FACTORIZATION IN GROUP ALGEBRAS

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The following describes a connection with some automatic continuity problems of the study of probability measures and random walks on groups. The connection is via some questions concerning factorization in certain ideals in group algebras.

For this, let G be a locally compact group and M(G) denote the algebra of all bounded Borel measures on G with convolution product and total variation norm. The closed ideal in M(G), consisting of those measures which are absolutely continuous with respect to Haar measure, will be identified with the (topological) group algebra $L^1(G)$ and the closed subalgebra of M(G), consisting of discrete measures, will be identified with the (discrete) group algebra $\ell^1(G)$. Further, the (algebraic) group algebra, $\mathbb{C}G$, will be identified with the subalgebra of $\ell^1(G)$ consisting of functions with finite support. Since $L^1(G)$ is an ideal in M(G), the convolution product defines a right module action on $L^1(G)$ by each of the subalgebras, $\mathbb{C}G$, $\ell^1(G)$ and $L^1(G)$. Three automatic continuity problems now arise, namely, whether module homomorphisms from $L^1(G)$ to an arbitrary right $\mathbb{C}G$-, $\ell^1(G)$- or $L^1(G)$-module X are continuous.

The $L^1(G)$-module problem was solved by B. E. Johnson, see [6].

THEOREM 1. *Let* T *be a module homomorphism from* $L^1(G)$ *to an arbitrary right* $L^1(G)$-*module* X. *Then* T *is continuous.*

Proof. The proof is included in order to motivate an approach to the $\ell^1(G)$-module problem later. Let $(F_n)_{n=1}^{\infty}$ be a sequence in $L^1(G)$ which converges to zero in norm. Then, by an extension of Cohen's factorization theorem given in [6] there are F and F'_n, n = 1,2,3,... in $L^1(G)$ such that

(i) $F_n = F*F'_n$, n = 1,2,3,...

(ii) $\lim_{n\to\infty} \|F'_n\| = 0.$

Hence $T(F_n) = T(F) \cdot F'_n$, because T is a module homomorphism, so $T(F_n) \to 0$ as $n \to \infty$.

Therefore T is continuous.

The $\mathbb{C}G$-module problem has been solved in various cases by several authors. Define a right $\mathbb{C}G$-module action on the one dimensional space $X = \mathbb{C}$ by

$$(1) \qquad x \cdot f = \Big(\sum_{g \in G} f(g) \Big) x, \quad (f \in \mathbb{C}G,\ x \in \mathbb{C}).$$

Then a $\mathbb{C}G$-module homomorphism from $L^1(G)$ to X is just a translation invariant linear functional on $L^1(G)$. There are discontinuous translation invariant linear functionals on $L^1(G)$ if G is σ-compact but not compact, see [16], or if G is infinite abelian, see [16] and [8].

We come now to the problem of continuity of $\ell^1(G)$-module homomorphisms from $L^1(G)$. It is closely related to the $\mathbb{C}G$-module problem as $\ell^1(G)$ is the closure of $\mathbb{C}G$ in $M(G)$. Hence every $\ell^1(G)$-module homomorphism is a $\mathbb{C}G$-module homomorphism. Conversely, it is easily seen that every continuous $\mathbb{C}G$-module homomorphism is an $\ell^1(G)$-module homomorphism. However, if the operator is not assumed continuous, then requiring it to be an $\ell^1(G)$-module homomorphism is a strictly stronger condition than requiring it to be translation invariant. In constrast to the $\mathbb{C}G$-module case, it may be shown that $\ell^1(G)$-module homomorphisms from $L^1(G)$ are automatically continuous, see [11]. This result has an application to the problem of automatic continuity of module derivations from group algebras. In [12] it is shown that, if $\ell^1(G)$-module homomorphisms from $L^1(G)$ are automatically continuous for all locally compact groups G, then there is a discontinuous derivation from $L^1(G)$ for some locally compact group G (if and) only if there is a discontinuous derivation from $\ell^1(G)$ for some discrete group G. It is further argued that one can restrict attention to $G = \mathbb{F}_\omega$, the free group on a countably infinite number of generators. The complete proof of continuity of $\ell^1(G)$-module homomorphisms will not be given here. Instead, two unsuccessful attempted proofs will be discussed. These attempts provide some partial results towards the complete proof and one of them leads to an investigation of probability measures on groups.

The first approach is to try to copy the proof that $L^1(G)$-module homomorphisms from $L^1(G)$ are continuous. For this to work, a factorization result similar to Cohen's factorization theorem is required. More specifically, we need to know that, given a sequence $(F_n)_{n=1}^{\infty}$ in $L^1(G)$ which converges to zero in norm, there is $F \in L^1(G)$, and a sequence $(f_n)_{n=1}^{\infty} \subset \ell^1(G)$, which converges to zero in norm, such that $F_n = F*f_n$ for every n. An indication that this might be so is provided by the fact that if $F_1, F_2, \ldots, F_n \in L^1(G)$ and $\epsilon > 0$ are given, then there is $F \in L^1(G)$, and $f_1, f_2, \ldots, f_n \in \ell^1(G)$, such that

(2) $$\|F_j - F*f_j\| < \epsilon \text{ and } \|F\|\|f_j\| \leq \|F_j\| \text{ for each } j = 1,2,\ldots,n.$$

This attempt to prove the continuity of $\ell^1(G)$-module homomorphisms fails because, for example, the sequence of functions $F_n(\theta) = \frac{1}{n}\cos n\theta$, $n = 1,2,\ldots$ on the circle group converges to zero in the L^1 norm but cannot be factored in the required way, see [12]. However, the above mentioned approximate factorization (2) does imply a much weaker factorization result which is useful for the proof that $\ell^1(G)$-module homomorphisms are automatically continuous.

THEOREM 2. *Let* G *be a compact group,* $(F_n)_{n=1}^{\infty}$ *be a sequence in* $L^1(G)$ *which converges to zero in norm and* $(\mathcal{F}_n)_{n=1}^{\infty}$ *be a sequence of finite sets of irreducible unitary representations of* G. *Then there is* $F \in L^1(G)$, *and* $(f_n)_{n=1}^{\infty} \subset \ell^1(G)$ *such that:*

(i) $\|f_n\| \to 0$ *as* $n \to \infty$; *and*

(ii) $\rho(F_n) = \rho(F*f_n)$ *for each* $\rho \in \mathcal{F}_n$, $n = 1,2,3,\ldots$.

The proof will be given in [11]. Although this result was inspired by Cohen's factorization theorem it is really quite different and less delicate as the factor F depends only on the sequence $(\mathcal{F}_n)_{n=1}^{\infty}$ and the rate at which $(\|F_n\|)_{n=1}^{\infty}$ converges to zero, it does not depend on the particular functions F_n.

The second approach is connected with the study of random walks on groups and produces some further factorization results. Showing that every $\ell^1(G)$-module homomorphism from $L^1(G)$ is continuous entails showing in particular that

$$(3) \qquad L_0^1(G) = L^1(G)\cdot \ell_0^1(G) \equiv \text{span}\,\{F*f \mid F \in L^1(G),\, f \in \ell_0^1(G)\},$$

where $L_0^1(G) = \{F \in L^1(G) \mid \int_G F = 0\}$ and $\ell_0^1(G) = \{f \in \ell^1(G) \mid \sum_{x\in G} f(x) = 0\}$.

It is easily seen that the factorization (3) holds if and only if every $\ell^1(G)$-module homomorphism from $L^1(G)$ to the one dimensional right $\ell^1(G)$-module with action defined by (1) is continuous.

The proof of (3) in [11] requires different techniques for different types of groups. For example, if G is a connected Lie group, use is made of its one parameter subgroups, whereas, if G is totally disconnected, then Theorem 2 above is required together with the fact that every totally disconnected group has a compact, open subgroup, see [5], Theorem II.7.7. The connection with random walks on groups arises from the proof of (3) in the case when G is compact. The idea for this proof is suggested by the proof of the following factorization result for $L_0^1(G)$.

In [13] it is shown that every element of $L_0^1(G)$ is a sum of four products of elements in $L_0^1(G)$. Given an element, $F \in L_0^1(G)$, it is first written as a product, $F = E_1*F'*E_2$, where $E_1, E_2 \in L^1(G)$ and $F' \in L_0^1(G)$. Then F' is split into its real and imaginary parts, which belong to $L_0^1(G)$, and they in turn are split into their positive and negative parts to yield $F = \sum_{k=1}^{4} U_k*(\delta_e-P_k)*V_k$, where P_k is a probability measure in $L^1(G)$ and U_k and V_k are scalar multiples of E_1 and E_2. The proof that F is a sum of four products is completed by factoring δ_e-P_k, which may be done by applying Cohen's factorization theorem or by noting that it has a square root. Now, by using factorization on one side only, we find that each F in $L_0^1(G)$ has the form $F = \sum_{k=1}^{4} U_k*(\delta_e - P_k)$. It follows that one way to prove (3) would be to show that $U*(\delta_e - P)$ belongs to $L^1(G)\cdot \ell_0^1(G)$ whenever P is a probability measure on G.

Define now, for each probability measure, μ, on G,

$$J_\mu = \{F - F*\mu \mid F \in L^1(G)\}^- = [L^1(G)*(\delta_e - \mu)]^-.$$

Then J_μ is a closed, left ideal in $L^1(G)$ which is contained in $L_0^1(G)$ and has a right bounded approximate identity, see [14]. This subspace is also a right Banach module over the subalgebra, A_μ, of $M(G)$ generated by δ_e and μ. Furthermore, the subalgebra $A_{\mu,0} = \{\nu \in A_\mu \mid \nu(G) = 0\}$ has a bounded approximate identity for J_μ. Since, if μ is discrete, $A_{\mu,0} = A_\mu \cap \ell_0^1(G)$ it follows that, if μ is discrete, J_μ is contained in $L^1(G)\cdot \ell_0^1(G)$. Therefore, we can prove (3) if we can show that

(4) for every probability measure, μ, on G there is a discrete probability measure, μ', such that $J_\mu \subseteq J_{\mu'}$.

Unfortunately, (4) does not hold for all probability measures on all groups, as will be shown later. However, it can be proved for amenable groups by applying the following

LEMMA 3. *Let* G *be a locally compact group and* $\mathcal{F}$ *be a norm closed, convex set of probability measures on* G *which is closed under convolution. Let* X *be a separable subspace of* $L^1(G)$ *such that:*

(i) $J_\mu \subseteq X$, $(\mu \in \mathcal{F})$;

(ii) *for every* $f \in X$ and $\epsilon > 0$, there is $\mu \in \mathcal{F}$ such that $d(f, J_\mu) < \epsilon$.

Then there is a $\mu' \in \mathcal{F}$ *such that* $X = J_{\mu'}$.

The proof is modelled on the proof of Cohen's factorization theorem, see [14]. Now let $\mathcal{J}$ denote the set of all ideals in $L^1(G)$ of the form J_μ for some μ. As (4) is, partly, a statement about inclusion relations in $\mathcal{J}$, regard $\mathcal{J}$ as partially ordered by inclusion. The following result is proved in [14].

THEOREM 4. *Let* G *be a separable group. Then:*

(i) *every* J_μ *is contained in a maximal element of* $\mathcal{J}$;

(ii) *if* G *is amenable, then there is a probability measure* μ, *which may be chosen to be discrete, such that* $L_0^1(G) = J_\mu$.

Part (ii) of this result provides a new proof of a conjecture of Furstenberg, see [2], which was proved in 1981 by Kaimanovich and Versik and independently by Rosenblatt, see [7] and [10]. That the measure μ in part (ii) may be chosen discrete is new and proves (3) and (4) for amenable groups. In particular, it proves (3) for compact G, which is what is required at one point in [11].

The connection with random walks on groups is through the ideals J_μ. A function, $\varphi \in L^\infty(G)$ is said to be μ-harmonic if $\mu*\varphi = \varphi$, where convolution by μ is defined to be the adjoint of the operator $f \mapsto f*\mu$ on $L^1(G)$, $L^\infty(G)$ being identified with the dual of $L^1(G)$. It is clear then that φ is μ-harmonic if and only if it annihilates J_μ and that the space of μ-harmonic functions, H_μ, is isomorphic to $(L^1(G)/J_\mu)'$. One of the basic results of the theory of random walks on groups is that H_μ is isomorphic to a space $L^\infty(\Omega,\eta)$, where Ω is a certain measurable G-space with quasi-invariant measure η. The G-space Ω is called the μ-boundary and its significance is that it may be adjoined to G in such a way that almost every trajectory of the random walk with transition probabilities given by μ hits a point of Ω "at infinity".

In [14] it is shown that $L^1(G)/J_\mu$ with the quotient norm and a certain order structure is an abstract L^1-space, from which it follows that $L^1(G)/J_\mu$ is isometrically isomorphic to $L^1(\Omega,\eta)$ for some measurable G-space Ω and quasi-invariant measure η. Of course this is just the predual of the above mentioned result about H_μ but it has another factorization of $L_0^1(G)$ as a consequence. A probability measure, μ, on G is called *nondegenerate* if it is absolutely continuous with respect to Haar measure and if the closed semigroup generated by the support of μ is G itself. Then

THEOREM 5. *If* μ *is a nondegenerate probability measure on* G, *then*

$$L_0^1(G) = [L^1(G)*(\delta_e-\mu)]^- + [(\delta_e-\mu)*L^1(G)]^-$$

Idea of proof. This is proved by applying the ergodic theorem, see [1] VIII:5.5, to the operator on $L^1(\Omega,\eta)$ of convolution by μ, see [14].

Theorem 5 expresses $L_0^1(G)$ as an algebraic sum of a closed left ideal with a right bounded approximate identity and a closed right ideal with a left bounded approximate identity. We have then the following results.

COROLLARY 6. *If* $(F_n)_{n=1}^\infty$ *is a sequence in* $L_0^1(G)$ *which converges to zero in norm, then there are* A, B $\in L_0^1(G)$, *and sequences* $(H_n)_{n=1}^\infty$, $(K_n)_{n=1}^\infty \subset L_0^1(G)$ *such that*

(i) $F_n = H_n*A + B*K_n$, $n = 1,2,3,\ldots$

(ii) $\lim_{n\to\infty} \|H_n\| = 0 = \lim_{n\to\infty} \|K_n\|$.

COROLLARY 7. *Every* $L_0^1(G)$*-bimodule homomorphism from* $L_0^1(G)$ *is continuous.*

Corollary 6 is a big improvement on the factorization of $L_0^1(G)$ in [13] because it reduces the number of products required from four to two and because it factors sequences. However, the proof of Theorem 5 is much longer and more sophisticated than the argument in [13], which is outlined above.

Some problems are suggested by Corollaries 6 and 7. First, Corollary 6 shows that every function in $L_0^1(G)$ is a sum of two products rather than four as was known previously, but it is still not known whether every function in $L_0^1(G)$ is a product. Secondly, in view of Corollary 7 it would be of interest to know whether or not every left $L_0^1(G)$-module homomorphism from $L_0^1(G)$ is continuous. If G is a discrete group, so that $L^1(G)$ has a unit, any left $L_0^1(G)$ may be extended to be a derivation from $L^1(G)$. This question is thus a special case of the question as to whether or not every derivation from $L^1(G)$ is continuous and it may be worthwhile to work on this special case. Thirdly, it would be useful to generalize Theorem 5 so that it applied to certain other ideals in $L^1(G)$.

To explain what is meant by this third problem, let N be a closed, normal subgroup of G. Then the quotient map $G \to G/N$ induces an algebra homomorphism $T:L^1(G) \to L^1(G/N)$. If μ is a probability measure on G with support contained in N, then it is easily seen that

$$[L^1(G)*(\delta_e-\mu)]^- + [(\delta_e-\mu)*L^1(G)]^- \subseteq \ker T.$$

What is required, is to find a probability measure μ supported in N such that the left hand side equals $\ker T$. Suppose that such a μ can be found when N is the commutator subgroup of G. Then $\ker T$ will be the algebraic sum of a closed left ideal with a right bounded approximate identity and a closed right ideal with a left bounded approximate identity and so Corollary 6 will hold with $\ker T$ in place of $L_0^1(G)$. Now let I be any codimension 2 ideal in $L^1(G)$. Then $\ker T \subseteq I$ and so $T(I)$ is a codimension 2 ideal in $L^1(G/N)$. Since G/N is abelian, every element of $T(I)$ is a product. As every element of $\ker T$ is a sum of two products, it follows that every element of I is a sum of three products. The hoped for generalization of Theorem 5 would thus greatly improve on what is known about factorization in codimension 2 ideals of group algebras, see [15].

There is one non-trivial case where it is possible to find a probability measure with the required property. Let $\mathbb{F}_2$ be the free group on two generators and N be its commutator subgroup. Then it is shown in [14] that the probability measure, μ, on N constructed by Furstenberg in [4] satisfies $[\ell^1(\mathbb{F}_2)*(\delta_e-\mu)]^- + [(\delta_e-\mu)*\ell^1(\mathbb{F}_2)]^- = \ker T$. The construction of μ relies on the fact that $\mathbb{F}_2/N \cong \mathbb{Z}^2$ and that random walks on $\mathbb{Z}^2$ are recurrent. It does not extend to groups with three or more generators because $\mathbb{F}_3$ modulo its commutator subgroup is isomorphic to $\mathbb{Z}^3$ and random walks on $\mathbb{Z}^3$ are transient.

Finally, some other work of Furstenberg [3] provides an example which shows that (4) does not hold for all probability measures on all groups. The details of this example will appear in [9] but an outline follows. For this we need some more information about the isomorphism between $L^1(G)/J_\mu$ and $L^1(\Omega,\eta)$. If μ is an absolutely continuous probability measure, then it belongs to $L^1(G)$ and is a right modular unit for J_μ. It follows that $\mu + J_\mu$ is mapped to a probability measure ν on Ω which is absolutely continuous with respect to η and satisfies $\mu * \nu = \nu$, where the convolution is determined by the action of G on Ω. This measure ν is said to be μ-stationary and, since μ is a modular unit for J_μ, it is clear that $J_\mu = \{F \in L^1(G) \mid F*\nu = 0\}$. In some cases it is possible to obtain more concrete information about (Ω,η) and ν. Furstenberg showed that, if G is $SL(2,\mathbb{R})$ and μ is absolutely continuous, then the μ-boundary, Ω, is either the unit circle, or the one dimensional projective space, $\mathbb{P}$. It may be shown, see [14], that, if the μ-boundary is $\mathbb{P}$, then J_μ is maximal in the sense of Theorem 4 above and we will need only to work with this case. The action of $SL(2,\mathbb{R})$ on $\mathbb{P}$ is the action induced from the action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ by identifying $\mathbb{P}$ with the set of all lines through the origin in $\mathbb{R}^2$. Parametrize $\mathbb{P}$ by θ, $-\frac{\pi}{2} < \theta \le \frac{\pi}{2}$, where θ is the angle between a line through the origin and the horizontal axis in $\mathbb{R}^2$. Then the measure η may be chosen to be Lebesgue measure on $(-\frac{\pi}{2},\frac{\pi}{2}]$ because this measure is quasi-invariant for the action of $SL(2,\mathbb{R})$ and $\mathbb{P}$ has a unique quasi-invariant measure class since it is transitive for the action of $SL(2,\mathbb{R})$.

With this choice of η, it may be shown that, if ν is μ-stationary for some absolutely continuous μ, then the Radon-Nikodym derivative, $\frac{d\nu}{d\eta}$, is a lower semicontinuous function on $\mathbb{P}$. Conversely, if F is any non-negative lower semicontinuous function with $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F d\eta = 1$ whose support is contained in a proper compact subset of $(-\frac{\pi}{2},\frac{\pi}{2}]$, then an absolutely continuous probability measure, μ, may be constructed on $SL(2,\mathbb{R})$ so that $L^1(SL(2,\mathbb{R}))/J_\mu \cong L^1(\mathbb{P},\eta)$ and the μ-stationary measure ν satisfies $\frac{d\nu}{d\eta} = F$, see [9].

Now let F be any non-negative function on $(-\frac{\pi}{2},\frac{\pi}{2}]$ whose support is contained in $[0,\frac{\pi}{2}]$ and which is continuous everywhere except at three points p_1,p_2,p_3. Assume further that $F(p) \to \infty$ as $p \to p_i$ and that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F d\eta = 1$. Then F is lower semicontinuous and so there is a probability measure, μ, on $SL(2,\mathbb{R})$ such that $L^1(SL(2,\mathbb{R}))/J_\mu \cong L^1((-\frac{\pi}{2},\frac{\pi}{2}],\eta)$ and the μ-stationary measure, ν, satisfies $\frac{d\nu}{d\eta} = F$. Suppose that μ' is a discrete probability measure on $SL(2,\mathbb{R})$ such that $J_\mu \subseteq J_{\mu'}$. Since J_μ is maximal we have that $J_\mu = J_{\mu'}$ and so ν will also be μ^1-stationary. It follows that any matrix, x, in the support of μ' must permute the points p_1,p_2 and p_3. Observe that each point p in $\mathbb{P}$ which is fixed by x corresponds to an eigenvector of x in $\mathbb{R}^2$. Hence, if x fixes all three of p_1,p_2 and p_3, then x is a scalar matrix, i.e. $x \in \left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right\}$. Since there are only six permutations of $\{p_1,p_2,p_3\}$, it follows that the support of μ' is contained in a subgroup of $SL(2,\mathbb{R})$ of order 12. It is then impossible for $L^1(SL(2,\mathbb{R})/J_\mu$ to be isomorphic to $L^1(\mathbb{P},\eta)$. Therefore, there is no discrete μ' with $J_{\mu'} \supsetneq J_\mu$.

REFERENCES

[1] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1957.

[2] H. Furstenberg, *Boundary Theory and Stochastic processes on homogeneous spaces*, Proc. Symp Pure Math. 26, Amer. Math. Soc. Providence, R. I. (1974), 193-22.

[3] H. Furstenberg, A Poisson formula for semi-simple Lie groups, *Ann. Math.*, 77 (1963), 335-386.

[4] H. Furstenberg, Random walks and discrete subgroups of Lie groups, *Adv. Prob. and Related Topics,* Dekker, New York, 1 (1971), 3-63.

[5] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, Volume I,* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[6] B. E. Johnson, Continuity of centralizers on Banach algebras, *J. London Math. Soc.*, 41 (1966), 639-640.

[7] V. A. Kaimanovich and A. M. Versik, Random walks on discrete groups: boundary and entropy, *Ann. of Probability,* 11 (1983), 457-490.

[8] P. Ludvik, Discontinous translation-invariant linear functionals on $L^1(G)$, *Studia Math.*, 56 (1976), 21-30.

[9] W. Moran and G. Willis, Boundaries of random walks on semisimple Lie groups when the transition probabilities are not absolutely continuous, to appear.

[10] J. Rosenblatt, Ergodic and mixing random walks on locally compact groups, *Math. Ann.*, 257 (1981), 31-42.

[11] G. Willis, The continuity of $\ell^1(G)$-module homomorphisms from $L^1(G)$, to appear.

[12] G. Willis, The continuity of derivations and module homomorphisms, *J. Austral. Math. Soc., (Series A)* 40 (1986), 299-320.

[13] G. Willis, Factorization in codimension one ideals of group algebras, *Proc. Amer. Math. Soc.,* 86 (1982), 599-601.

[14] G. Willis, Probability measures on groups and some related ideals in group algebras , *J. Funct. Analysis*, to appear.

[15] G. Willis, Factorization in codimension two ideals of group algebras, *Proc. Amer. Math. Soc.,* 89 (1983), 95-100.

[16] G. S. Woodward, Translation-invariant linear forms on $C_0(G)$, $C(G)$, $L^p(G)$ for non compact groups, *J. Funct. Analysis,* 16 (1974), 205-220.

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