REMARKS ON SEMI-SIMPLE REFLEXIVE ALGEBRAS

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1. INTRODUCTION AND PRELIMINARIES

Below, we discuss the problem: For which subspace lattices $\mathcal{L}$ on a complex Banach space X is $\operatorname{Alg}\mathcal{L}$ semi-simple? Our first proposition gives a lattice-theoretic necessary condition on $\mathcal{L}$ in order that $\operatorname{Alg}\mathcal{L}$ be semi-simple, and an improvement of a result of Lambrou [5] follows from it. The converse of this proposition, though not true in general, is shown to be valid for two large classes of subspace lattices, each of which contains every completely distributive subspace lattice amongst its members. Each of these two classes also contains every pentagon. The Alg of every pentagon is shown to be semi-simple. For certain subspace lattices which belong to both these aforementioned classes we obtain a result (Theorem 3) which has another result of Lambrou [5] as a corollary. Examples are given to show that, in general, the semi-simplicity of $\operatorname{Alg}\mathcal{L}$ cannot be a purely lattice-theoretic property of $\mathcal{L}$; more precisely, we show that it is possible to have $\mathcal{L}_1$ and $\mathcal{L}_2$ lattice-isomorphic with $\operatorname{Alg}\mathcal{L}_1$ semi-simple and $\operatorname{Alg}\mathcal{L}_2$ not. We also show that any double triangle on X with elements $K, L \neq X$ satisfying $K + L = X$ has a semi-simple Alg.

We use the notation, terminology and definitions of the preceding article [10]. Some of these are repeated here for the convenience of the reader. In particular, throughout X denotes a complex non-zero Banach space. Also, by a *subspace lattice on X* we mean a collection $\mathcal{L}$ of subspaces of X satisfying (i) $(0), X \in \mathcal{L}$, and (ii) for every family $\{M_\gamma\}$ of elements of $\mathcal{L}$, both $\cap M_\gamma$ and $\vee M_\gamma$ belong to $\mathcal{L}$. For every subspace lattice $\mathcal{L}$ on X we define $\operatorname{Alg}\mathcal{L}$ by

$$\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(X) : TM \subseteq M\,, \quad \text{for every } M \in \mathcal{L}\}\,.$$

For every $\mathcal{L}$, $\operatorname{Alg}\mathcal{L}$ is a unital Banach algebra and the class of operator algebras of the form $\operatorname{Alg}\mathcal{L}$ for some subspace lattice $\mathcal{L}$ (on X) is the class of *reflexive* operator algebras (on X). On any subspace lattice $\mathcal{L}$ on X, we define the map $L \mapsto L_-$ of $\mathcal{L}$ into itself by

$$L_- = \vee\{K \in \mathcal{L} : L \not\subseteq K\}.$$

The conventions $\cap\emptyset = X$ and $\vee\emptyset = (0)$ are observed (then $(0)_- = (0)$).

DEFINITION. Let $\mathcal{L}$ be a subspace lattice on X. Define the subset $\mathcal{J} \subseteq \mathcal{L}$ by

$$\mathcal{J} = \{L \in \mathcal{L} : L \neq (0) \quad \text{and} \quad L_- \neq X\}.$$

In what follows, we will refer to this subset $\mathcal{J}$ often. Its importance lies in its intimate relation with the set of rank one operators in $\operatorname{Alg}\mathcal{L}$. The following lemma will be used repeatedly.

LEMMA 1. ([9], see also [7, 10]) *If $\mathcal{L}$ is a subspace lattice on X, the rank one operator $f^* \otimes e$ belongs to $\operatorname{Alg}\mathcal{L}$, if and only if $e \in L$ and $f^* \in (L_-)^\perp$ for some $L \in \mathcal{J}$.*

Recall that if $e \in X$ and $f^* \in X^*$, the operator $f^* \otimes e \in \mathcal{B}(X)$ is defined by $(f^* \otimes e)(x) = f^*(x)e$. Note that, for every $T \in \mathcal{B}(X)$, $T(f^* \otimes e) = f^* \otimes Te$ and $(f^* \otimes e)T = T^*f^* \otimes e$ where $T^* \in \mathcal{B}(X^*)$ denotes the adjoint of T. Also note that $(f^* \otimes e)^2 = f^*(e)(f^* \otimes e)$.

A subspace lattice $\mathcal{L}$ on X is *distributive* if $K \cap (L \vee M) = (K \cap L) \vee (K \cap M)$ holds identically in $\mathcal{L}$. We call $\mathcal{L}$ $\vee$-*distributive* if $K \cap (\vee L_\gamma) = \vee(K \cap L_\gamma)$ holds for every $K \in \mathcal{L}$ and every family $\{L_\gamma\}$ of elements of $\mathcal{L}$. This notion is obviously stronger than distributivity. Stronger still is the notion of *complete distributivity*. We will need the following two characterizations. (These were mentioned in [10] and proofs can be found in [9]. The latter proofs are based on a characterization of complete distributivity due to Raney [11].) A subspace lattice $\mathcal{L}$ on X is completely distributive, if and only if

$$L = \vee\{M \in \mathcal{L} : L \not\subseteq M_-\}, \quad \text{for every } L \in \mathcal{L},$$

if and only if

$$L = \cap\{M_- : M \in \mathcal{L} \quad \text{and} \quad M \not\subseteq L\}, \quad \text{for every } L \in \mathcal{L}.$$

The abbreviation ABSL will be used for atomic Boolean subspace lattice (see [10]). Every ABSL is completely distributive (see [9]) and we will need the fact that, for any ABSL $\mathcal{L}$ on X, the subset $\mathcal{J}$ is precisely the set of atoms of $\mathcal{L}$; moreover $K_- = K'$ (the Boolean complement of K) for every atom K of $\mathcal{L}$ [9].

Recall that a complex unital Banach algebra $\mathcal{A}$ is *semi- simple* if and only if it has no non-zero left ideals (or no non-zero right ideals) consisting entirely of quasinilpotent elements. Also, $\mathcal{A}$ is *semi-prime* if and only if it has no non-zero left ideal whose square is zero. Clearly, $\mathcal{A}$ is semi-simple implies $\mathcal{A}$ is semi-prime.

2. SOME RESULTS

Our first two theorems concern two classes of subspace lattices on X. The first class consists of those subspace lattices $\mathcal{L}$ which satisfy $\vee\{L \in \mathcal{L} : L \in \mathcal{J}\} = X$ and the second, those which satisfy $\cap\{L_- : L \in \mathcal{J}\} = (0)$. Note that since $\vee\emptyset = (0)$ and $\cap\emptyset = X$, $\mathcal{J}$ is non-empty for any member of either class. We will show (Example 2) that neither of these classes includes the other. Our theorems show that, for each of these classes, the semi-simplicity of $\operatorname{Alg}\mathcal{L}$, for any member $\mathcal{L}$, is a lattice-theoretic property of $\mathcal{L}$. (In these theorems we may, of course, take $\mathcal{L} = \operatorname{Lat}\mathcal{A}$ where $\mathcal{A}$ is a reflexive algebra; then $\operatorname{Alg}\mathcal{L} = \mathcal{A}$.) First we prove a result of some independent interest.

PROPOSITION 1. *Let $\mathcal{L}$ be a subspace lattice on X. If $\operatorname{Alg}\mathcal{L}$ is semi-prime, then either $\mathcal{J} = \emptyset$, or for every $L \in \mathcal{J}$, $L \cap L_- = (0)$ and $L \vee L_- = X$.*

Proof. *Let $\operatorname{Alg}\mathcal{L}$ be semi-prime, and suppose that $\mathcal{J}$ is non-empty. Let $L \in \mathcal{J}$. Let $e \in L$ and $f^* \in (L_-)^\perp$ be arbitrary non-zero vectors. By Lemma 1, the operator $R = f^* \otimes e$ belongs to $\operatorname{Alg}\mathcal{L}$. By semi-primeness, $RTR \neq 0$ for some operator $T \in \operatorname{Alg}\mathcal{L}$ (otherwise the left ideal of $\operatorname{Alg}\mathcal{L}$ generated by R has square zero). Now $RTR = f^*(Te)R$ so $f^*(Te) \neq 0$. It follows that $L \cap L_- = (0)$ (since $f_1^*(T_1e_1) = 0$ for every $e_1 \in L \cap L_-$, $T_1 \in \operatorname{Alg}\mathcal{L}$ and $f_1^* \in (L_-)^\perp$, because $T_1e_1 \in L \cap L_-$) and that $L \vee L_- = X$ (since $f_2^*(T_2e_2) = 0$ for every $f_2^* \in (L \vee L_-)^\perp$, $T_2 \in \operatorname{Alg}\mathcal{L}$ and $e_2 \in L$, because $T_2e_2 \in L$). This completes the proof.*

An element L of a subspace lattice $\mathcal{L}$ is called *comparable* if, for every $M \in \mathcal{L}$, either $M \subseteq L$ or $L \subseteq M$. Notice that $L_- \subseteq L$ for every comparable element L of $\mathcal{L}$. It follows that the necessary condition for the semi-primeness of Alg $\mathcal{L}$ given in the above proposition is stronger than the condition that $\mathcal{L}$ has no comparable elements except (0) and X; in fact it is strictly stronger as Example 1 below shows. Since semi-primeness is weaker than semi-simplicity, the above proposition improves the following result due to Lambrou.

COROLLARY. (M.S. Lambrou [5]) *Let $\mathcal{L}$ be a subspace lattice on X. If* Alg $\mathcal{L}$ *is semi-simple, then $\mathcal{L}$ has no comparable elements except* (0) *and* X.

EXAMPLE 1. Let H be a non-zero complex Hilbert space and on $H \oplus H \oplus H$ consider the subspace lattice $\mathcal{L}_0$ given by

$$\mathcal{L}_0 = \{(0), (0) \oplus H \oplus (0), H \oplus (0) \oplus (0), H \oplus H \oplus (0), (0) \oplus H \oplus H, H \oplus H \oplus H\}.$$

The Hasse diagram of $\mathcal{L}_0$ is given in Figure 1.

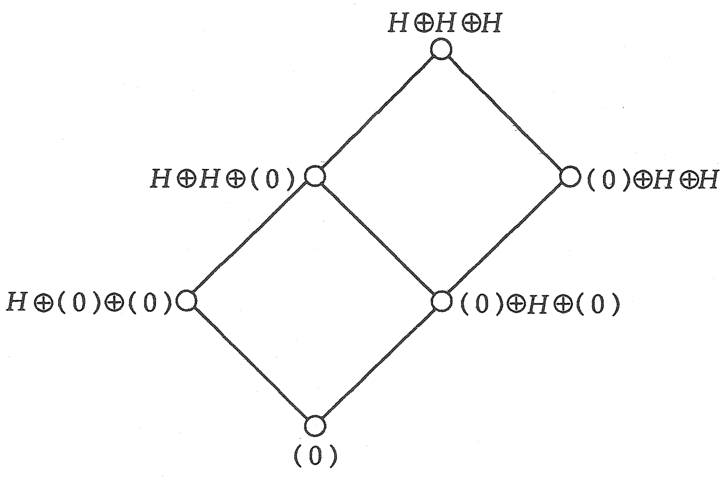

Figure 1

Clearly, $\mathcal{L}_0$ has no comparable elements except (0) and $H \oplus H \oplus H$, but the condition "$\mathcal{J} = \emptyset$ or, for every $L \in \mathcal{J}$, $L \cap L_- = (0)$ and $L \vee L_- = H \oplus H \oplus H$" is false because $L = (0) \oplus H \oplus H$ belongs to $\mathcal{J}$ and $L_- = H \oplus H \oplus (0)$ so $L \cap L_- \neq (0)$. Of course, by Proposition 1, Alg $\mathcal{L}_0$ is not semi-prime (so not semi-simple).

We show later (Example 3) that the converse of Proposition 1 is false by providing an example where $\mathcal{J} = \emptyset$ and $\operatorname{Alg}\mathcal{L}$ is not semi-prime. However, our first two theorems show that the converse is true for any subspace lattice belonging to either of the two classes described in the opening paragraph.

THEOREM 1. *Let $\mathcal{L}$ be a subspace lattice on X satisfying $\vee\{L : L \in \mathcal{J}\} = X$. The following are equivalent.*

(1) $\operatorname{Alg}\mathcal{L}$ *is semi-simple*,

(2) $\operatorname{Alg}\mathcal{L}$ *is semi-prime*,

(3) *For every $L \in \mathcal{J}$, $L \cap L_- = (0)$ and $L \vee L_- = X$*,

(4) *For every $L \in \mathcal{J}$, $L \cap L_- = (0)$.*

Proof. By the definitions, (1) $\Rightarrow$ (2). Since $\mathcal{J}$ cannot be empty, (2) $\Rightarrow$ (3) by Proposition 1. Obviously (3) $\Rightarrow$ (4).

Assume that (4) holds. We show that every non-zero right ideal of $\operatorname{Alg}\mathcal{L}$ contains a non-zero idempotent (so non-quasinilpotent) operator. Let $\mathcal{K}$ be a non-zero right ideal of $\operatorname{Alg}\mathcal{L}$. Let $J \in \mathcal{K}$ be non-zero. Since $\vee\{L \in \mathcal{L} : L \in \mathcal{J}\} = X$, $JL \neq (0)$ for some element $L \in \mathcal{J}$. Now $JL \subseteq L$ so $JL \not\subseteq L_-$ (since $L \cap L_- = (0)$). Thus there exists a vector $e \in L$ such that $Je \notin L_-$. By the Hahn-Banach theorem, there exists a vector $f^* \in (L_-)^\perp$ such that $f^*(Je) = 1$. By Lemma 1, $f^* \otimes e \in \operatorname{Alg}\mathcal{L}$. Thus the non-zero operator $J(f^* \otimes e)$ belongs to $\mathcal{K}$. We have

$$(J(f^* \otimes e))^2 = (f^* \otimes Je)^2 = f^*(Je)(J(f^* \otimes e)) = J(f^* \otimes e)\,.$$

This completes the proof.

THEOREM 2. *Let $\mathcal{L}$ be a subspace lattice on X satisfying $\cap\{L_- : L \in \mathcal{J}\} = (0)$. The following are equivalent.*

(1) $\operatorname{Alg}\mathcal{L}$ *is semi-simple*,

(2) $\operatorname{Alg}\mathcal{L}$ *is semi-prime*,

(3) *For every $L \in \mathcal{J}$, $L \cap L_- = (0)$ and $L \vee L_- = X$*,

(4) *For every* $L \in \mathcal{J}$, $L \vee L_- = X$.

Proof. By the definitions (1) $\Rightarrow$ (2). Since $\mathcal{J}$ cannot be empty, (2) $\Rightarrow$ (3) by Proposition 1. Obviously (3) $\Rightarrow$ (4).

Assume that (4) holds. We show that every non-zero left ideal of $\operatorname{Alg}\mathcal{L}$ contains a non-zero idempotent (so non-quasinilpotent) operator. Let $\mathcal{K}$ be a non-zero left ideal of $\operatorname{Alg}\mathcal{L}$.

For any family $\{L_\gamma\}$ of subspaces of X, the weak* closure of the linear span of $\{L_\gamma^\perp\}$ equals $(\cap L_\gamma)^\perp$. In particular, if $\cap L_\gamma = (0)$, then the linear span of $\{L_\gamma^\perp\}$ is weak* dense in X^*.

Let $J \in \mathcal{K}$ be non-zero. Then $J^* \neq 0$. Since J^* is weak* continuous, and $\cap\{L_- : L \in \mathcal{J}\} = (0)$ implies that the linear span of $\{(L_-)^\perp : L \in \mathcal{J}\}$ is weak* dense in X^*, $J^*(L_-)^\perp \neq 0$ for some element $L \in \mathcal{J}$. Let $f^* \in (L_-)^\perp$ satisfy $J^*f^* \neq 0$. Now $J^*f^* \in (L_-)^\perp$ so, since $L \vee L_- = X$, $J^*f^* \notin L^\perp$. Hence $J^*f^*(e) = 1$ for some vector $e \in L$. By Lemma 1, $f^* \otimes e \in \operatorname{Alg}\mathcal{L}$. Thus the non- zero operator $(f^* \otimes e)\, J$ belongs to $\mathcal{K}$. We have

$$((f^* \otimes e)\, J)^2 = (J^*f^* \otimes e)^2 = J^*f^*(e)((f^* \otimes e)\, J) = (f^* \otimes e)\, J\,.$$

This completes the proof.

The following example shows that the class of subspace lattices $\mathcal{L}$ satisfying $\vee\{L \in \mathcal{L} : L \in \mathcal{J}\} = X$ is not included in the class satisfying $\cap\{L_- : L \in \mathcal{J}\} = (0)$ and vice-versa.

EXAMPLE 2. Let H be an infinite-dimensional, complex, separable Hilbert space and let $A \in \mathcal{B}(H)$ be a positive injective operator which is not invertible. By a result of von Neumann (see [1, Theorem 3.6]) there exists a positive injective operator $B \in \mathcal{B}(H)$ such that the ranges of A and B have only the zero vector in common. Let K and L be the subspaces of $H \oplus H \oplus H$ given by

$$K = \{(x, y, Ax) : \quad x, y \in H\}\,,$$

$$L = \{(x, y, By) : \quad x, y \in H\}\,.$$

Let $\mathcal{L}$ be the subspace lattice on $H \oplus H \oplus H$ with elements (0), $(0) \oplus H \oplus (0)$, $H \oplus (0) \oplus (0)$, $H \oplus H \oplus (0)$, K, L and $H \oplus H \oplus H$. Figure 2 below is a Hasse diagram of $\mathcal{L}$; Figure 3 is a Hasse diagram of $\mathcal{L}^{\perp} = \{M^{\perp} : M \in \mathcal{L}\}$.

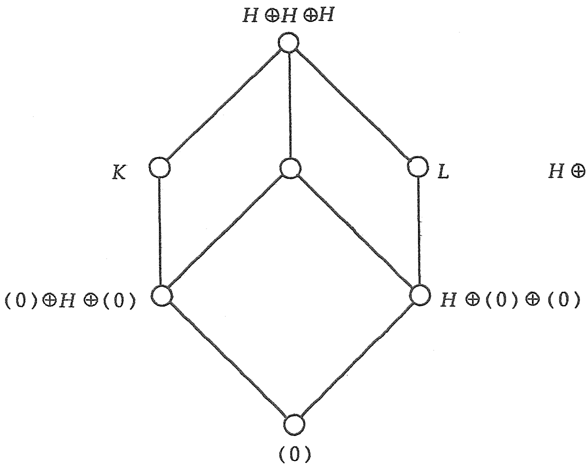

Figure 2

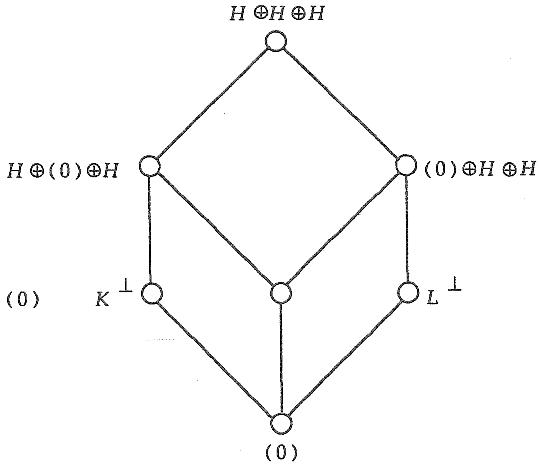

Figure 3

(Note that $(x, y, Ax) = (u, v, Bv)$ gives $Ax = Bv$, so $Ax = Bv = 0$, since the ranges of A and B have only the zero vector in common. This shows that $K \cap L = (0)$.) For $\mathcal{L}$, $\mathcal{J} = \{(0) \oplus H \oplus (0), H \oplus (0) \oplus (0)\}$ so $\mathcal{L}$ does not belong to the first class. But $\mathcal{L}$ does belong to the second class since $((0) \oplus H \oplus (0))_{-} = L$, $(H \oplus (0) \oplus (0))_{-} = K$ and $K \cap L = (0)$.

On the other hand, $\mathcal{L}^{\perp}$ belongs to the first class but not to the second.

Note that, by Theorems 1 and 2, both $\mathcal{L}$ and $\mathcal{L}^{\perp}$ have semi-simple Alg's.

The two classes of subspace lattices described in the opening paragraph of this section are far from being disjoint. Indeed, by the two characterizations of complete distributivity mentioned in the introduction, every completely distributive subspace lattice belongs to both classes; so does every pentagon.

In abstract lattice theory, the pentagon and the double triangle play special roles (this terminology is due to Halmos). These lattices have Hasse diagrams, respectively, as follows.

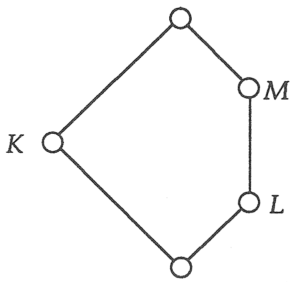

Figure 4

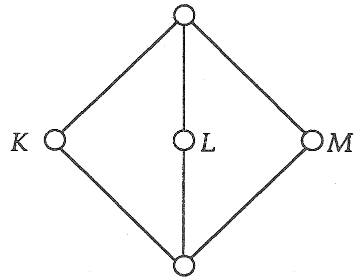

Figure 5

We will comment soon on the Alg of a double triangle subspace lattice (notice that here $\mathcal{J} = \emptyset$), but notice that, indeed, every pentagon subspace lattice on X does belong to both of the aforementioned classes (for a pentagon, $\mathcal{J} = \{K, L\}$ and $K_- = M$, $L_- = K$ if, as in Figure 4, K, L and M are the elements which are neither (0) nor X with $L \subseteq M$). Moreover, by Theorem 1 (or 2), the Alg of any pentagon is semi-simple.

The following characterizes those $\vee$-distributive subspace lattices which belong to both of the aforementioned classes and have semi-simple Alg's.

THEOREM 3. *Let $\mathcal{L}$ be a $\vee$- distributive subspace lattice on X satisfying $\vee\{L \in \mathcal{L} : L \in \mathcal{J}\} = X$ and $\cap\{L_- : L \in \mathcal{J}\} = (0)$. The following are equivalent.*

(1) *For every $K \in \mathcal{J}$, $K \cap K_- = (0)$,*

(2) *$\mathcal{L}$ is an* ABSL,

(3) *For every $K \in \mathcal{J}$, $K \vee K_- = X$,*

(4) Alg $\mathcal{L}$ *is semi-simple,*

(5) Alg $\mathcal{L}$ *is semi-prime.*

Proof. The implication (1) $\Rightarrow$ (2) will be proved last.

(2) $\Rightarrow$ (3): Assume that (2) holds. Then, as remarked in the introduction, $\mathcal{J}$ is the set of atoms of $\mathcal{L}$ and $K_- = K'$ for every $K \in \mathcal{J}$. Thus (3) holds.

(3) $\Rightarrow$ (4): This follows from Theorem 2.

(4) $\Rightarrow$ (5): This is obvious, by the definitions.

(5) $\Rightarrow$ (1): This follows from Theorem 1.

(1) $\Rightarrow$ (2): Assume that (1) holds. We need only show that $\mathcal{L}$ is atomic and complemented.

First we show that every element of $\mathcal{J}$ is an atom of $\mathcal{L}$. Let $K \in \mathcal{J}$ and let $L \in \mathcal{L}$ satisfy $(0) \subseteq L \subset K$. Then $K \not\subseteq L$ so, by the definition of K_-, $L \subseteq K_-$. Hence $L \subseteq K \cap K_- = (0)$, so $L = (0)$. Thus K is an atom of $\mathcal{L}$.

Next we show that $\mathcal{L}$ is complemented. Of course, (0) has a complement in $\mathcal{L}$. Let $M \in \mathcal{L}$ be non-zero. We show that M has complement $M' = \cap\{K_- : K \in \mathcal{J}$ and $K \subseteq M\}$.

Let $K \in \mathcal{J}$. If $K \subseteq M$, then $M \cap M' \subseteq M' \subseteq K_-$, so $M \cap M' \subseteq K_-$. If $K \not\subseteq M$, then $M \subseteq K_-$ by the definition of K_-, and again $M \cap M' \subseteq K_-$. Thus $M \cap M' \subseteq \cap\{K_- : K \in \mathcal{J}\} = (0)$, so $M \cap M' = (0)$.

Since $\vee\{K \in \mathcal{L} : K \in \mathcal{J}\} = X$, to show that $M \vee M' = X$ it is enough to show that $K \subseteq M \vee M'$, for every $K \in \mathcal{J}$. Let $K \in \mathcal{J}$. If $K \subseteq M$, then certainly $K \subseteq M \vee M'$. Suppose that $K \not\subseteq M$. We show that $K \subseteq M'$. Let $W \in \mathcal{J}$ satisfy $W \subseteq M$. Since $K \not\subseteq M$ and $W \subseteq M$, $W \neq K$. Hence, since W and K are atoms of $\mathcal{L}$, $W \not\subseteq K$. By the definition of W_-, it follows that $K \subseteq W_-$. This shows that $K \subseteq M'$. Thus $K \subseteq M \vee M'$ and $M \vee M' = X$.

Finally we show that $\mathcal{L}$ is atomic. Let $L \in \mathcal{L}$ be non- zero. If $K \in \mathcal{J}$ and $K \not\subseteq L$, then $L \subseteq K_-$. Thus L must contain an element of $\mathcal{J}$ (if not, $L \subseteq \cap\{K_- : K \in \mathcal{J}\} = (0)$, giving $L = (0)$). Since $\mathcal{J}$ consists of atoms, L contains an atom of $\mathcal{L}$. Now $X = \vee\{K \in \mathcal{L} : K \in \mathcal{J}\}$ so, by $\vee$-distributivity, $L = L \cap X = \vee\{K \cap L : K \in \mathcal{J}\}$. For every $K \in \mathcal{J}$, $(0) \subseteq K \cap L \subseteq K$ and since K is an atom this implies that $K \cap L = (0)$ or K. Thus $L = \vee\{K : K \in \mathcal{J}$ and $K \subseteq L\}$ and it follows that L is the closed linear span of the atoms it contains. Hence $\mathcal{L}$ is atomic and the proof is complete.

COROLLARY. (M.S. Lambrou [5,6,8]) *Let $\mathcal{L}$ be a completely distributive subspace lattice on X. The following are equivalent.*

(1) $\operatorname{Alg}\mathcal{L}$ *is semi-simple*,

(2) $\operatorname{Alg}\mathcal{L}$ *is semi-prime*,

(3) $\mathcal{L}$ *is an* ABSL,

(4) *For every* $K \in \mathcal{J}$, $K \cap K_- = (0)$.

In this corollary, we may add

(5) *For every* $K \in \mathcal{J}$, $K \vee K_- = X$

to the list of equivalent conditions. For example, by this corollary, the Alg of any totally ordered subspace lattice with more than two elements is not semi-simple (such a subspace lattice is an example of a commutative subspace lattice whose Alg is not semi-simple, given that the underlying space is a Hilbert space; so is $\mathcal{L}_0$ of Example 1).

As promised earlier, the following example proves that the converse of Proposition 1 is false, inasmuch as it shows that there exists a subspace lattice with $\mathcal{J} = \emptyset$ whose Alg is not semi-prime. More significantly, it also shows that, in general the semi-primeness (respectively, semi-simplicity) of $\operatorname{Alg}\mathcal{L}$ is not just a purely lattice-theoretic property of $\mathcal{L}$: that is, it is possible to have two lattice-isomorphic subspace lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ (even on the same space) with $\operatorname{Alg}\mathcal{L}_1$ semi-prime (respectively, semi- simple) and $\operatorname{Alg}\mathcal{L}_2$ not.

A subspace lattice $\mathcal{L}$ on X is called *medial* [2] if $K \cap L = (0)$ and $K \vee L = X$ for every pair of distinct elements $K, L \notin \{(0), X\}$. It is clear that $\mathcal{J} = \emptyset$ for every medial subspace lattice with at least five elements.

EXAMPLE 3. In the following, for any operator S, $G(S)$ denotes its graph.

Let H be a complex Hilbert space with $2 \leq \dim H \leq \infty$, and let $T \in \mathcal{B}(H)$ be a non-zero operator with square zero. On $H \oplus H$ let $\mathcal{L}_2$ be the medial subspace lattice given by

$$\mathcal{L}_2 = \{(0), G(0), G(I), G(T+\alpha I), G(T+\beta I), (0) \oplus H, H \oplus H\},$$

where α and β are any distinct elements of $\mathbf{C}\backslash\{0,1\}$. Then

$$\operatorname{Alg}\mathcal{L}_2 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathcal{B}(H) \quad \text{and} \quad AT = TA \right\}.$$

Since $TAT = AT^2 = 0$ whenever $AT = TA$, the left ideal of $\operatorname{Alg}\mathcal{L}_2$ generated by $\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}$ has square zero. Thus $\operatorname{Alg}\mathcal{L}_2$ is not semi- prime.

On the other hand, there exists a subspace lattice $\mathcal{L}_1$ on $H \oplus H$ which is lattice-isomorphic to $\mathcal{L}_2$ and which satisfies $\operatorname{Alg}\mathcal{L}_1 = \mathbb{C}I$ [2]. Of course, $\operatorname{Alg}\mathcal{L}_1$ is semi-simple.

In the above example, $\mathcal{L}_1$ and $\mathcal{L}_2$ are medial subspace lattices on the given space $H \oplus H$, each with seven elements, with $\operatorname{Alg}\mathcal{L}_1$ semi-simple and $\operatorname{Alg}\mathcal{L}_2$ not semi-prime. Can subspace lattices with these properties, but with six, not seven, elements be found? The answer is affirmative. Indeed, if we omit $G(T + \beta I)$ from $\mathcal{L}_2$, the resulting medial subspace lattice, $\mathcal{M}_2$ say, satisfies $\operatorname{Alg}\mathcal{M}_2 = \operatorname{Alg}\mathcal{L}_2$, so $\operatorname{Alg}\mathcal{M}_2$ is not semi-prime. If we define the medial subspace lattice $\mathcal{M}_1$ by

$$\mathcal{M}_1 = \{(0), G(0), G(I), G(-I), (0) \oplus H, H \oplus H\},$$

then

$$\operatorname{Alg}\mathcal{M}_1 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathcal{B}(H) \right\},$$

and $\operatorname{Alg}\mathcal{M}_1$ is semi-simple, since $\mathcal{B}(H)$ is. (Incidentally, if the given space $H \oplus H$ is finite-dimensional, there is no subspace lattice $\mathcal{M}$ on it, isomorphic to $\mathcal{M}_2$ and satisfying $\operatorname{Alg}\mathcal{M} = \mathbb{C}I$ [2]; if H is infinite-dimensional and separable such an $\mathcal{M}$ does exist [3].) Concerning semi-simplicity and semi-primeness, what can be said about the Alg's of medial subspace lattices with five elements, that is, double triangles? It is not known whether or not there exists a double triangle whose Alg is not semi-prime. Our final theorem shows how to obtain examples with semi-simple Alg's. First we prove a proposition. As remarked earlier, for any double triangle we have $\mathcal{J} = \emptyset$, so by Lemma 1 no rank one operator belongs to its Alg. In what follows $\langle Qe, (I - Q)\, e\rangle$ denotes the linear span of Qe and $(I - Q)\, e$.

PROPOSITION 2. *Let $\mathcal{T} = \{(0), K, L, M, X\}$ be a double triangle subspace lattice on X with Hasse diagram given by Figure 5. Let the vector sum $K + L$ be closed and let $Q \in \mathcal{B}(X)$ denote the projection onto K along L. For every pair of non- zero vectors $e \in M$, $f^* \in M^\perp$ the operator*

$$R = Q(f^* \otimes e)\, Q - (I - Q)(f^* \otimes e)(I - Q)$$

is a rank two operator of $\operatorname{Alg}\mathcal{T}$ *with range* $\langle Qe, (I-Q)\,e\rangle$.

Proof. Let $e \in M$ and $f^* \in M^\perp$ be non-zero vectors and let R be the operator defined as in the above statement. Since Q and $I-Q$ are both idempotent, $RQ = Q(f^* \otimes e)Q$ and $R(I-Q) = -(I-Q)(f^* \otimes e)(I-Q)$. The former gives $RK \subseteq K$ since the range of Q is K; the latter gives $RL \subseteq L$ since the range of $I-Q$ is L.

If $x \in M$, then since $f^*(x) = 0$, we have

$$\begin{aligned}
Rx &= Q(f^* \otimes e)Qx - (I-Q)(f^* \otimes e)(I-Q)x \\
&= Q(f^* \otimes e)Qx - (I-Q)(f^* \otimes e)x + (I-Q)(f^* \otimes e)Qx \\
&= (f^* \otimes e)Qx - f^*(x)(I-Q)e \\
&= (f^* \otimes e)Qx \\
&= f^*(Qx)e\,,
\end{aligned}$$

so $Rx \in M$. Thus $R \in \operatorname{Alg}\mathcal{T}$. Clearly $\langle Qe, (I-Q)\,e\rangle$ contains the range of R. But Qe and $(I-Q)\,e$ both belong to the range of R. For, since $f^* \notin K^\perp$ there exists a vector $y \in K$ such that $f^*(y) = 1$. Also, since $f^* \notin L^\perp$ there exists a vector $z \in L$ such that $f^*(z) = -1$. Then

$$Ry = Q(f^* \otimes e)Qy = Q(f^* \otimes e)y = f^*(y)Qe = Qe$$

and

$$Rz = -(I-Q)(f^* \otimes e)(I-Q)z = -(I-Q)(f^* \otimes e)z = -f^*(z)(I-Q)e = (I-Q)e\,.$$

Hence the range of R is $\langle Qe, (I-Q)\,e\rangle$. It only remains to show that Qe and $(I-Q)\,e$ are linearly independent, and this is fairly clear.

COROLLARY. *With R and Q as in the statement of Proposition 2, for every operator* $J \in \operatorname{Alg}\mathcal{T}$ *we have*

$$JQ = QJ \quad \textit{and} \quad (JR)^2 = f^*(QJe)[JR]\,.$$

Proof. Let $J \in \operatorname{Alg}\mathcal{T}$. Since J leaves K invariant, $JQ = QJQ$. Since J leaves L invariant, $J(I-Q) = (I-Q)\,J(I-Q)$. These two equalities give $JQ = QJ$.

With $E = f^* \otimes e$ we have

$$\begin{aligned} JR &= JQEQ - J(I-Q)\,E(I-Q) \\ &= QJEQ - (I-Q)\,JE(I-Q) \end{aligned}$$

so

$$\begin{aligned} (JR)^2 &= [QJEQ]^2 + [(I-Q)\,JE(I-Q)]^2 \\ &= [Q^*f^* \otimes QJe]^2 + [(I-Q)^*f^* \otimes (I-Q)\,Je]^2 \\ &= Q^*f^*(QJe)[QJEQ] + (I-Q)^*f^*((I-Q)\,Je)[(I-Q)\,JE(I-Q)] \\ &= f^*(QJe)[QJEQ] + f^*((I-Q)\,Je)[(I-Q)\,JE(I-Q)]\,. \end{aligned}$$

But $f^*(QJe) + f^*((I-Q)\,Je) = f^*(Je) = 0$, since $Je \in M$ and $f^* \in M^\perp$. Thus $(JR)^2 = f^*(QJe)[JR]$, as required.

THEOREM 4. *Let $\mathcal{T} = \{(0), K, L, M, X\}$ be a double triangle subspace lattice on X with Hasse diagram given by Figure 5. If the vector sum $K + L$ is closed, then* $\operatorname{Alg}\mathcal{T}$ *is semi-simple.*

Proof. Let the vector sum $K+L$ be closed and let $Q \in \mathcal{B}(X)$ be the projection onto K along L. Now QM is a linear manifold and $QM \subseteq K$. In fact QM is dense in K. For this, it is enough to show that if $g^* \in X^*$ and $QM \subseteq \ker g^*$, then $g^* \in K^\perp$. Suppose that $QM \subseteq \ker g^*$. Then $g^*(Qx) = 0$ for every $x \in M$, so $Q^*g^* \in M^\perp$. However, Q^* has range $L^\perp$, so $Q^*g^* \in M^\perp \cap L^\perp = (0)$. Thus $g^* \in \ker Q^* = K^\perp$.

Similarly $(I-Q)\,M$ is dense in L.

To show that $\operatorname{Alg}\mathcal{T}$ is semi-simple it is enough to show that for every non-zero operator $J \in \operatorname{Alg}\mathcal{T}$, there exists a rank two operator R of $\operatorname{Alg}\mathcal{T}$ such that JR is a non-zero idempotent.

Let $J \in \operatorname{Alg}\mathcal{T}$ be non-zero. By the preceding corollary, $JQ = QJ$. We cannot have $JM = (0)$. For if JM were (0), then for every vector $z \in M$ we would have $JQz = QJz = 0$ and $J(I-Q)\,z = (I-Q)\,Jz = 0$, so $QM \subseteq \ker J$ and $(I-Q)\,M \subseteq \ker J$.

But QM is dense in K and $I-Q)\,M$ is dense in L, so we would have $K \vee L = X \subseteq \ker J$, which is a contradiction. Thus $JM \neq (0)$, so $Je \neq 0$ for some vector $e \in M$. Now $Je \in M$ and $Je \neq 0$, so $Je \notin L$ (since $L \cap M = (0)$) and $QJe \neq 0$. Since $QJe \in K$, $QJe \notin M$. Hence, by the Hahn-Banach theorem, there exists a vector $f^* \in M^{\perp}$ such that $f^*(QJe) = 1$.

Put $R = Q(f^* \otimes e)Q - (I-Q)(f^* \otimes e)(I-Q)$. By Proposition 2, R is a rank two operator of $\operatorname{Alg}\mathcal{T}$. Finally, JR is non-zero and idempotent. For, by Proposition 2, R has range $\langle Qe, (I-Q)\,e\rangle$ and $JQe = QJe \neq 0$, so $JR \neq 0$. Also, by the preceding corollary, $(JR)^2 = f^*(QJe)[JR] = JR$. This completes the proof.

Note that the preceding theorem does not extend to medial subspace lattices with more than five elements. Indeed, $\mathcal{L}_2$ in Example 3 has the property that $K + L$ is a closed vector sum for every pair of elements K and L, but $\operatorname{Alg}\mathcal{L}_2$ is not semi-simple. (The medial subspace lattice $\mathcal{M}_2$ obtained by omitting $G(T+\beta I)$ from $\mathcal{L}_2$ also has this property, and $\operatorname{Alg}\mathcal{M}_2 = \operatorname{Alg}\mathcal{L}_2$.)

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