right modules have, by definition, the form $E \hat{\otimes} A_+$, and for the latter free bimodules have the form $A_+ \hat{\otimes} E \hat{\otimes} A_+$ (with obvious actions of A) .

Now let us notice that for every $X \in$ **A-mod** there exists an epimorphism of some free left A-module on X , which is a retraction in **Ban** - that is, it has a right inverse which is a continuous operator. Actually, defining $\pi_X : A_+ \hat{\otimes} X \to X$ by $a \otimes x \mapsto a\cdot x$ ("outer product map"), the right inverse operator is given by $x \mapsto e \otimes x$.

Let X be a given left A-module. Let us represent it as a factor module of some projective A-module, say P_0 , in such a way that the quotient map $d_{-1} : P_0 \to X$ is a retraction in **Ban**. Then let us take the module $K_1 := \operatorname{Ker} d_{-1}$ and apply to it the same procedure as to X ; then do the same with the kernel of the quotient map $d_0 : P_1 \to K_1$ and so on. What follows is actually a compact form of describing this procedure (and a "dual" procedure as well).

DEFINITION. Let X , Y be given A-modules, and let

$$0 \longleftarrow X \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \dots \qquad (0 \longleftarrow X \longleftarrow \mathcal{P})$$

$$0 \longrightarrow Y \longrightarrow J_0 \longrightarrow J_1 \longrightarrow \dots \qquad (0 \longrightarrow Y \longrightarrow \mathcal{J})$$

be a complex over X , and a complex under Y , respectively. We call $0 \leftarrow X \leftarrow \mathcal{P}$ a *projective resolution of* X if all P_k ; $k \geq 0$, are projective, and we call $0 \to Y \to \mathcal{J}$ an *injective resolution of* Y if all J_k ; $k \geq 0$, are injective.

Every $X \in$ **A-mod** has a projective, as well as an injective, resolution. As an example, if we use the "canonical" morphisms π_X , then π_{K_1} and so on (see above), we get the so called *bar-resolution*:

$$0 \leftarrow X \xleftarrow{\pi_X} A_+ \hat{\otimes} X \leftarrow A_+ \hat{\otimes} K_1 \leftarrow A_+ \hat{\otimes} K_2 \leftarrow \dots \qquad B(X)$$

Its particular case $B(A_+)$ (that is, when $X = A_+$) provides an obviously projective - moreover, free - resolution of A_+ not only in **A-mod** , but in **A-mod-A** as well. We call it the *bimodule bar-resolution of* A_+ .

We come at last to a result which is at the core of all "Banach" homology.

THEOREM. *Let* X , Y *be given A-modules, let* $0 \leftarrow X \leftarrow \mathcal{P}$ *be a projective resolution of* X , *and let* $0 \to Y \to \mathcal{J}$ *be an injective resolution of* Y . *Then n-th cohomologies of the complexes* ${}_A h(\mathcal{P},Y)$ *and* ${}_A h(X,\mathcal{J})$ *coincide for all* $n \geq 0$. *In particular each of these cohomologies does not depend on the choice of projective (respectively, injective) resolution.*

Accordingly to "classical" tradition, we denote the n-th cohomology of each of the complexes ${}_A h(\mathcal{P},Y)$ and ${}_A h(X,\mathcal{J})$ by $\mathrm{Ext}_A(X,Y)$; actually it is complete seminormed space. In the case X , Y ∈ **A-mod-A** , the corresponding spaces will be denoted by $\mathrm{Ext}^n_{A-A}(X,Y)$; $n \geq 0$.

It is not difficult to express the notions of projectivity and injectivity in the language of Ext.

THEOREM. 1) X *is projective if and only if* $\mathrm{Ext}^1_A(X,Y) = 0$ *for all* Y ∈ A-mod, *if and only if* $\mathrm{Ext}^n_A(X,Y) = 0$ *for all* Y ∈ A-mod *and for all* $n > 0$.

2) Y *is injective if and only if* $\mathrm{Ext}^1_A(X,Y) = 0$ *for all* X ∈ A-mod, *if and only if* $\mathrm{Ext}^n_A(X,Y) = 0$ *for all* X ∈ A-mod *and for all* $n > 0$.

Of course, this theorem has its obvious analogue for bimodules. Now it is time to connect this second part of the talk with the first one. The key result in this direction is the following.

THEOREM. *Let* A *be a Banach algebra, and let* X *be a Banach A-bimodule. Then, up to a topological isomorphism,* $\mathcal{H}^n(A,X) = \mathrm{Ext}^n_{A-A}(A_+,X)$ *for all* $n \geq 0$.

(Here and afterwards A_+ is considered as an A-bimodule with obvious actions.)

OUTLINE OF PROOF. If we compute the abovementioned Ext with the help of the bimodule bar-resolution $\mathbf{B}(A_+)$, we come to considering the cohomology of the complex ${}_A\mathbf{h}_A\ (\mathbf{B}(A_+)\ ,\ X)$. The latter is isomorphic to the standard complex $\mathcal{C}(A,X)$.

In several important cases the groups $\mathcal{H}^n(A,\cdot)$ coincide with some one-sided (and not just with two-sided, as above) Ext spaces. For given Y , Z ∈ **A-mod** let us consider the Banach space $\mathcal{B}(Y,Z)$ of continuous linear operators between Y and Z as an A-bimodule with actions $[a\cdot y](y) := a\cdot(\varphi(y))$ and $[\varphi\cdot a](y) := \varphi(a\cdot y)$.

THEOREM. *Up to a topological isomorphism* , $\mathcal{H}^n(A,\mathcal{B}(Y,Z)) = \mathrm{Ext}_A(Y,Z)$.

OUTLINE OF PROOF. If we compute the latter Ext with the help of the resolution B(Y) , we come the cohomology of a complex which is isomorphic to $\mathcal{C}(A,\mathcal{B}(Y,Z))$.

Now we should like to show by an example how the last theorem can work. Let A be a Banach operator algebra in a Banach space E (with some norm $\|\cdot\|_A \geq \|\cdot\|_{B(E)}$. In this case E naturally becomes a left Banach A-module with the action $y\cdot x := y(x)$, and $\mathcal{B}(E,E)$ ∈ **A-mod-A** is just $\mathcal{B}(E)$ with exterior multiplications coinciding with interior ones. In virtue of the previous theorem we immediately get $\mathcal{H}^n(A,\mathcal{B}(E)) = \mathrm{Ext}^n_A(E,E)$, and these cohomology groups are zero for all $n > 0$ provided E is either projective or injective. But these things really can happen:

THEOREM (Kaliman, Selivanov). *Suppose that* A *contains all finite rank operators. Then* E *is a projective* A*-module. As a corollary,* $\mathcal{H}^n(A,\mathcal{B}(E)) = 0$ *for all* $n > 0$.

OUTLINE OF PROOF. The map $A_+ \to E : a \mapsto a(x_0)$, where x_0 is a fixed element in E, is a retraction in **A-mod**.

In particular, for $A = \mathcal{B}(E)$ and $n = 1$ we get the well known result of Kaplansky that all derivations of $\mathcal{B}(E)$ are inner. Considering the same A and $n = 2,3$ we obtain that the Banach algebra $\mathcal{B}(E)$ is stable under small perturbations. On the other hand, we have the following result, due to Golovin [46].

THEOREM. *Let* A *be a nest algebra (see, e.g.* [39]*) in a Hilbert space* H. *Then* H *is an injective left* A*-module. As a corollary,* $\mathcal{H}^n(A,\mathcal{B}(H)) = 0$ *for all* $n > 0$.

(The corollary was proved earlier by direct methods by Nielsen [33] and, independently, Lance [34].)

Now we shall discuss the simplest and at the same time most severe condition of homological triviality of Banach algebras.

DEFINITION. A Banach algebra A is called *contractible* if $\mathcal{H}^1(A,X) = 0$ for all $X \in$ **A-mod-A**. (Actually, if A is contractible, then $\mathcal{H}^n(A,X) = 0$ for all $X \in$ **A-mod-A** and for all $n > 0$.)

Using the connection between $\mathcal{H}^n(A,\cdot)$ and $\mathrm{Ext}_{A-A}(\cdot,\cdot)$ and the criterion of projectivity in terms of Ext (see above) we obtain the following result.

THEOREM. *Let* A *be a Banach algebra. Then the following are equivalent:*

(i) *A is contractible;*

(ii) *A_+ is a projective A-bimodules.*

(iii) *A is unital, and A itself is a projective A-bimodule.*

Since $A_+ \hat{\otimes} A_+$ is a free A-bimodule, one can check the contractibility of a given Banach algebra with the help of the following assertion.

THEOREM. A *is contractible if and only if the product map* $\pi : A_+ \hat{\otimes} A_+ \to A_+ : a \otimes b \mapsto ab$ *is a retraction in* ***A-mod-A*** .

If A is unital, one can remove subscript "+" in this theorem.

The simplest example of a contractible algebra is the algebra $\mathcal{M}$ of (all) $n \times n$ matrixes. In this case a right inverse morphism to π is given by the rule $a \mapsto a \sum_{k=1}^{n} e_{k1} \otimes e_{1k}$, where e_{jk} denotes the matrix with 1 in "jk-th" place and with 0 in other places. This morphism is well known in pure algebra. It is not convenient, however, if we consider $\mathcal{M}_n$ with the norm of $\mathcal{B}(H)$. If $\dim H = n$ and we wish to have a right inverse morphism ρ to π with $\|\rho\| = 1$, we take $a \mapsto \frac{a}{n} \cdot \sum_{j,k=1}^{n} e_{kj} \otimes e_{jk}$ (see, e.g., [12]).

Since the direct sum of two contractible algebras is obviously again a contractible algebra, the contractibility of $\mathcal{M}_n$ implies the contractibility of all finite-dimensional, semisimple Banach algebras. Do other contractible Banach algebras exist? This problem remains open. Actually it is a part of the following more detailed question.

Let us first make an observation. It is not difficult to see that $A_+ \hat{\otimes}_A X \simeq X$ in **A-mod** , and that, for any $X \in$ **A-mod**, $P \hat{\otimes}_A X$ is projective in **A-mod** provided P is projective in **A-mod-A** . As an immediate corollary, we get the following result.

THEOREM. *If* A *is contractible then every left (and every right as well) Banach A-module is projective.*

So, we obtain the following hierarchy of possible properties of a Banach algebra A:

(1) A is semisimple and finite-dimensional

$\Rightarrow$ (2) A is contractible

$\Rightarrow$ (3) every left Banach A-module is projective

$\Rightarrow$ (4) every irreducible left Banach A-module is projective.

QUESTION. Is it true that all these logical arrows, or at least some of them, can be reversed?

REMARK. The answer to the pure algebraic prototype of this question is known. Namely, (1)$\Leftrightarrow$(2) and (3)$\Leftrightarrow$(4), but (2)$\not\Leftarrow$(3). As an example, the algebra $\mathbb{C}(t)$ of rational functions of one variable t has only projective modules because it is a field, but it is not contractible because it is infinite-dimensional. Certainly, it has no Banach algebra norm.

Nevertheless in the Banach algebra context the answer to the given question is positive if we lay some rather mild conditions on the Banach space geometry of the algebras and modules considered. The most advanced result of this kind is, perhaps, as follows.

THEOREM (Selivanov [28]). *Suppose that either every irreducible Banach left A-module or* A/RadA *has, as a Banach space, the (Grothendieck) approximation property. Then properties* (1)-(4) *of* A *are equivalent.*

Let us recall that it is known that every irreducible left module over a C^*-algebra is, as a Banach space, topologically isomorphic to some Hilbert space.

COROLLARY. *For every* C^**-algebra* A, *the properties* (1)-(4) *are equivalent.*

Of course, Selivanov's theorem is also applicable to all commutative Banach algebras (because their irreducible modules are one-dimensional), and to all $L^1(G)$, where G is a locally compact group (because these algebras are semisimple and have the approximation property), but there exist much easier proofs of the corresponding results.

Coming to the end of the discussion of the contractibility property - our first property of homological triviality - let us remark that outside the framework of Banach structures the situation turns out to be rather unexpected from the "classical" pure algebraic point of view. Considering similar kinds of properties of topological algebras, Taylor [9] discovered a phenomenon, which in our (equivalent) language is as follows.

THEOREM (as to the proof, see, e.g., [42]). *Let* A *be a commutative Arens-Michael algebra* (*in other terms, a complete, locally multiplicatively convex topological algebra*). *Then* A *is contractible if and only if* A *is, up to topological isomorphism, just* $\mathbb{C}^M$ *with the topology of pointwise convergence, for some set* M.

(Let us notice that it is easy to prove that $\mathbb{C}^M$ is indeed contractible, because of $\mathbb{C}^M \hat{\otimes} \mathbb{C}^M \simeq \mathbb{C}^{M\times M}$. We recommend the reader display a right inverse to $\pi : A \hat{\otimes} A \to A$ for such an A).

As to algebras which are not Arens-Michael algebras, Taylor has shown that for every compact Lie group G the algebra of distributions $\mathcal{E}'(G)$ (with the convolution as multiplication) is also contractible, [11].

We now turn to a second, and perhaps the most important concept of homological triviality of algebras - that of amenability. We shall try to show that this concept, which was introduced by Johnson in his memoir of 1972, is actually one of the most important of avatars in functional analysis of the general concept of flatness.

To begin with, let us recall the following well known fact. Suppose that A is a Banach algebra, and X is a left Banach A-module which is dual, as a Banach space, to some Banach space X_*. Then two properties of X are equivalent: 1) for every $a \in A$ the operator $X \to X : x \mapsto a \cdot x$ is continuous (not only relative to the norm in X , but also relative to the weak* topology in X ; 2) X_* possesses the (necessarily uniquely determined) structure of a right Banach A-module such that $<a \cdot x , x_*> = <x , x_* \cdot a>$ for all $a \in A$, $x \in X_*$, $x_* \in X_*$. In both cases X is called a *dual left* A-module (dual to the right A-module X_* , if one wants to be more precise), and X_* is called its *predual right* A-module. By obvious analogy, one can define a *dual right* A-module (and its *predual left* A-module) and, combining both definitions, *dual A-bimodule* (and its *predual A-bimodule*).

It is not difficult to show that if P is a projective left (right, bi-) module, then its dual P^* is an injective module of corresponding type. The converse is false: for example, as we soon shall see, none of the cyclic left Banach modules over C[0,1] are projective, but all their dual modules are injective. This observation justifies the following.

DEFINITION. A (bi)module F (of arbitrary type) over A is called *flat* if its dual is injective.

There exists also an equivalent definition of flatness, which corresponds to the well known concept of flatness in pure algebra. Namely, an A-module F , say a left one, is flat if and only if for every admissible complex $\mathcal{X}$ of right Banach A-modules the complex of Banach spaces $\mathcal{X} \hat{\otimes}_A F$ is exact. However - and this feature differs in our functional-analytic context from the situation in algebra - both definitions of flatness are here equally important.

In which cases are the most "widespread" classes of modules - ideals of A and its factor modules (=cyclic A-modules) necessarily flat? To clarify the situation, let us first ask the similar question about projectivity. It is rather easy to prove the following.

THEOREM. *Let* I *be a closed left ideal in* A .

(1) *If* I *has a right unit, then* I *and* A_+/I *are projective left* A-*modules.*

(2) *If* A_+/I *is projective and, in addition,* I , *as a subspace of* A (*or, equivalently, in* A_+), *is complemented* (*that is, has a Banach complement*), *then* I *has a right unit.*

Turning to the more complicated question, concerning flatness, we discover that the role of the unit is now being played by a bounded approximate unit (we shall, as usual, use the abbreviation b.a.u.). Let us recall that a subspace E_0 of a Banach space E is called *weakly complemented* if $(E/E_0)^*$, being considered as a subspace in E^*, is complemented. Every complemented subspace is certainly weakly complemented, but the converse is false (e.g. the pair $c_0 \subset c_b$).

THEOREM ([7], [15]). *Let* I *be a closed left ideal in* A.

(1) *If* I *has a right b.a.u., then* I *and* A_+/I *are flat left* A*-modules.*

(2) *If* A_+/I *is flat and, if in addition,* I, *as a subspace, of* A *(or, equivalently of* A_+*) is weakly complemented, then* I *has a right b.a.u..*

The first papers where dual A-modules (more precisely, dual A-bimodules) appeared, were these of Kadison and Ringrose, [6] (1971), and Johnson, [9] (1972). These authors had observed that the computation of the cohomology groups $\mathcal{H}^n(A,X)$ is, as a rule, much easier in the case of dual X. The main reason for this phenomenon is the compactness of the unit bull of X in the weak*-topology; it often facilitates the solution of an equation $\delta g = f$ in the standard complex by choosing the limit of some convergent subnet in a suitable bounded net of cochains. So, restricting himself to this "dual cohomology" (cohomology with dual coefficients), Johnson gave the following.

DEFINITION. A Banach algebra A is called *amenable* (sometimes we shall say, for precision, *amenable-after-Johnson*) if $\mathcal{H}^1(A,X) = 0$ for every *dual* Banach A-bimodule X. (Again, similar to the case of the contractibility, one can prove that if A is amenable, then $\mathcal{H}^n(A,X) = 0$ for all dual X and for all $n > 0$.)

The choice of the term "amenable" for these algebras by Johnson will be explained later. Now we proceed to a new way of expressing of the cohomology groups in the language of Ext for the case of dual cohomology.

THEOREM. *Let* A *be a Banach algebra,* $X = (X_*)^*$ *a dual Banach* A*-bimodule.*

Then, up to a topological isomorphism, $\mathcal{H}^n(A,X) = \mathrm{Ext}^n_{A-A}(X_*,A_+^*)$ *for all* $n \geq 0$.

This latter formula is the corollary of $\mathcal{H}^n(A,X) = \mathrm{Ext}^n_{A-A}(A_+,X)$ which was discussed earlier, and of the general formula $\mathrm{Ext}^n_A(Y,Z^*) = \mathrm{Ext}^n_{A^{op}}(Z,Y^*)$ for $Y \in$ **A-mod** , $Z \in$ $\mathbf{A^{op}}$**-mod** , (see, e.g., [42]) where one must replace A by A^{env} .

Using the new formula and the criterion of injectivity in terms of Ext, one can obtain the following.

THEOREM. A *is amenable if and only if* A_+ *is a flat A-bimodule, if and only if* A *has a b.a.u. and* A *itself is a flat A-bimodule.*

Recalling the criterion of contractibility in terms of $\pi : A_+ \hat{\otimes} A_+ \to A_+$, and noting that $(A_+ \otimes A_+)^*$ is always an injective A-bimodule, one can easily obtain the following.

THEOREM. A *is amenable if and only if the morphism* $\pi^* : A_+^* \to (A_+ \hat{\otimes} A_+)^*$, *which is dual to the product map, is a coretraction in* **A-mod-A** .

(If A has a b.a.u., one can remove the subscript "+" in this theorem).

Let us notice that A_+ is actually a factor-A-bimodule of $A_+ \hat{\otimes} A_+$ with π as natural projection. We denote $\mathrm{Ker}\pi$ by I^Δ ; it is left ideal in $A^{env} = A_+ \hat{\otimes} A_+^{op}$, the so-called *diagonal ideal.* Combining some of the preceding results, we get the following intrinsic description of amenability.

THEOREM. *The Banach algebra* A *is amenable if and only if the diagonal ideal* I^Δ *in* A^{env} *has a right b.a.u..*

As an example, we point out that Drury [3] has proved the existence of a b.a.u. in a wide class of ideals in Varopoulos algebras (that is, in $C(\Omega_1) \hat{\otimes} C(\Omega_2)$) . In particular, $I^\Delta \subset C(\Omega)^{env} = C(\Omega) \hat{\otimes} C(\Omega)$, where Ω is a compact set, certainly has a b.a.u.. We immediately obtain that $C(\Omega)$ is amenable. (This result is due to Johnson [9] and, independently, by Kadison and Ringrose [6]; it was proved by other means).

Recently an interesting parallel result was added to this last theorem. Bade, Curtis and Dales have introduced the class of so-called *weakly amenable* Banach algebras; one can define them as those A for which $\mathcal{H}^1(A,A^*) = 0$, [48].

THEOREM (Grønbæk [49]). *A commutative Banach algebra* A *is weakly amenable if and only if* I^Δ *coincides with its topological square* $(I^\Delta)^2 = \mathrm{sp}\{xy : x,y \in I^\Delta\}^-$.

One can prove (similar to the case of projectivity; see above) that the tensor product $F \hat{\otimes}_A X$, where F is a flat A-bimodule and X is an arbitrary left A-module, is a flat left A-module. By putting $F := A_+$, we immediately get:

THEOREM. *If* A *is amenable then every left* (*and every right as well*) *Banach A-module is flat.*

This means exactly that for amenable A every admissible short complex of left Banach A-modules of the form

$$0 \longrightarrow X^* \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

splits. Another proof of this result is obtained in the recent paper of Curtis and Loy [51].

Now let us discuss several concrete classes of Banach algebras and characterize some functional analytic properties which correspond to amenability. We shall begin with group algebras. The following well-known theorem of Johnson, [9] (1972), actually explains the choice of the term "amenable". We recall that a locally compact group G is called *amenable* (the name was coined by Day in the middle of the fifties) if there exists a left-invariant mean on $L^\infty(G)$ - or, equivalently, on one of some other standard function space on G , say $C_b(G)$.

THEOREM [9]. *The Banach algebra* $L^1(G)$ *is amenable if and only if* G *itself is amenable* (*as a locally compact group*).

The rough idea of one of many possible approaches to the proof (it differs from original) can be expressed as follows. It is easy to observe, that for $A = L^1(G)$ the morphism π^* in one of the abovementioned criteria of amenability takes the form $\pi^* : L^1(G) \to L^1(G\times G)$ where $\pi^* f(s,t) = f(st)$, $s,t \in G$. One can show that a left inverse ρ to π^* can be given by the formula $[\rho(u)](s) = M(\bar{v}(s))$, where $[\bar{v}(s)](t) := u(st,t^{-1})$ and $M : L^\infty(G) \to \mathbb{C}$ is our hypothesized invariant mean.

Which algebras are amenable among uniform Banach algebras? Here is the answer:

THEOREM. A *uniform algebra* A *with a spectrum* (*that is, maximal ideal space*) Ω *is amenable if and only if it is just* $C(\Omega)$.

The "$\Leftarrow$" part was already mentioned. The converse is a non-trivial result of Sheinberg, [21] (1977), which is at last well-known, and has been cited several times during different talks at this conference.

And what does amenability mean for C^*-algebras? As the first steps in this direction, the most important are due to Johnson, who has proved that every GCR-algebra (in other terms, postliminal C^*-algebra or C^*-algebra of type I) is amenable, [9] (1972), and to Kadison and Ringrose, who have proved that every AF-algebra (that is, approximatively finite-dimensional C^*-algebra) has the same property, [6] (1971). However, the general problem of characterizing amenable C^*-algebras in terms of functional analysis was completely solved only in 1982, by the combined efforts of several mathematicians. As it turned out, this problem was closely connected with another concept of "topological" cohomology groups which differs from that of Kamowitz and with another concept of amenability which differs from that of Johnson. Our next aim is to discuss these concepts.

Hitherto A was an arbitrary Banach algebra. Now let us concentrate on the special case when A is an operator C^*-algebra; that is, a uniformly closed self-adjoint subalgebra of $\mathcal{B}(H)$ for some fixed Hilbert space H . We recall that, besides the uniform (=norm) topology there are other (at least seven) important topologies on $\mathcal{B}(H)$ and hence on its subalgebras. We need, however, only one of them: the so-called ultra-weak topology. By definition, it is the weak*-topology of $\mathcal{B}(H)$ which is considered, by virtue of the Schatten-von Neumann theorem, as dual to the space the $\mathcal{N}(H)$ of nuclear operators on H . Actually, the ultra-weak topology in $\mathcal{B}(H)$ (and in A as well) can be defined with the help of the family $\{\|\cdot\|_S ; S \in \mathcal{N}(H)\}$ of seminorms, where for T in $\mathcal{B}(H)$ (or in A), $\|T\|_S = |\mathrm{Trace}\ ST|$.

Because of reasons of rather historical character, an operator from A to some dual Banach space $E = (E_*)^*$ is called *normal* if it is continuous not only relative to the norm topologies in A and E but also relative to the ultra-weak topology in A and the weak*-topology in E . A multilinear operator from $A\times\ldots\times A$ to the same E is called *normal* if it is separately normal in each variable.

Now let A be our operator C^*-algebra, and $X = (X_*)^*$ be a dual Banach A-bimodule. In this case the Banach space $C^n(A,X)$ of all cochains has a closed subspace consisting of all normal cochains; we denote it by $C^n_w(A,X)$. Generally speaking, these new spaces do not form a subcomplex in the standard cohomological complex $\mathcal{C}(A,X)$: because there are exterior multiplications in the formula for δ^n (see above), the property of a cochain to be normal can be lost after applying the coboundary operator. Therefore Kadison and Ringrose, [6] (1971), introduced the following special class of dual bimodules.

DEFINITION. A dual bimodule X over an operator C^*-algebra A is called *normal*, if for every $x \in X$, the operators $A \to X : a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are normal.

One can easily verify that if X is normal, then the spaces $C^n_w(A,X)$ indeed form a subcomplex in $\mathcal{C}(A,X)$; we denote it by $\mathcal{C}_w(A,X)$. Now the second variant of cohomology groups for operator algebras can be defined as follows.

DEFINITION (Kadison and Ringrose, [6] 1971). The n-th cohomology of the complex $\mathcal{C}_w^n(A,X)$ is called the *n-dimensional normal cohomology group of* (the operator C^*-algebra) *A with coefficients in* (the normal A-bimodule) X, and it is denoted by $\mathcal{H}_w^n(A,X)$.

It appears that the groups $\mathcal{H}_w^n(A,X)$ were invented as a useful means to compute the "usual" (or, as we shall call them later, *continuous*) groups $\mathcal{H}^n(A,X)$. As a matter of fact, in some situations it is easier to compute normal than continuous cohomology. At the same time the following deep theorem was proved.

THEOREM (Johnson, Kadison, Ringrose, [13] 1972). *Let* $A \subseteq \mathcal{B}(H)$ *be an operator* C^*-*algebra,* $\bar{A}$ *its ultra-weak closure in* $\mathcal{B}(H)$ (*that is, the smallest von Neumann algebra which contains* A), *and* X *a normal* $\bar{A}$-*bimodule. Then for all* $n \geq 0$, $\mathcal{H}_w^n(\bar{A},X) = \mathcal{H}^n(A,X)$, *where* X *is considered as a* (*necessarily normal*) A-*bimodule.*

Later we shall show that this theorem can be presented in a slightly stronger form and, in particular, that one can dispense with the assumption that X is a normal $\bar{A}$-bimodule (and require only that it be a normal A-bimodule).

Here is an important result which was obtained by the same three authors with the help of the preceding theorem. We recall that a von Neumann algebra is called *hyperfinite* if it is the ultra-weak closure of some net of its finite-dimensional self-conjugate subalgebras, ordered by inclusion.

THEOREM [13]. *Let* A *be a hyperfinite von Neumann algebra, and let* X *be a normal* A-*bimodule. Then* $\mathcal{H}_w^n(A,X) = \mathcal{H}^n(A,X) = 0$ *for all* $n > 0$.

In 1976 the now famous article of Connes [22] on the classification of injective factor appeared. Among many other things, Connes paid attention there to a class of von Neumann algebras which he called "amenable as von Neumann algebras". We shall call them, and also some other operator C^*-algebras, slightly otherwise.

DEFINITION. An operator C^*-algebra A is called *amenable-after-Connes* if $\mathcal{H}^1_w(A,X) = 0$ for every normal A-bimodule X.

(Actually, if A is amenable-after-Connes, then $\mathcal{H}^n_w(A,X) = 0$ for all normal X and for all $n > 0$. But this fact is more difficult to prove than the corresponding facts concerning contractibility and amenability-after-Johnson). A little later, [29], Connes discovered a deep connection between the class of algebras which has just been defined, and the class of injective von Neumann algebras, which was so important in his paper [23], already cited. Let us recall that he called a von Neumann algebra $A \subseteq \mathcal{B}(H)$ *injective* (it is indeed an injective object in a suitable category) if there exists a projection $\mathcal{B}(H) \to A$ of norm 1. Connes has shown that many apparently quite different ways lead to the same class, and in particular, a von Neumann algebra is injective if and only if it is hyperfinite. Using this result and the theorem of Johnson, Kadison and Ringrose cited above, he established in 1978 the following theorem.

THEOREM [29]. *A von Neumann algebra in amenable-after-Connes if and only if it is injective (in his sense).*

(Actually Connes proved his theorem with some extra conditions on the algebra in question, which were later removed by Elliott [31]).

Thus it was shown that the normal cohomology groups (at least, with normal coefficients) are of significant independent interest. Moreover, in the same paper Connes has proved the "$\Rightarrow$" part of the following result, which establish a connection between both types of amenability.

THEOREM. *A C^*-algebra A is amenable-after-Johnson if and only if its enveloping von Neumann algebra A^{**} is amenable-after-Connes.*

The "$\Leftarrow$" part of this theorem which appeared far less obvious, was proved in

1982 by Haagerup [36], as a byproduct of his eventual solution of the difficult problem of the description of amenable-after-Johnson C^*-algebras as nuclear C^*-algebras (these things will be discussed later). Effros [50], in a paper of 1988 has given a direct and more simple proof of the assertion. Now we proceed to some recent results, proved in 1988, which, in particular, will permit us to show that the "$\Leftarrow$" part of the previous theorem is an immediate corollary of a description of amenable - this time after-Connes - C^*-algebras in appropriate terms of "full" homology.

First of all we shall show that the normal cohomology groups, notwithstanding their definition with the help of a non-normed topology, can also be expressed in the language of the "Banach" Ext spaces which were introduced above. For this aim let us consider a rather important object, which is connected to a given operator C^*-algebra A. It is the A-bimodule which is predual to the Banach A-bimodule $\bar{A}$ (that is, the ultra-weak closure of A); we denote it by $\bar{A}_*$. It is easy to see that $\bar{A}_*$ is actually the closed sub-A-bimodule of A^* consisting of all normal functionals. For example, if $A = \mathcal{K}(H)$ (the algebra of compact operators on H), then $\bar{A} = \mathcal{B}(H)$ and $\bar{A}_* = A^* = \mathcal{N}(H)$; if $A = C[0,1] \subseteq \mathcal{B}(L^2[0,1])$, then $\bar{A} = L^\infty[0,1]$, $A^* = M[0,1]$ and $\bar{A}_* = L^1[0,1]$. In order to better present the following result, let us at first recall the formula $\mathcal{H}^n(A,X) = \mathrm{Ext}^n_{A-A}(X_*,A^*)$ for all dual Banach A-bimodules $X = (X_*)^*$ (see above).

THEOREM [52]. *Let* A *be an operator* C^**-algebra,* $X = (X_*)^*$ *a normal* A-*bimodule. Then, up to a topological isomorphism,*

$$\mathcal{H}^n_w(A,X) = \mathrm{Ext}^n_{A-A}(X_*\,,\,\bar{A}_*) \textit{ for all } n \geq 0\,.$$

The idea of the proof is to compute the mentioned Ext with the help of a special injective resolution of the bimodule $\bar{A}_*$. To begin with, let us consider the injective A-bimodule $(A\times\ldots\times A)^*$ consisting of all n-linear continuous functionals on A, with the actions $(a \cdot f)(a_1,\ldots,a_n) := f(a_1,\ldots,a_{n-1},a_n a)$ and $(f \cdot a)(a_1,\ldots,a_n) := f(aa_1,a_2,\ldots,a_n)$. Further, for any subset $\alpha \subseteq \{1,\ldots,n\}$ let us consider the closed sub-A-bimodule $(A\times..\times A)_\alpha$ consisting of those functionals which are normal relative to the variables with indices in α. By using some properties of the universal representation of our C^*-algebra A, it is not difficult to prove that $(A\times...\times A)_\alpha$ is a retract of $(A\times...\times A)^*$ in **A-mod-A** ; hence it is also injective. In particular, the bimodule $(A\times...\times A)_*$ consisting of n-linear functionals which are normal relative to each variable (the case $\alpha = \{1,...,n\}$) , and the bimodule $(A\times...\times A)_1$ consisting of functionals which are normal in the first variable (the case $\alpha = \{1\}$) , are injective A-bimodules.

Now let us consider the complex

$$0 \to \bar{A}_* \xrightarrow{\pi_*} (A\times A)_* \xrightarrow{\epsilon_0} (A\times A\times A)_* \longrightarrow \ldots \longrightarrow \underbrace{(A\times..\times A)}_{n}{}_* \xrightarrow{\epsilon_{n-2}} \ldots \qquad (St_*)$$

where π_* is given by $\pi_* g(a,b) := g(ab)$ and ϵ_{n-2} is given by $(\epsilon_{n-2}f)(a_1,...,a_{n+1}) := f(a_1a_2,a_3,...,a_{n+1}) - f(a_1,a_2a_3,...,a_{n+1}) +...+(-1)^n f(a_1,...,a_{n-1},a_na_{n+1})$. This is admissible and hence it is an injective resolution of the A-bimodule $\bar{A}_*$. Therefore $\mathrm{Ext}^n_{A-A}(X_*,\bar{A}_*)$ can be computed as nth cohomology of the complex ${}_A\mathbf{h}_A(X_*,St_*)$. The last step of the proof is to establish, with the help of the conjugate associativity (we mean formulae like $\mathcal{B}(E\times F,G) = \mathcal{B}(E,\mathcal{B}(F,G))$, that the latter complex is isomorphic to the "normal" standard complex $\mathcal{C}^n_w(A,X)$.

Now let us return to the theorem of Johnson, Kadison and Ringrose. In the framework of "full" homology, it can be included as a principal particular case in the following result.

Theorem [52]. *Let* A *be an operator* C^**-algebra,* $X = (X_*)^*$ *a dual Banach A-bimodule, which is normal either from the left, or from the right (that means that either the operators* $A \to X : a \mapsto a \cdot x$ *are normal for all* $x \in X$ *or the same is true with* $a \mapsto x \cdot a$*). Then* $\mathrm{Ext}^n_{A-A}(X_*,A^*) = \mathrm{Ext}^n_{A-A}(X_*,\bar{A}_*)$ *for all* $n \geq 0$.

Indeed, for any normal X, as we just have seen, the right Ext, *is* $\mathcal{H}_w^n(A,X)$, and for any dual X, as it was indicated earlier, the left Ext *is* $\mathcal{H}^n(A,X)$. Hence, for any normal X, $\mathcal{H}_w^n(A,X) = \mathcal{H}^n(A,X)$.

As to the proof of the theorem, it is based on computing the left Ext with the help of an injective resolution

$$0 \to A^* \xrightarrow{\pi_*} (A\times A)^* \xrightarrow{\epsilon_0'} \ldots \to (A\times\ldots\times A)^* \xrightarrow{\epsilon_n'} \ldots \qquad (St^*)$$

and on computing the right Ext with the help of an injective resolution

$$0 \to \bar{A}_* \xrightarrow{\pi_1} (A\times A)_1 \xrightarrow{\epsilon_0''} \ldots \to (A\times\ldots\times A)_1 \xrightarrow{\epsilon_n''} \ldots \qquad (St_1).$$

(The bimodules $(A\times\ldots\times A)^*$ and $(A\times\ldots\times A)_1$ were defined above; ϵ_n' and ϵ_n'' act just as ϵ_n in St_*; π_1 acts just as π^* and π_*).

As a result, we proceed to the cohomology of the complexes ${}_A\mathbf{h}_A(X_*,St^*)$ and ${}_A\mathbf{h}_A(X_*,St_1)$, and those, as it is easy to observe, actually coincide.

Now a natural question arises: what is hidden behind the question mark in the "ratio"

$$\frac{\text{contractibility}}{\text{projectivity of } A} = \frac{\text{amenability-a.-J.}}{\text{injectivity of } A^*} = \frac{\text{amenability-a.-C.}}{?}$$

THEOREM. [52] *An operator C^*-algebra* A *is amenable-after-Connes if and only if the A-bimodule* $\bar{A}_*$ *is injective.*

As a direct corollary, we obtain from here a short proof of the assertion that amenability-after-Connes of A implies triviality of $\mathcal{H}_w^n(A,X)$ - that is, $Ext_{A-A}^n(X_*,\bar{A}_*)$ - for all normal X and all $n > 0$.

The proof of this theorem is essentially more difficult than the description of the corresponding "full homology background" of contractible and amenable-after-Johnson Banach algebras. The main obstacle is that the condition of amenability-after-Connes does not permit one *a priori* to apply the criterion of the injectivity in terms of Ext: there are too few A-bimodules which are predual to normal bimodules. (Let us notice in this connection that, though every von Neumann algebra B such that $A \subseteq B \subseteq \mathcal{B}(H)$ is a normal A-bimodule, neither A^* nor A^{**} are normal. But the most inconvenient circumstance in this relation is that the A-bimodule $((A\times A)_*)^*$ is also apparently not normal (cf. [50]).

Nevertheless, it happens to be possible to represent $\bar{A}_*$ as a retract of the (certainly injective, as we remember) A-bimodule $(A\times A)_*$ with the help of some strong medicine of the deep lemma 2.3 of the paper of Effros [50]. (And this lemma, in its turn, is founded on a non-trivial achievement of Haagerup - the so-called Grothendieck-Pisier-Haagerup inequality for bilinear functionals on C^*-algebras.)

The corresponding criterion for the amenability-after-Connes in terms of a canonical morphism (recall that the latter was π for contractibility and π^* for amenability -after-Johnson) is as follows.

THEOREM. *An operator C^*-algebra* A *is amenable-after-Connes if and only if the morphism* $\pi_* : \bar{A}_* \to (A\times A)_* : g \mapsto f : f(a,b) := g(ab)$ (cf. above) *is a coretraction in* **A-mod-A** .

We are able at last to give a short proof of the Haagerup's implication "A^{**} is amenable-after-Connes only if A is amenable-after-Johnson" (see above). Indeed, the previous theorem being applied to A^{**} implies that $\pi_*:A^{**})_* \to (A^{**}\times A^{**})_*$ possesses a left inverse morphism of A^{**}-bimodules, which is a morphism of A-bimodules into the bargain. But $(A^{**})_*$ is just A^* and $(A^{**}\times A^{**})_*$, by virtue of a theorem of Johnson, Kadison and Ringrose on the extension of normal multilinear operators, is just $(A\times A)^*$. We only have to observe that, by identifying these A^{**}-, and hence A-, bimodules, we transform π_* into the canonical morphism $\pi^* : A^* \to (A\times A)^*$.

Now let us return to the problem of describing C^*-algebras, which are amenable(-after-Johnson), in intrinsic terms of C^*-algebras theory. If A and B are C^*-algebras then generally speaking, there exist many C^*-norms in $A \otimes B$ (by the way, these include a maximal and minimal among them). However, for a large class of C^*-algebras A, there is only one C^*-norm on $A \otimes B$ for every C^*-algebra B ; it was Lance [20] who has coined the name "nuclear" for these C^*-algebras, taking a view obviously parallel to the definition of nuclear spaces by Grothendieck. One can come to the same nuclear C^*-algebras by many apparently quite different ways. For us, in particular, it is important that Choi and Effros [25] have established the following deep connection between nuclearity and injectivity in the sense of Connes: a C^*-algebra A is nuclear if and only if its enveloping von Neumann algebra A^{**} is injective.

Now we have already discussed the following equivalences:

$$\begin{array}{ccc} \text{amenability-after-Johnson of } A & & \text{nuclearity of } A \\ \Updownarrow & & \Updownarrow \\ \text{amenability-after-Connes of } A^{**} & \Longleftrightarrow & \text{injectivity of } A^{**} \end{array}$$

As a corollary, we have

THEOREM (Connes, Haagerup). *A C^*-algebra A is amenable-after-Johnson if and only if it is nuclear.*

(Actually this theorem was proved earlier than when the short proof of the equivalence on the left became known. It was Connes who proved "$\Rightarrow$" , and gave a conjecture that the converse was also true (1978, [29]). However, several attempts to prove this converse had failed before Haagerup [36] managed to succeed in 1982).

So, we have already discussed contractible algebras, amenable algebras and - in the framework of operator algebras - amenable-after-Connes algebras. There is one more condition of homological triviality which is still to be discussed.

To begin with, let us recall that if a Banach algebra A has a unit, then it is contractible if and only if it is projective as an A-bimodule. It is easy to observe that in the case of non-unital A the latter property gives rise to a wider class of algebras.

DEFINITION. A Banach algebra A (unital or non-unital) is called *biprojective* if A it is a projective Banach A-bimodule.

It is not difficult to establish an equivalent definition in terms of a corresponding canonical morphism:

THEOREM. A *is biprojective if and only if the product map* $\pi : A \hat{\otimes} A \to A$ *is a retraction in* ***A-mod-A***.

It was the following property of biprojective algebras which was actually the initial stimulus to their selection and investigation.

THEOREM. *If a Banach algebra* A *is biprojective then* $\mathcal{H}^n(A,X) = 0$ *for all Banach A-bimodules* X *and for all* $n \geq 3$.

This theorem was proved by the present speaker [16] for all biprojective algebras and by Johnson [14] (in equivalent terms) for all amenable biprojective algebras about the same time. Let us mention in this connection that a Banach algebra A is called *biflat* if the A-bimodule A is flat. By analogy with the proof of the previous theorem - for the detailed exposition see, e.g., [42] - one can prove the following: if A is biflat, then $\mathcal{H}^n(A,X) = 0$ for all *dual* A-bimodules X and for all $n \geq 3$.

Now let us turn to examples. The simplest non-contractible biprojective algebra is perhaps ℓ_1 : the canonical morphism $\pi : \ell_1 \hat{\otimes} \ell_1 \simeq L^1(\mathbb{N}\times\mathbb{N}) \to L^1(\mathbb{N}) = \ell_1$ obviously has a right inverse which sends "row" $(\xi_1,..,\xi_n,..)$ to "matrix"

$$\begin{bmatrix} \xi_1 & & O \\ & \xi_2 & \\ O & & \ddots \end{bmatrix}.$$

As to some popular classes of Banach algebras, the situation is as follows:

THEOREM ([18], see, e.g.,[42]). *The algebra* $C_0(\Omega)$ *of all continuous functions on a locally compact space* Ω *, which vanish at the infinity, is biprojective if and only if* Ω *is discrete.*

THEOREM ([18], see, e.g., [42]). *An algebra* $L^1(G)$ *where* G *is a locally compact group, is biprojective if and only if* G *is compact.*

THEOREM [12], [32] *A* C^**-algebra* A *is biprojective if and only if its primitive spectrum is discrete, and all its irreducible representations are finite-dimensional (that is, our* A *is a* c_0*-sum of some family of algebras of the form* $\mathcal{B}(H)$ *for Hilbert spaces* H *of, generally speaking, different finite dimensions).*

Our last (but not least, as we shall now see) example is as follows. Let E be a Banach space, E^* be its dual and let $E \hat{\otimes} E^*$ be considered as a Banach algebra with the multiplication which is defined by $(x \otimes f)(y \otimes g) := [g(x)](y \otimes f)$. We recall that in the case, when E has the approximation property, $E \hat{\otimes} E^*$, up to a isometric isomorphism, is just the algebra $\mathcal{N}(E)$ of all nuclear operators in E with the nuclear norm. For $A = E \hat{\otimes} E^*$, let us fix $x_0 \in E$, $f_0 \in E^*$ with $f_0(x_0) = 1$, and put $\rho : A \to A \hat{\otimes} A : x \otimes f \mapsto (x_0 \otimes f) \otimes (x \otimes f_0)$. Since $\pi\rho = 1$ and ρ is a morphism in **A-mod-A**, we have established that $E \hat{\otimes} E^*$ is biprojective.

REMARK. So for a Hilbert space H, $\mathcal{N}(H)$ is certainly biprojective; at the same time it is not amenable because it has no b.a.u.. On the other hand, $\mathcal{K}(H)$ is certainly not biprojective because it has no finite-dimensional irreducible representation (see the criterion above); at the same time it is amenable - actually it was one of the first examples of amenable algebras [6], [9]. Finally, $\mathcal{B}(H)$, being injective-after-Connes, is amenable-after-Connes; at the same time it is neither biprojective nor amenable-after-Johnson (the latter because, as Wassermann [24] has proved, it is not a nuclear C^*-algebra).

As to the algebra $\mathcal{N}(E)$ for a "good" E, it happens to be something more than simply one of the examples. Selivanov [32] (and this is, perhaps, the deepest result in his thesis) has shown that these algebras are actually the blocks from which an arbitrary biprojective Banach algebra belonging to a rather wide class is built. With some degree of simplification, his theorem is as follows: every biprojective semisimple Banach algebra which has the approximation property, is a topological direct sum of so-called algebras of nuclear operators of some dual pairs of Banach spaces [19], [27]. (We get the "usual" algebra $\mathcal{N}(E)$ in the case of the standard dual pair (E,E^*)).

The last topic of our talk concerns rather important numerical characteristics of Banach algebras and modules - that is, their so-called homological dimensions. Informally, the homological dimension of a (bi)module X over a Banach algebra A is a number (or ∞) which shows to what degree this module is "homologically worse" than the projective modules. In order to give a formal definition, we shall say that a given complex of the form

$$0 \longleftarrow X \longleftarrow X_0 \longleftarrow \dots \longleftarrow X_n \longleftarrow \dots$$

has a length n, if $X_n \neq 0$ and $X_k = 0$ when $k > n$.

DEFINITION. Let X be a Banach A-(bi)module. The minimal length of a projective resolution if X is called the *homological dimension* of X, and is denoted by $dh_A X$ for $X \in$ **A-mod** and by $dh_{A-A} X$ for $X \in$ **A-mod-A**.

(We put $dh_{(\cdot)} X = \infty$ if it has no projective resolution of finite length).

Since a projective P has a resolution of form $0 \leftarrow P \leftarrow P \leftarrow 0$, (bi)modules of homological dimension zero are just projective (bi)modules.

An alternative, and somewhat more instructive, definition can be given as follows. Let us recall that a projective resolution of X is a compact form of describing the process of representing first X, and then kernels of corresponding quotient maps as factor (bi)modules of projective (bi)modules (with additional assumptions about such representations, which were described earlier). In terms of this process, the homological dimension of X is just the first number n of the step when the kernel K_n of corresponding quotient map $P_{n-1} \to K_{n-1}$ is itself a projective A-(bi)module. This number does not depend on our choice of projective (bi)modules and quotient maps participating in the process.

The third equivalent definition, this time in terms of Ext, is as follows: $\mathrm{dh}_A X := \min\{n : \mathrm{Ext}_A^k(X,Y) = 0$ for all $Y \in$ **A-mod** and for all $k > n\}$. (The same is true for A-bimodules as well).

Now we shall indicate several results, which show the connection of homological dimension with some problems of topology and analysis. Let us begin with the discussion of the homological dimensions of (closed) ideals in $C(\Omega)$ for a compact set Ω. The first fact to be established was as follows:

THEOREM [5, 1970]. *An ideal* I *in* $C(\Omega)$ *is a projective* $C(\Omega)$*-module* (*that is,* $\mathrm{dh}_{C(\Omega)} I = 0$) *if and only if the spectrum of* I (*in other words, the complement to the hull of* I *in* Ω) *is paracompact.*

So, if Ω is metrisable and compact, then all ideals in $C(\Omega)$ are projective $C(\Omega)$-modules. On the contrary, for $\Omega = \beta\mathbb{N}$ and for every $t \in \beta\mathbb{N}\backslash\mathbb{N}$, the maximal ideal $I_t = \{f \in C(\Omega) : f(t) = 0\}$ is not projective (here $C(\Omega)$ is, of course, just c_b). (Applying some of these ideas to non-commutative C^*-algebras, Lykova [43] has proved that every closed left ideal in a separable C^*-algebra is projective; on the contrary an infinite-dimensional von Neumann algebra always contains a non-projective, closed left ideal.)

The previous theorem has shown the existence of compact sets Ω and ideals $I \subset C(\Omega)$ with positive homological dimension. But what can this dimension actually be? The next step was made by Moran [26] in 1977; he took as Ω the space of ordinals from 0 to $\aleph_\omega$ (limit of $\aleph_n$, $n \in \mathbb{N}$) and has shown that $dh_{C(\Omega)} I = \infty$ for $I = \{f \in C(\Omega) : f(\aleph_\omega) = 0\}$. Finally, after some time, Krichevetz ([45], 1986), constructed an example of a compact set Ω such that the homological dimension of its maximal ideals can be any integer from 0 to n. (Actually, his compact is the n-th Cartesian power of the one-point compactification of a sufficiently large discrete topological space).

Our next example is connected with complex analysis, more exactly with the old circle of problems about the existence of so-called analytic structure in subsets of the spectrum Ω of a given commutative Banach algebra A. Let us recall that a subset Δ of Ω is called an *analytic n-disc* in Ω if there exists a homeomorphism between Δ and the unit n-disc $\mathbb{D}^n \subset \mathbb{C}^n$ such that the functions $a(t)$; $t \in \Omega$ become analytic on Δ after identifying the latter with $\mathbb{D}^n$.

If we take the polydisc algebra $\mathcal{A}(\bar{\mathbb{D}}^n)$, then every point of the subset $\mathbb{D}^n$ of its spectrum $\bar{\mathbb{D}}^n$ naturally has a neighbourhood which is an analytic n-disc. Now let us notice that at the same time the maximal ideal in $\mathcal{A}(\bar{\mathbb{D}}^n)$ which corresponds to such a point, has homological dimension n-1. The following theorem shows that this example reflects the general situation. The proof relies heavily on results of T. T. Read [8].

THEOREM (Pugach [41]). *Let* A *be a commutative Banach algebras with spectrum* Ω, I *a maximal ideal such that* $dh_A I = n - 1$.

(1) (*linear*) $\dim I/I^2$, *where* I^2 *is the topological square of* I, *is not bigger than* n [*and actually it can be any integer among* 0,1,..,n-A.H.]

(2) [*most essential part* - A.H.] *if* $\dim I/I^2 = n$ ("*non-degenerate case*"), *then there exists a neighbourhood of* I *in* Ω *which is an analytic n-disc.*

In particular, if I is a projective A-module with $\dim I/I^2 = 1$, then I is an inner point of some analytic disc in Ω; this fact was established previously by Pugach in [37]. But in [41] he also proved the following interesting counterpart to this result if I with $\dim I/I^2 = 1$ is "only" flat : I belongs to some analytic disc Δ in Ω, but, generally speaking, Δ is not obliged to be a neighbourhood of I.

The third example is the so-called augmentation ideal I_0 in $L^1(G)$, where G is a locally compact group; it consists of all $f \in L^1(G)$ with $\int_G f(s)ds = 0$ (we mean, as usual, integration with respect to the left-invariant Haar measure on G). It is easy to observe that compactness of G implies $dh_{L^1(G)} I_0 = dh_{L^1(G)} L^1(G)/I_0 = 0$. Then it was proved in [17] that for the case of a locally compact, non-compact commutative group G, every maximal ideal $I \subset L^1(G)$ satisfies, $dh_{L^1(G)} I > 0$. But the most important is apparently the following result, which was obtained by Sheinberg in 1973 [21].

THEOREM. *Let* G *be an amenable, locally compact, non-compact group. Then* $dh_{L^1(G)} I_0 = dh_{L^1(G)} L^1(G)/I_0 = \infty$.

(In fact, as was noticed by Sheinberg himself, the theorem is valid for an arbitrary G which contains a non-compact, closed, amenable subgroup).

Now let us proceed from the homological dimension of individual modules over a given algebra to the so-called homological dimensions of the algebra itself. There are several variants of the notion of homological dimension of an algebra A, which show how "homologically nice" is this or that class of A-(bi)modules. Here we shall restrict ourselves to two of them. Let A be a Banach algebra.

DEFINITION. The number (or ∞) $\sup\{dh_A X : X \in A\text{-mod}\}$ is called the *left projective global homological dimension of* A.

DEFINITION. The number (or ∞) $dh_{A-A} A_+$ is called the *projective homological* *bidimension of* A.

For brevity, we shall use the terms "global dimension of A" and "bidimension of A" , and denote them respectively by dg A and db A . It is obvious, that in terms of Ext,

$$\text{dg } A = \min\{n : \text{Ext}_A^k(X,Y) = 0 \text{ for all } X,Y \in \textbf{A-mod} \text{ and for all } k > n\}, \text{ and}$$

$$\text{db } A = \min\{n : \mathcal{H}^n(A,X) = 0 \text{ for all } X \in \textbf{A-mod-A} \text{ and for all } k > n\}.$$

These equivalent definitions imply immediately that $\text{dg } A \leq \text{db } A$ for every A . It seems to be unknown as to whether strict inequality can actually happen for Banach algebras (in pure algebra corresponding examples do exist).

Which values can in fact be attained by homological dimensions for different classes of Banach algebras - concrete or more or less general?

We have observed that for Banach algebras with "good geometry" and, in particular, for commutative Banach algebras, the equality $\text{dg } A = \text{db } A = 0$ holds if and only if A is classically semisimple (that is, $A = \mathbb{C}^n$ in the commutative case). As for other values, the following old result shows that for function algebras 1 is "forbidden" but 2 is "permitted".

THEOREM ([12], [16], 1972). *Let* A *be an infinite-dimensional Banach function algebra. Then* dg A, db A > 1 . *If, in addition, A is biprojective, then* $\text{dg } A = \text{db } A = 2$.

(Parts of this result were cited above).

So, for such algebras as c_0 or ℓ_1 , both of their homological dimension are equal to 2 ; it is worth mentioning, that $\text{dh}_{c_0} c_b = \text{dh}_{\ell_1} c_b = 2$ [16], where c_b is considered as a c_0 - (or ℓ_1) module with coordinatewise outer multiplication. As to non-commutative algebras, $\text{dg } A = \text{db } A = 2$ for such A as, say, $L^1(G)$ and $C^*(G)$

for compact G , and also $\mathcal{N}(H)$ for a Hilbert space H . It is also known that for every infinite-dimensional CCR-algebra A, dg A , db A ≥ 2 (Lykova [44]).

Further, for every *even* $n \in \mathbb{N}$ there are function Banach algebras A with dg A = db A = n ; for example, it is the case with $A = \underbrace{c \hat{\otimes} \ldots \hat{\otimes} c}$ (n factors) , where $c = (c_0)_+$ is the algebra of all convergent sequences, (Krichevetz [38]). But we do not know the answer to the following.

QUESTION. Does there exist a function Banach algebra A with dgA = n and/or dbA = n for some *odd* positive integer n > 1? In particular, is it true for n = 3?

REMARK. For every $n \in \mathbb{N}$, Banach algebras A with dgA = dbA = n do certainly exist, but they are, generally speaking, neither commutative nor semisimple. For example, dg A = db A = 1 for the algebra A consisting of 2×2 matrixes of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$; $a , b \in \mathbb{C}$. To speak precisely, this algebra provides a positive answer to a question, raised by Effros and Kishimoto [47]: do there exist non-amenable Banach algebras A with $\mathcal{H}^2(A,X) = 0$ for all dual A-bimodules X ? We think, however, that the authors of [47] implicitly had in view some more concrete class of algebras. In any case, the answer is unknown to us, if one restricts to function algebras or to C^*-algebras.

Concluding our talks, we should like to put again one old question about two of "the most popular" algebras, which has been explicitly stated on several occasions (see, e.g., [14], [30], [35], [42]).

QUESTION. What are the homological dimensions (or at least some one of them) of C[0,1] ? $\mathcal{K}(H)$?

(The apparent simplicity of these algebras must not deceive: in trying to answer the question, we have to work with their tensor powers, and these things are far more complicated.)

Let us notice that Johnson had also asked in [35] about db A for $A = \ell_1(\mathbb{Z})$

(that is, for the Wiener algebra). But the theorem of Sheinberg [21], which was cited above, provides immediately that $\mathrm{dg}\ \ell_1(\mathbb{Z}) = \mathrm{db}\ \ell_1(\mathbb{Z})_1) = \infty$.

To conclude, the present speaker thanks his audience for their patience and humbly asks them to forgive his unbearable English (in fact, it is his first experience of this kind).

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