# SUPERSYMMETRIC QUANTUM MECHANICS AND THE INDEX THEOREM

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The basic features of supersymmetric quantum mechanics are reviewed and illustrated by examples from physics and geometry (the hydrogen atom, and massless fields in curved space). Using a discrete approximation to the path integral in the associated supersymmetric quantum mechanics, the Atiyah-Singer Index Theorem is derived for the twisted Dirac operator. Specializations of this in four dimensions include the Gauss-Bonnet theorem, and the Hirzebruch signature theorem. The relationship of the index theorem to anomalies, and their cancellation in the standard model and beyond, is briefly discussed.

## INTRODUCTION

The Atiyah-Singer Index Theorem [1] is a classical result in differential geometry of about twenty years' standing. However, its importance for field theory in various settings has become increasingly appreciated over the years. In particle physics, it is relevant for unified models (anomalies, instantons, monopoles), gravitation and Kaluza-Klein theories (for example, massless modes in higher dimensions), and string theories (anomaly removal, and topological aspects of compactification).

However, applications of the index theorem in other areas are legion: for example crystal defects, fermion fractionization, Berry's phase, liquid helium, and condensed matter physics generally, wherever topological considerations are important. No review can give justice to all these many facets of the index theorem [2]. By way of illustration of just one application, the role of the index theorem in govern-

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ing anomalies in gauge theories (the "standard model" and beyond) will briefly be described below (§2).

The connection between supersymmetry and the index theorem is less than ten years old, dating from ideas of Witten [3] and more recently from attempts by physicists to provide proofs by path integral and other methods [4–7]. This review aims to give a pedagogical introduction to supersymmetric quantum mechanics and to establish its relevance to the index theorem (§1). Finally, and as an alternative to existing work [4–6], a discrete approximation is set up for the path integral representation of the supersymmetric quantum mechanics equivalent of the index, and used to provide a heuristic derivation [7] of the index theorem itself (§3).

It should be pointed out that the whole story of supersymmetry and the index theorem is a precursor to the exciting recent developments in "topological quantum field theory" [8]. In particular Witten [9] following Atiyah [10] has formulated the "Donaldson invariants" of four-manifolds [11] as correlation functions of a certain supersymmetric *quantum field theory*. From this perspective this review should perhaps be subtitled "topological (supersymmetric) quantum mechanics".

## §1. SUPERSYMMETRIC QUANTUM MECHANICS

### SSQM and the Witten Index

Consider a quantum mechanical system described by a hamiltonian operator $H$ acting on a Hilbert space $\mathcal{H}$ [12]. The system is called supersymmetric if there are operators $Q$, $Q^\dagger$ such that

$$\{Q, Q^\dagger\} = H \tag{1}$$

$$\{Q, Q\} = 0 = \{Q^\dagger, Q^\dagger\}, \tag{2}$$

(from which

$$[Q, H] = 0 = [Q^\dagger, H]).$$

Furthermore, $\mathcal{H}$ is $\mathbb{Z}_2$-graded, i.e. there is a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. If $F = 0, 1$ is the corresponding projection operator, then $Q$ is assumed to be "odd":

$$Q(-1)^F = -(-1)^F Q. \tag{3}$$

In terms of the orthogonal decomposition of $\mathcal{H}$ we can express $F$, $Q$, $Q^\dagger$ and $H$ in $2 \times 2$ block form as follows:

$$\begin{aligned} F &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \\ Q &= \begin{pmatrix} 0 & 0 \\ A & 1 \end{pmatrix}, \quad H = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}. \end{aligned} \tag{4}$$

The above definitions are inherited from the relativistic regime [13] where there is a spinor multiplet $\{Q_\alpha, \alpha = 1, \ldots, N\}$ of supersymmetry charges. Their anticommutators close, amongst other things, on the momentum 4-vector. (The whole structure is an example of a Lie superalgebra generalization of the space-time symmetry, e.g. the Poincaré algebra.) If the three-momentum vanishes, then the remaining component $P^0$ is precisely the total energy as in (1).

Moreover, nonrelativistically spin and orbital transformations are independent, so one may focus on a particular pair of components of the $\{Q_\alpha\}$, as in (1), (2), and neglect the rest. Finally, in the general case there is a natural candidate for $(-1)^F$: namely $\exp(2\pi i J_z)$, where $J_z$ is the generator of rotations about the $z$ axis. By the spin-statistics theorem, $J_z$ is half-odd-integral for fermions, but integral for bosons, so the states with $F = 0, 1$ do indeed deserve to be called "bosonic", and "fermionic", respectively.

In the nonrelativistic case (1) can be generalized by an additional term on the right-hand side. This can be either an overall constant (a trivial redefinition of the zero of energy), or a constraint, which vanishes on physical states [14]. This generalization is not required in the present context.

In what follows, we shall assume that the system is regularized in such a way that $H$ has a discrete spectrum. For the applications in differential geometry, involving elliptic operators on compact manifolds, this is appropriate. It is still useful in general, if the quantities to be calculated are independent of the regularization.

With these preliminaries, let us look at some of the properties of supersymmetric systems [3, 12, 15, 16]. The following are easy consequences of the definitions (1) and (2):

#1 The energy of the system is zero or positive.

(For $\langle\psi|H|\psi\rangle = \langle\psi|(QQ^\dagger+Q^\dagger Q)|\psi\rangle = \|Q|\psi\rangle\|^2+\|Q^\dagger|\psi\rangle\|^2 \geq 0$ for any ket$|\psi\rangle$).

#2 The state(s) of zero energy are just those annihilated by $Q$ and $Q^\dagger$.

(For $Q|0\rangle = 0 = Q^\dagger|0\rangle \Leftrightarrow H|0\rangle = 0$ as above.)

#3 If $E > 0$, states occur as degenerate boson-fermion pairs (related by $Q$ and $Q^\dagger$, which commute with $H$.)

(For if $H|E\rangle = E|E\rangle$ then we can consider its projection $|E,b\rangle \neq 0$ say, i.e. $(-1)^F|E,b\rangle = +|E,b\rangle$. $|E,f\rangle = (Q/\sqrt{E})|E,b\rangle$, necessarily nonzero by (#2), and such that $Q^\dagger|E,f\rangle = \sqrt{E}|E,b\rangle$, $(-1)^F|E,f\rangle = -|E,f\rangle$.)

Thus the general appearance of the spectrum of the supersymmetric hamiltonian is as shown in fig. 1: states of energy $E > 0$ occur in doublet representations of the supersymmetry algebra (1), (2), while singlet representations necessarily have $E = 0$.

An important question in modelling physical systems is the possibility of spontaneous breaking of any (continuous) symmetries of the model. (In fact, supersymmetric field theories with *unbroken* supersymmetry seem to be ruled out phenomenologically, since they would require elementary particles to occur as boson-fermion pairs degenerate in mass and differing in spin by $\frac{1}{2}\hbar$ (since the $\{Q_\alpha\}$ generators are spinors in the relativistic case).) It is a textbook result that a symmetry is unbroken if and only if the vacuum is annihilated by the appropriate generator, in this case $Q|0\rangle = 0 = Q^\dagger|0\rangle$.

Witten [15] pointed out that, in turn, a sufficient condition for *supersymmetry*

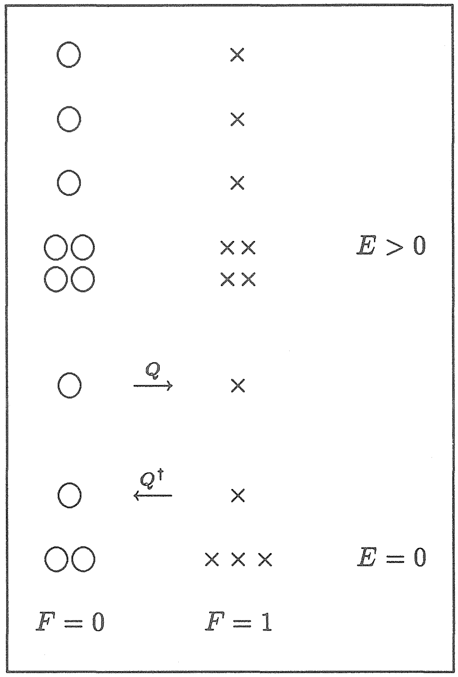


Fig. 1. Spectrum of Supersymmetric Hamiltonian.

to be unbroken is that the number of zero-energy bosonic minus fermionic states,

$$I = n_b^{E=0} - n_f^{E=0} \,, \tag{5}$$

be nonzero: for if this is so, then there are at least $|I|$ states of zero energy, all annihilated by $Q$ and $Q^\dagger$ (by #2 above), whence supersymmetry is unbroken.[1]

The property of $I$ crucial for what follows is that it is invariant under smooth perturbations of the parameters of the supersymmetric system (for example, of the masses and couplings of the participating fields; and as mentioned, taking the volume in which the fields are defined smoothly to infinity). In the applications to differential geometry, $H$ will become a functional of various external gauge and gravitational fields, (i.e. connections on appropriate manifolds), so that $I$ is in fact a true topological invariant.

[1] The case $I = 0$ could arise either via (A) $n_b^{E=0} = n_f^{E=0} \neq 0$ (supersymmetry unbroken) or (B) $n_b^{E=0} = n_f^{E=0} = 0$ (supersymmetry broken).

The definition (5) becomes much more constructive when the implied trace of $(-1)^F$ on the zero energy subspace is extended to the whole of $\mathcal{H}$. As it stands such an alternating sum is undefined, but it becomes well-behaved when supplied with a convergence factor $e^{-\tau H}$ for $\tau > 0$. From property #3 above, for $E > 0$ the contributions from the bosonic and fermionic states will precisely cancel in pairs.

Finally note from the block diagonal form (4) of the operators, and with property #2 above in mind, that the quantity $n_b^{E=0}$ is precisely the dimension of the subspace of states annihilated by $A$, and similarly for $n_f^{E=0}$ and $A^\dagger$. Thus we can write the index formally as

$$I(A) \equiv \dim \ker A - \dim \ker A^\dagger \tag{6}$$

where the $A$ has been appended to emphasize that this coincides precisely with the normal mathematical definition of the index of the operator $A$ in this case.

Putting this together with the above remarks, we have the form most useful for computations,

$$I(A) = \mathrm{tr}(-1)^F e^{-\tau H}, \quad \tau > 0. \tag{7}$$

It is in this form[2] that one can attempt a quantum mechanical derivation of the index (§3) (in particular by letting $\tau \to \infty$), following [4–7]. However, to complete this section we quote some examples of supersymmetric quantum mechanical systems, from physics and from geometry.

### Example: the Hydrogen Atom

It is of interest to see a supersymmetric formulation of such a well-known textbook case as the quantum-mechanical Kepler problem (other solvable potentials could be taken equally well [17]). Consider a system defined on the direct sum $\mathcal{H} \oplus \mathcal{H}$

[2] The factor $(-1)^F$ could be absorbed by introducing the graded trace or "supertrace" rather than the ordinary trace.

of two copies of the usual Hilbert space of radial wavefunctions. In the notation of (4) we take

$$\sqrt{2m}A_\ell = P_r - i\hbar((\ell+1)/R - 1/(\ell+1)a)$$

and find

$$H = \begin{pmatrix} H_\ell + \mathcal{E}(\ell+1)^2 & 0 \\ 0 & H_{\ell+1} + \mathcal{E}/(\ell+1)^2 \end{pmatrix},$$

$$2mH_\ell = P_r^2 + \ell(\ell+1)\hbar^2/R^2 - 2me^2/R$$

is the usual radial hamiltonian for angular momentum $\ell$

$$\Big(R \to r\,, \quad P_r \to -i\hbar\Big(\partial/\partial r + \tfrac{1}{r}\Big)\,, \quad [R, P_r] = i\hbar\Big)\,;$$

and $\mathcal{E} \cong 13.6$ eV is the Rydberg:

$$\mathcal{E} = mc^2(e^2/\hbar c)^2 \equiv e^2/a.$$

The only acceptable ground-state wavefunction

$$\psi(r) \propto (r/a)^\ell e^{-r/(\ell+1)a}$$

is that annihilated by $A_\ell$ (not $A_\ell^\dagger$); this immediately gives the ground-state energy as $-\epsilon/(\ell+1)^2$. Moreover, since the $n$'th level of $H_\ell$ is supersymmetrically paired with the $(n-1)$'th level of $H_{\ell+1}$, the entire bound-state spectrum is thereby fixed (see fig. 2).

### Example: Massless Fields in Curved Space

In field theory applications it is frequently important to have information on which particles moving in a curved space-time (i.e. with external gauge and gravitational fields) will be massless. This could be for the direct reason that the degrees of freedom relevant to the presently known particle spectrum are truly massless, on the scale of some Planckian unification mass; or for technical reasons, that these modes require special treatment computationally. In any case, one requires the zero

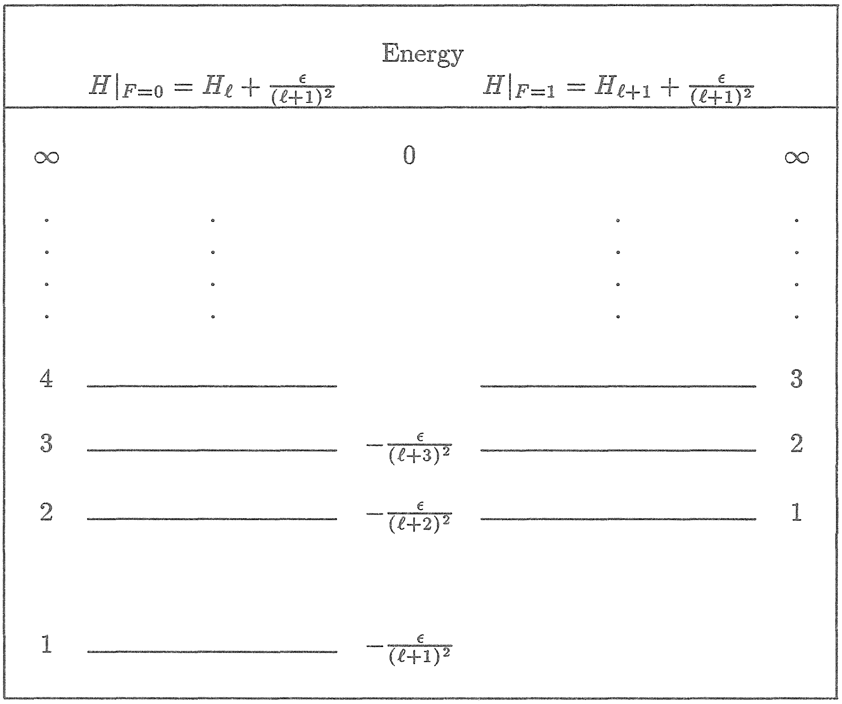


Fig. 2. Supersymmetry of the Hydrogen Spectrum.

eigenvalues of an appropriate wave equation for fields of various spin (represented by fields with different types of tensor structure).

For example, integer spin fields (bosons) are described by scalar fields $\phi(x)$, vector fields $A_\mu(x)$, symmetric tensor fields $g_{\mu\nu}(x)$ and so on. The correct wave equation for higher-spin fields is to some extent a matter of taste [18], but clearly should be some generalization of the Laplacian (or d'Alembertian, if one remembers the Minkowskian signature of the metric). In fact the precise choice is not crucial as far as topological invariants are concerned. A natural choice does exist for the case of totally antisymmetrical covariant tensors (rank 0 (scalar), rank 1 (vector), $\ldots$, up to $n$ in $n$ dimensions), and the Laplacian for the collection of such fields can be written in a supersymmetrical way (see below).

For fields of half-integer spin (fermions), one must use spinors, the simplest case being spin-$\frac{1}{2}$. Spinors are introduced via the Dirac $\gamma$-matrices satisfying the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}1, \quad \gamma_\mu^\dagger = \gamma_\mu, \quad \mu, \nu = 1, \ldots, n$$

where $g_{\mu\nu}$ is the Riemannian metric. The massless spinors are the zero solutions of the Dirac equation

$$i\not{D}\psi(x) = 0,$$

when the Dirac operators is

$$i\not{D} = i\gamma^\mu D_\mu = i\gamma^\mu(\nabla_\mu + A_\mu), \tag{8}$$

with $A_\mu$ the (Lie algebra-valued) gauge potential, and $\nabla_\mu$ the appropriate covariant derivative defined using the Riemannian connection.

In even-dimensional spacetime, there is an additional quantity

$$\Gamma = (i)^{n/2}\epsilon_{\mu_1\ldots\mu_n}\gamma^{\mu_1}\ldots\gamma^{\mu_n}/n!,$$

which satisfies

$$\Gamma\gamma_\mu = -\gamma_\mu\Gamma, \quad \Gamma^2 = 1$$

and allows spinors to be separated into pieces of positive and negative "chirality",

$$\psi(x) = \tfrac{1}{2}(1+\Gamma)\psi(x) + \tfrac{1}{2}(1-\Gamma)\psi(x) = \psi_+ + \psi_-,$$
$$\Gamma\psi_\pm = \pm\psi_\pm.$$

Clearly then, $\not{D}_\pm = \frac{1}{2}(1 \pm \Gamma)\not{D}$ map spinors of one chirality on to spinors of opposite chirality. Thus for the square of the Dirac operator,

$$\not{D}^2 = \not{D}_+\not{D}_- + \not{D}_-\not{D}_+ = \{\not{D}_+, \not{D}_-\},$$
$$\not{D}_\pm^2 = 0, \tag{9}$$

because

$$\not{D}_\pm^2 = \tfrac{1}{2}(1 \pm \Gamma)\not{D}\tfrac{1}{2}(1 \pm \Gamma)\not{D} = \tfrac{1}{4}(1 \pm \Gamma)(1 \mp \Gamma)\not{D}^2 = 0,$$

and the supersymmetric structure is evident (cf. (1), (2)).

The physical language of classical fields used above can be transcribed succinctly into natural geometrical constructions [19]. This is useful to emphasize the supersymmetric structure more precisely, and for the formal statement of the index theorem to be given below. Given a smooth orientable compact manifold $M$ without boundary, of dimension $n$. Consider the bundles of totally antisymmetric covariant tensors of rank $p$, $\Lambda^p(T^*M)$, $p = 1, \ldots, n$. The exterior derivative $d$ acts on $\Omega^p$, the corresponding space of smooth sections of $\Lambda^p(T^*M)$ ($p$-forms), to give $(p+1)$-forms

$$d : \Omega^p \to \Omega^{p+1},$$

with [19]

$$d\omega = \frac{1}{p!}\partial_\mu \omega_{\mu_1 \ldots \mu_p} dx^\mu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}.$$

Correspondingly there is the differential operator $\delta$,

$$\delta : \Omega^p \to \Omega^{p-1},$$

with [19]

$$\delta\omega = +(1/(p-1)!)\nabla^\mu \omega_{\mu\mu_1 \ldots \mu_{p-1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p-1}}.$$

Here $\nabla^\mu$ is the covariant derivative defined using the Riemannian connection. $\delta$ is in fact the adjoint of $d$ with respect to the inner product on $p$ forms

$$(\alpha, \beta) = \int_M (\alpha \cdot \beta) dV.$$

The Laplacian, $\Delta$, is simply the second-order operator

$$\Delta : \Omega^p \to \Omega^p$$

defined by

$$\Delta = d\delta + \delta d. \tag{10}$$

Since both $d$ and $\delta$ are nilpotent,

$$d^2 = \delta^2 = 0,$$

a supersymmetric structure is evident, once the analogue of "fermion number" is refined. The natural grading is to split the total bundle

$$\Lambda^* = \bigoplus_{p=0}^{n} \Lambda^p(T^*M)$$

into spaces of tensors of even and odd degree:

$$\Lambda^* = \Lambda^+ \oplus \Lambda^-;$$

then it is the operators $(d+\delta)_\pm$ restricted to the corresponding smooth sections which provide the analogues of the supersymmetry generators: in the notation of (4),

$$Q = \begin{pmatrix} 0 & 0 \\ (d+\delta)_+ & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & (d+\delta)_- \\ 0 & 0 \end{pmatrix},$$
$$\Delta = \begin{pmatrix} d_-\delta_+ + \delta_- d_+ & 0 \\ 0 & d_+\delta_- + \delta_+ d_- \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (11)$$

The index is the alternating sum of the number, $B_p$, of zero modes of the Laplacian ("harmonic forms"), which will be recognized as the Euler number of $M$. Witten [3] went further and provided a quantum mechanical derivation of the Morse inequalities for the $B_p$ themselves.

In the spinor case, $M$ must be chosen so as to admit a spin structure. Then one can construct the spin bundle with its decomposition

$$S = S_+ \oplus S_-$$

into bundles of opposite chirality; furthermore if there is an internal gauge group then there are the associated twisted bundles $S_\pm \otimes V$ where $V$ carries a representation of the gauge group. The Dirac operator acts on the smooth sections

$$\not{D}_\pm : \Gamma^\infty(S_\pm \otimes V) \to \Gamma^\infty(S_\mp \otimes V)$$

and since

$$(\psi, i\not{D}_+\chi) = \int_M \psi^\dagger \tfrac{1}{2}(1+\Gamma) i\not{D}\chi dV$$

$$= \int_M ((i\not{D})^\dagger \tfrac{1}{2}(1+\Gamma)^\dagger \psi)^\dagger \chi dV = \int_M (i\not{D}\tfrac{1}{2}(1+\Gamma)\psi)^\dagger \chi dV$$
$$= \int_M (\tfrac{1}{2}(1-\Gamma) i\not{D}\psi)^\dagger \chi dV$$
$$= (i\not{D}_-\psi, \chi)$$

the chiral projections are adjoints as required. Thus in terms of the notation of (4) we have

$$H = (i\not{D})^2 = \begin{pmatrix} i\not{D}_-\not{D}_+ & 0, \\ 0 & i\not{D}_+\not{D}_- \end{pmatrix}, \quad (-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \Gamma,$$
$$Q = \begin{pmatrix} 0 & 0 \\ i\not{D}_+ & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & i\not{D}_- \\ 0 & 0 \end{pmatrix}. \tag{12}$$

## §2. THE INDEX THEOREM AND APPLICATIONS

The Atiyah-Singer Index Theorem [1, 19, 20] formally relates to elliptic operators on sequences of vector bundles over some manifold $M$. The most important case, and which contains many other results as special cases (for example the harmonic $p$-forms for integer spin), is that of the Dirac operator. This is the case which we shall take up below (§3) and develop a path integral formulation for; in this section we begin by giving a concise statement of the theorem, and evaluate some special cases. The remaining discussion is intended to bring out the importance of the result in field theory applications, in questions of anomaly cancellation, in the standard model and beyond.

### The Atiyah-Singer Index Theorem for the Twisted Dirac Operator

Let $M$ be a compact manifold without boundary as before, with Riemannian curvature 2-form

$$\mathcal{R}^\mu{}_\nu = \tfrac{1}{2} R^\mu{}_{\nu\rho\sigma} dx^\rho \wedge dx^\sigma,$$

and consider an associated vector bundle $V$ with curvature 2-form

$$\mathcal{F} = \tfrac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

taking values in the Lie algebra of the structure group $G$ of the appropriate principal bundle. The index theorem for this case states

$$I(\not{D}_+) = \int_M \hat{A}(\mathcal{R})\mathrm{ch}(\mathcal{F}). \tag{13}$$

Here $\hat{A}(\mathcal{R})$ and $\mathrm{ch}(\mathcal{F})$, the Dirac genus and Chern character respectively, are polynomials in the characteristic classes of $M$ (nontrivial, closed $p$-forms) expressible in terms of the curvatures as

$$\begin{aligned} \hat{A}(\mathcal{R}) &= \det{}^{-1/2}\left[\frac{\sinh(i\mathcal{R}/2\pi)}{i\mathcal{R}/2\pi}\right] \\ \mathrm{ch}(\mathcal{F}) &= \mathrm{tr}[\exp(i\mathcal{F}/2\pi)], \end{aligned} \tag{14}$$

where the matrix operations are with respect to the tensor labels of $\mathcal{R}^{\mu}{}_{\nu}$, and the fibre space $V$ respectively.

Computationally we should evaluate the right hand side of (13) by expanding (14) in power series and extracting the $n$-form part as the integrand, which gives a local function of some invariant combination of $R^{\mu}{}_{\nu\rho\sigma}$ and $F_{\mu\nu}$ to be integrated over $M$. The power of the index theorem (13) is displayed by the fact that it relates solutions of differential equations on $M$, that is local information, on the left-hand side, with global, topological invariants on the right-hand side. In particular, they depend only on the characteristic classes of the bundles involved.

### Specializations [7]

If we take $G$ to be the structure group $SO(n)$ of the frame bundle of the Riemannian manifold, then the sections acted on by the Dirac operator will carry representations of the local Lorentz group, i.e. they will be spinor-vector, spinor-tensor ... etc. higher-spin fields. Thus we take

$$F_{\mu\nu} \equiv \tfrac{1}{2} R_{\mu\nu}{}^{\alpha\beta} J_{\alpha\beta}$$

where $J_{\alpha\beta}$ are the antihermitian generators of $SO(n)$,

$$[J_{\alpha\beta}, J_{\gamma\delta}] = \delta_{\beta\gamma}J_{\alpha\delta} - \delta_{\alpha\gamma}J_{\beta\delta} - \delta_{\beta\delta}J_{\alpha\gamma} + \delta_{\alpha\delta}J_{\beta\gamma}$$

in the appropriate representation.

In four dimensions $SO(4) \cong SO(3) \times SO(3)$ and irreducible representations are labelled $(a,b)$, $a,b = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, corresponding to the spins of the commuting $SO(3)$ factors generated by

$$J_i^{\pm} = \tfrac{1}{4}\epsilon_{ijk} J_{jk} \pm \tfrac{1}{2} J_{i4}. \tag{15}$$

Denoting the associated bundles $V$ for brevity also as $(a,b)$, we have for example $S_+ = (\frac{1}{2}, 0)$, $S_- = (0, \frac{1}{2})$, and the twisted Dirac operator will be acting on smooth sections of $(\frac{1}{2}, 0) \otimes (a,b) = (a + \frac{1}{2}, b) \oplus (a - \frac{1}{2}, b)$.

Since

$$\mathrm{tr}(J_i J_j) = -\tfrac{1}{12} A(A^2 - 1)\delta_{ij}, \quad A = 2a + 1$$

within an irreducible representation of $SO(3)$ with spin $a$, we deduce for $SO(4)$ (from (15))

$$\mathrm{tr}(J_{\alpha\beta} J_{\gamma\delta}) = \frac{AB}{12}[(\delta_{\beta\gamma}\delta_{\alpha\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta})(A^2 + B^2 - 1) - \epsilon_{\alpha\beta\gamma\delta}(A^2 - B^2)],$$

where $A = 2a + 1$, $B = 2b + 1$, whence

$$\begin{aligned} I(a,b) &= \int_M \left(1 + \frac{\mathrm{tr}(R \wedge R)}{12(4\pi)^2}\right)\left(\mathrm{tr}(1) - \frac{R^{\alpha\beta} \wedge R^{\gamma\delta}\mathrm{tr}(J_{\alpha\beta} J_{\gamma\delta})}{2!(4\pi)^2}\right) \\ &= \frac{AB}{24}[(A^2 + B^2 - 1)P(M) + 2(A^2 - B^2)\chi(M)] \end{aligned}$$

where

$$P(M) = \frac{1}{8\pi^2}\int R^{\mu\nu} \wedge R_{\mu\nu}$$

and

$$\chi(M) = \frac{1}{32\pi^2}\int \epsilon_{\mu\nu\rho\sigma} R^{\mu\nu} \wedge R^{\rho\sigma}$$

are the Euler and Pontrjagin numbers of $M$, respectively. The first few cases are

$$I(0,0) = P/24$$

(the index of the ordinary Dirac operator, related to the spin-$\frac{1}{2}$ axial anomaly (see below));

$$I(\tfrac{1}{2},0) + I(0,\tfrac{1}{2}) = P/3,$$

(the index of an operator: $C^\infty((1,0)\oplus(0,0)\oplus(\frac{1}{2},\frac{1}{2})) \to C^\infty((0,1)\oplus(0,0)\oplus(\frac{1}{2},\frac{1}{2}))$, the Hirzebruch signature theorem);

$$I(\tfrac{1}{2},0) - I(0,\tfrac{1}{2}) = \chi$$

(the index of an operator: $C^\infty((1,0)\oplus(0,1)\oplus 2(0,0)) \to C^\infty(2(\frac{1}{2},\frac{1}{2}))$, the Gauss-Bonnet theorem); and

$$I(\tfrac{1}{2},\tfrac{1}{2}) - I(0,0) = 21P/24,$$

(the index of an operator: $C^\infty(1,\frac{1}{2}) \to C^\infty(\frac{1}{2},1)$, related to the spin-$\frac{3}{2}$ axial anomaly).

Thus, assuming that the index of an elliptic operator is a topological invariant, we have derived, just from the twisted Dirac case, index formulae for several different bundles (including, paradoxically, results such as the Gauss-Bonnet and Hirzebruch signature theorem for integer-spin fields). By contrast, Christensen and Duff [18] considered index formulae for various higher-spin fields, case by case by heat kernel methods.

### The Index Theorem and Chiral Anomalies

As emphasized in the introduction, the index theorem has implications for so many different topics in theoretical physics that it is necessary to make a selection for the purposes of review. The discovery and still developing understanding of anomalies in quantum field theory has had a profound effect in shaping the evolution of the modern gauge theory approach to the description of elementary particles and their interactions, and the following discussion is intended to bring out the connections with the index theorem, and to illustrate the vital issues of anomaly cancellation in some relevant cases.

In the earlier discussion of supersymmetric quantum mechanics, we considered solutions of wave equations for different types of field, describing massless particles of various spins moving in curved space. The same setup provides a vehicle for consideration of anomalies, but it is essential to consider the matter fields to be quantized in the presence of classical background gauge and gravitational fields. The simplest case is that of massless spin-$\frac{1}{2}$ fermions, described by the action

$$S = \int_M (\bar{\psi} i \not{D} \psi) dV, \tag{16}$$

where $\not{D}$ is the Dirac operator as in (8).

Noether's theorem in classical field theory states that if an action functional is invariant with respect to some field transformation, then there is a corresponding conserved current. In the case of (16), in addition to local gauge and general coordinate invariance, there is a symmetry with respect to "chiral" phase transformations[3] ($\theta$ a real constant)

$$\psi \to e^{i\theta\gamma_{D+1}}\psi, \quad \bar{\psi} \to \bar{\psi} e^{i\gamma_{D+1}\theta}, \tag{17}$$

and classically conserved current,

$$\partial^\mu (\bar{\psi}\gamma_\mu \gamma_{D+1}\psi) = 0. \tag{18}$$

In the quantized theory the current is ill-defined because of singularities in the products of field operators at the same space-time point. Set the external fields to zero and define

$$J_\mu(x) = \operatorname*{lt}_{y \to x} \bar{\psi}(x)\gamma_\mu\gamma_{D+1}\psi(y)$$

for an appropriate (regulated) limiting process. Consider the vacuum expectation value

$$\begin{aligned}\langle J_\mu(x)\rangle &= \operatorname*{lt}_{y \to x} \langle \bar{\psi}(x)\gamma_\mu\gamma_{D+1}\psi(y)\rangle \\ &= -\operatorname*{lt}_{y \to x} \operatorname{tr}(\gamma_\mu\gamma_{D+1} S(y-x))\end{aligned}$$

[3] In this section a Minkowski metric is used and the dimension of space-time is $D = 2n$. Thus in a suitable basis $\bar{\psi} = \psi^\dagger\gamma_0$ and $\gamma_{D+1}^\dagger = \gamma_{D+1}$, $\gamma_{D+1}^2 = 1$ is the analogue of the Euclidean $\Gamma$.

for the appropriate Green's function $S(x, y) = \langle\psi(x)\bar{\psi}(y)\rangle$. Then

$$\begin{aligned}\langle\partial^\mu J_\mu(x)\rangle &= -\operatorname*{lt}_{y\to x} \operatorname{tr}(\not{\partial}_x \gamma_{D+1} S(y-x)) \\ &= -\operatorname*{lt}_{y\to x} \operatorname{tr}(\gamma_{D+1} \not{\partial}_y S(y-x)) \\ &\equiv \operatorname*{lt}_{y\to x} \operatorname{tr}(\gamma_{D+1}\delta(y-x)).\end{aligned}$$

Undefined though this is, one further spatial integration will convert it into a trace of $\gamma_{D+1}$ regarded as an operator on both spin and spatial degrees of freedom. Recalling that $\gamma_{D+1} = (-1)^F$ from (12), it will then correspond precisely to an (unregulated) form of the Witten index [5]. A similar heuristic argument can be made if the external fields are nonvanishing.

Careful field-theoretical computations (see [21] and original references therein) confirm that the axial current divergence indeed is proportional to the local density of the Atiyah-Singer index. Thus in flat space [21]

$$\begin{aligned}\partial^\mu J_\mu &= K_n \epsilon_{\mu_1\mu_2\dots\mu_{2n}} \operatorname{tr}(F^{\mu_1\mu_2\cdots} F^{\mu_{2n-1}\mu_{2n}} + \text{perms}), \\ K_n &= i^n / 2^{2n-1}\pi^n n!,\end{aligned} \tag{19}$$

as expected from (13), which also provides the correct gravitational curvature contributions with the same relative normalization (for field-theoretical computations, see [22] and original references therein).

(17), (18) and (19) have generalizations for nonabelian chiral transformations

$$\psi \to e^{i\theta\gamma_{D+1}}\psi, \quad \bar{\psi} \to \bar{\psi} e^{i\gamma_{D+1}\theta} \tag{20}$$

leading to the classically conserved current

$$D^\mu(\bar{\psi}\gamma_\mu\gamma_{D+1}\theta\psi) = 0 \tag{21}$$

(where $D^\mu$ is the appropriate covariant derivative, and $\theta$ is now an element of the Lie algebra of the gauge group in the fermion representation).

The nonabelian anomaly (i.e., the anomalous form of (21)) can be established by field-theoretical calculations analogous to those for the abelian anomaly, (19). However it has been shown that starting formally from the abelian case in $(D+2)$-dimensions leads uniquely to a consistent form of the nonabelian anomaly in $D$ dimensions [21]. The mathematical background for this state of affairs is the index theorem for parametrized families of Dirac operators [22, 23, 24].

The distinction between the abelian and nonabelian cases is best explained in terms of differential forms. For the abelian case (19) one has

$$*\delta J \propto C_D \tag{22}$$

where $C_D$ is the $D$'th Chern class (and $*$ maps a 0-form to the corresponding volume form). Since $C_D$ is exact it can always be written locally as

$$C_D = d\omega^0_{D-1}$$

where $\omega^0_{D-1}$ is the Chern-Simons form. For example,

$$\omega^0_3 = \mathrm{tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A)$$

is the Chern-Simons Lagrangian arising in the topological quantum field theory of knots [8].

In the nonabelian case the anomalous form of (21) is

$$*\delta_A J^\theta \propto d\omega^\theta_D \tag{23}$$

where $\omega^\theta_D$ is the secondary Chern-Simons form related to $C_{D+2}$:

$$C_{D+2} = d\omega^0_{D-1},$$
$$\delta_\theta \omega^0_{D-1} = \omega^\theta_D;$$

where $\delta_\theta$ measures the infinitesimal response to a gauge transformation of $A$ generated by $\theta$, namely

$$A \to A + (d\theta + [A, \theta]) + O(\theta^2).$$

In four dimensions the secondary Chern-Simons form is

$$\omega_3^\theta = \mathrm{tr}\theta(A \wedge dA + \tfrac{1}{2} A \wedge A \wedge A)$$

which should be distinguished carefully from $\omega_3^0$ above.

The significance of the anomalies (22), (23) is that the $O(\hbar^1)$ terms in the quantized action no longer admit the symmetries of the classical ($O(\hbar^0)$) action. In the case of four-dimensional Yang-Mills theory, the abelian chiral anomaly leads to the phenomena of degenerate vacua, instanton tunnelling, and so on; the non-abelian chiral anomaly implies a breakdown of gauge invariance which spoils the Green's function identities vital to implement renormalization. Put another way, it would be necessary to include counterterms for interactions not present in the original Lagrangian, which lead in turn to further nonrenormalizable infinities. In higher dimensions, and for gravitational anomalies (see below), the criterion of non-renormalizability is not relevant, but the anomalies breaking gauge invariance are still thought to lead to inconsistencies in quantization and unitarity [22, 24].

In order to illustrate these points, we conclude with a discussion of anomaly cancellation in the standard electroweak model; of general gauge and gravitational anomalies in $D$ dimensions; and a sketch of the Green-Schwarz mechanism for anomaly removal in 10-dimensional superstrings.

### Anomaly Cancellation in the Standard Electroweak Model

The colour, weak isospin $I$ and hypercharge $Y$ (with electric charge, $Q$) assignments of a single quark and lepton generation of the standard $SU(3)_{colour} \times SU(2)_I \times U(1)_Y$ electroweak model (the neutrino, electron, and up and down quarks, regarded as left-handed Dirac fields) are shown in fig. 3.

From the discussion above and equation (23) of the nonabelian anomaly (e.g. the explicit form of $\omega_3^1$ given above), the nonabelian anomaly is proportional to a group-theoretical factor

$$\mathrm{tr}(ABC + \mathrm{perms}) = 3\mathrm{tr}(A\{BC\})$$

| $Q$ | Particle | Colour | I | Y |
|---|---|---|---|---|
| $\binom{0}{-1}$ | $\binom{\nu}{e^-}$ | singlet | $\frac{1}{2}$ | $-1$ |
| $\binom{\frac{2}{3}}{-\frac{1}{3}}$ | $\binom{u}{d}$ | triplet | $\frac{1}{2}$ | $\frac{1}{3}$ |
| $+1$ | $e^+$ | singlet | 0 | $+2$ |
| $-\frac{2}{3}$ | $\bar{u}$ | antitriplet | 0 | $-\frac{4}{3}$ |
| $+\frac{1}{3}$ | $\bar{d}$ | antitriplet | 0 | $+\frac{2}{3}$ |

Fig. 3. Quantum Numbers of One Lepton and Quark Generation.

for any generators $A$, $B$, $C$ of the gauge group. As far as the weak isospin is concerned, the nonsinglet multiplets are isodoublets, respresented by the standard Pauli matrices $\sigma^1$, $\sigma^2$ and $\sigma^3$. Thus the contributions to $\mathrm{tr}(Y^3)$, $\mathrm{tr}(\sigma^i Y^2)$, $\mathrm{tr}(Y\{\sigma^i,\sigma^j\})$ and $\mathrm{tr}(\sigma^i\{\sigma^j,\sigma^k\})$ should all cancel. Amazingly the first trace is explicitly zero; for the second the tracelessness of $\sigma^i$ within each isomultiplet of constant $Y$ ensures its vanishing. Also since $\{\sigma^i,\sigma^j\} = 2\delta^{ij}$, the tracelessness of $\sigma^i$ is also sufficient for the last trace, while for the third, by the same token, the tracelessness of $Y$ for the nonisosinglet multiplets is sufficient.

Similar considerations apply if colour generators are included (it is important to note that the anti-triplet generators are the negative transposes of the triplet generators, and there are an equal number of each), and so the standard set of particles in fig. 3 is adjudged to be anomaly-free. (Of course, the full story on anomalies necessitates the examination of various third-order Dynkin invariants for the relevant gauge groups [25], but this demonstration has the virtue of being concrete and simple).

### General Anomalies and the Green-Schwarz Mechanism

In addition to gauge anomalies, higher-dimensional theories with matter fields in appropriate complex representations also have gravitational anomalies ("Lorentz" or "Einstein") in certain dimensions [22]. In this case the symmetry with respect to local Lorentz transformations is violated by $O(\hbar)$ terms in the corresponding current divergence. Equivalently, translation invariance is violated, leading to the Einstein form of the anomaly [22].

The situation is summarized in fig. 4 where the spacetime dimensions $D$ at which $\mathcal{F}$ and $\mathcal{R}$ curvature terms are present (for chosen matter fields) are quoted for the different anomalies. The explicit forms of the anomalies are controlled by the relevant $(D+2)$-form Chern classes, as indicated by the discussions in brackets (for the gravitational as well as the nonabelian cases (23)). In fact [22, 26] *no* combination of Dirac (spin-$\frac{1}{2}$), Rarita-Schwinger (spin-$\frac{3}{2}$) and $D/2$-form (spin-1) matter fields can ensure anomaly cancellation in greater than ten dimensions.

| Contribution from | Chiral Abelian | Chiral Nonabelian | Lorentz (and Einstein) |
|---|---|---|---|
| $\mathcal{F}$ | D = 2k | D = 2k (2k+2) | D = 4k+2 (4k+4) |
| $\mathcal{R}$ | D = 4k | D = 4k (4k+2) | D = 4k+2 (4k+4) |

Fig. 4. Dimensions $D$ for which various gauge and gravitational anomalies occur and their controlling $(D+2)$-forms

On the other hand in the ten-dimensional superstring theory the matter content is specified by the effective low-energy point field theory ($N = 1$ supergravity plus Yang-Mills theory). The total anomaly is specified by a 12-form

$$I_{12} = \mathrm{tr}(\wedge\mathcal{F})^6 + (\cdots)\mathrm{tr}\,\mathcal{R}\wedge\mathcal{R}\,\mathrm{tr}(\wedge\mathcal{F})^4 + \ldots + (\cdots)\mathrm{tr}(\wedge\mathcal{R})^6 \tag{24}$$

with known coefficients [27]. A putative factorization of $I_{12}$ of the form

$$I_{12} = (\mathrm{tr}\mathcal{R}\wedge\mathcal{R} + k\mathrm{tr}\mathcal{F}\wedge\mathcal{F})\wedge X_8$$

$$\equiv (C_4(R) + kC_4(F)) \wedge X_8 \tag{25}$$

is imposed, where $X_8$ is some invariant 8-form and the $C_4$ and the four-form Chern classes. This leads by the descent equations (cf. (22), (23)) to the ten-dimensional form of the anomaly,

$$I_{10} = \int_{10} 2(\omega_2^1(R) + k\omega_2^1(F)) \wedge X_8 + 4(C_4(R) + kC_4(F))X_6^1,$$

where the secondary form $X_6^1$ is related to $X_8$ by descent in the same way as the Chern-Simons secondaries $\omega_2^1$ are related to the Chern classes $C_4$. But this means that there exists [27] a unique local counterterm which cancels the anomaly:

$$S_C = \int_{10} 4\big(\omega_3^0(R) + k\omega_3^0(F)\big) \wedge X_7^0 - 6BX_8.$$

The $B$ field (which the anomaly removal mechanism is designed to utilize) is the 2-form gauge potential of ten-dimensional $N = 1$ supergravity whose gauge transformation reads

$$\delta_\theta B = \omega_2^1(\theta, R) + k\omega_2^1(\theta, F)$$

as is known (and essential) from other considerations. The detailed form of (24), (25) leads uniquely [27] to the only possible gauge groups $SO(32)$ or $E_8 \times E_8$. For example, the $\mathrm{tr}(\wedge\mathcal{R})^6$ coefficient of (24), required to vanish for (25), is $\propto$ (dim $G -$ 496), and the Lie algebras of $SO(32)$ and $E_8 \times E_8$ are indeed 496-dimensional!

## §3. PATH INTEGRAL DERIVATION OF THE INDEX THEOREM

As mentioned in the introduction, the realization [3] that the mathematical formalism of the index theorem allows the relevant elliptic operators to be regarded as the hamiltonian and supersymmetry generators of an associated quantum mechanical system, has led to proofs of the index theorem using quantum mechanical methods: steepest descent methods in [3, 6], and general path integral considerations in [4]. Detailed derivations for the $U(1)$ chiral anomaly have been given in [5] using supersymmetric heat-kernel methods.

The appropriate quantum mechanics for the hamiltonian $(i\not{D})^2$ is in terms of operators

$$[X^\mu, P^\nu] = i\hbar g^{\mu\nu}, \quad \{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta},$$

for bosons and fermions, respectively, in the Schrödinger representation

$$X^\mu \to x^\mu, \quad P_\mu \to -i\hbar\partial_\mu.$$

In the path-integral derivations [4, 5] one interprets $\mathrm{tr}e^{-\tau H}$ as the partition function of an appropriate continuum model (a type of supersymmetric nonlinear $\sigma$ model in $0+1$ dimensions). The evaluation then proceeds along standard lines, viz. isolation of zero modes (in this case, constant paths), and with the non-constant modes (e.g. decomposed in Fourier components) contributing to certain infinite-dimensional determinants. For rigorous definitions of path integral measures for fermions, and applications to a proof of the Gauss-Bonnet form of the index theorem, see Rogers [28].

As an alternative, the discussion to be given below [7] uses a more naïve approach in that it works with an approximation to the path integral, in terms of a finite number $2DN$ of degrees of freedom, but, rather than going to the continuum limit, $2D(N-1)$ integrals are performed for the particular hamiltonian in question, giving the index correctly up to $O(1/N)$, with the correct normalization.

The derivation is heuristic but straightforward, and provides further insight into the supersymmetric structure of the index theorem. We commence with a discussion of path integrals via fermionic (and bosonic) coherent states [29].

### Fermionic Path Integrals

Consider the 2-state system $\{|0\rangle, |1\rangle\}$ generated by a fermionic creation and annihilation operators

$$|1\rangle = a^\dagger|0\rangle, \quad a|0\rangle = 0, \quad \{a, a^\dagger\} = 1.$$

By analogy with the bosonic case, introduce the following coherent state basis

$$|\varphi\rangle = e^{-\varphi a^\dagger}|0\rangle = |0\rangle - \varphi|1\rangle = |0\rangle + |1\rangle\varphi$$
$$\langle\bar\varphi| = \langle 0|e^{-a\bar\varphi} = \langle 0| - \langle 1|\bar\varphi = \langle 0| + \bar\varphi\langle 1|$$

where $\varphi$ and $\bar\varphi$ are Grassmann variables[4] which anticommute with $a$ and $a^\dagger$.

Then the following properties are easily verified:

$$\langle\bar\varphi|\chi\rangle = e^{\bar\varphi\chi} = 1 + \bar\varphi\chi \equiv \chi(\bar\varphi),$$
$$\int d\varphi d\bar\varphi |\varphi\rangle e^{-\bar\varphi\varphi}\langle\bar\varphi| = 1,$$

for the overlap and overcompleteness of the coherent states. Here Grassmann integration is defined by the rules

$$\int d\varphi = \int d\varphi = 0$$
$$\int d\varphi\varphi = \int d\bar\varphi\bar\varphi = 1 = \int d\varphi d\bar\varphi\bar\varphi\varphi.$$

In the coherent state basis, operators $A$ may be regarded as acting on functions of the coherent state variables

$$A\varphi(\bar\chi) = \langle\bar\chi|A|\varphi\rangle = \int d\eta d\bar\eta d^{-\bar\eta\eta}\langle\bar\chi|A|\eta\rangle\langle\bar\eta|\varphi\rangle$$
$$= \int d\eta d\bar\eta e^{-\bar\eta\eta} A(\bar\chi,\eta)\varphi(\bar\eta).$$

For an operator $A = \Sigma\alpha_{mn}(a^\dagger)^m a^n$ in normal-ordered form, the integral kernel $A(\bar\chi,\eta)$ is simply related to the normal kernel $\alpha(\bar\chi\eta) = \Sigma\alpha_{mn}\bar\chi^m\eta^n$:

$$A(\bar\chi,\eta) = e^{\bar\chi\eta}\alpha(\bar\chi,\eta).$$

Finally

$$\text{tr}A \equiv \langle 0|A|0\rangle + \langle 1|A|1\rangle = -\int d\bar\varphi d\varphi e^{\bar\varphi\varphi}\langle\bar\varphi|A|\varphi\rangle,$$
$$\text{tr}(-1)^F A \equiv \langle 0|A|0\rangle - \langle 1|A|1\rangle = +\int d\bar\varphi d\varphi e^{-\bar\varphi\varphi}\langle\bar\varphi|A|\varphi\rangle.$$

[4] Elements of a complex, infinite-dimensional Grassmann algebra. Thus $\varphi^2 = \bar\varphi^2 = 0$, $\varphi\bar\varphi = -\bar\varphi\varphi$, and the exponential is a linear function of $\bar\varphi\chi$!

All of these formulae generalize in the obvious way to multifermion spaces [29].

In the problem at hand we initially evaluate the matrix element $\langle\bar{\varphi}|e^{-\tau H}|\bar{\varphi}\rangle$ and convert it into a supertrace as above. The matrix element itself is interpreted as the amplitude for evolution (in Euclidean time) of a system from a state $|\varphi\rangle$ at $t=0$ to a final state $\langle\bar{\varphi}|$ at $t=\tau$.

We break up the interval into $N$ subintervals of $\epsilon=\tau/N$ and introduce complete sets of states

$$\int d\varphi_k d\bar{\varphi}_k e^{-\bar{\varphi}_k\varphi_k}|\varphi_k\rangle\langle\bar{\varphi}_k|,$$

$\varphi_k, \bar{\varphi}_k$ at time $t=k\epsilon$, $k=1,\ldots,N-1$.

Writing $\bar{\varphi}\equiv\bar{\varphi}_N$ and $\varphi\equiv\varphi_0$ we have

$$\begin{aligned}\langle\bar{\varphi}_N|e^{-\tau H}|\varphi_0\rangle = \int\prod_{k=1}^{N-1} d\varphi_k d\bar{\varphi}_k\{&\langle\bar{N}|e^{-\epsilon H}|N-1\rangle e^{-\bar{\varphi}_{N-1}\varphi_{N-1}} \\ &\times\langle N-1|e^{-\epsilon H}|N-2\rangle\langle\ldots|1\rangle e^{-\bar{\varphi}_1\varphi_1}\langle 1|e^{-\epsilon H}|0\rangle\}\end{aligned}$$

and assuming $H$ is in normal form,

$$\langle\bar{\varphi}|e^{-\epsilon H}|\varphi\rangle \cong e^{\bar{\varphi}\varphi-\epsilon H(\bar{\varphi},\varphi)}$$

(where the error comes by assuming the exponential is also in normal form, i.e. at $O(\epsilon^2)$), we have

$$\begin{aligned}\langle\bar{\varphi}_N|e^{-\tau H}|\varphi_0\rangle \cong \int\prod_1^{N-1} & d\varphi_k d\bar{\varphi}_k \\ &\times\exp\left\{\bar{\varphi}_N\bar{\varphi}_{N-1}-\sum_1^{N-1}[\bar{\varphi}_k(\varphi_k-\varphi_{k-1})-\epsilon H(\bar{\varphi}_k,\varphi_{k-1})]\right\}.\end{aligned}$$

Finally the supertrace will provide an extra $-\bar{\varphi}_N\varphi_0$ in the exponential. Thus if we identify $\varphi_0=\varphi_N$, $\bar{\varphi}_0=\bar{\varphi}_N$ then we have the uniform expression

$$\mathrm{tr}((-1)^F e^{-\tau H}) \cong \int_{PBC}\prod_1^N d\varphi_k d\bar{\varphi}_k \exp -S_E$$

$$S_E=\sum_{k=1}^N \bar{\varphi}_k(\varphi_k-\varphi_{k-1})+\epsilon H(\bar{\varphi}_k,\varphi_{k-1}) \underset{N\to\infty}{\to} \int_0^\tau dt(\bar{\varphi}\dot{\varphi}+H(\bar{\varphi},\varphi)) \qquad (26)$$

in the continuum limit.

Almost identical formulae can be given for bosonic systems in a coherent-state basis corresponding to annihilation and creation operators

$$a = \frac{1}{\sqrt{2}}(X + iP), \quad a^\dagger = \frac{1}{\sqrt{2}}(X - iP)$$

for each canonically conjugate position $X$ and momentum $P$. However, in the following we stick to the basis of plane wave momentum states $\langle p|x\rangle \propto e^{ix\cdot p}$, and the derivation of the path integral is identical to the original Feynman discussion [29].

In fact for Hamiltonians of the general form

$$H = \tfrac{1}{2}(P + f(X))^2 + V(X),$$

the integration over momentum variables can be performed, leading to

$$\begin{aligned}
\mathrm{tr}e^{-\tau H} &\cong (2\pi\epsilon)^{-N/2} \int_{PBC} \prod_1^N dX_k \exp -S_E \\
S_E &= \sum_{k=1}^{N} [\tfrac{1}{2}(X_k - X_{k-1})^2/\epsilon + \tfrac{1}{2}(X_k - X_{k-1})(f_k + f_{k-1}) + \tfrac{1}{2}(V_k + V_{k-1})] \\
&\to \int_0^\tau dt(\tfrac{1}{2}\dot{x}^2 + \dot{x}f + V(x)) \qquad (27)
\end{aligned}$$

where $V_k \equiv V(x_k)$, etc. Finally, we should note the general formulae for Gaussian integration:

$$\begin{aligned}
\int dX \exp(-\tfrac{1}{2}x^T A x + z^T x) &= \det\left(\frac{2\pi}{A}\right)^{1/2} \exp(-z^T A^{-1} z), \\
\int d\varphi d\bar{\varphi} \exp(-\bar{\varphi} A \varphi + \bar{v} A \varphi + \bar{\varphi} u) &= \det{}^{\pm 1} A \exp \bar{v} A^{-1} u. \qquad (28)
\end{aligned}$$

**A Discrete Approximation to the Index (Flat Space)** [7]

For simplicity let us consider the index for the Dirac operator in flat space. Then we have

$$\begin{aligned}
\tfrac{1}{2}(i\not{D})^2 &= \tfrac{1}{2}(\gamma^\mu(i\partial_\mu + iA_\mu))^2 \\
&= \tfrac{1}{4}\{\gamma^\mu, \gamma^\nu\}(i\partial_\mu + iA_\mu)(i\partial_\nu + iA_\nu) \\
&+ \tfrac{1}{4}[\gamma^\mu, \gamma^\nu](i\partial_\mu + iA_\mu)(i\partial_\nu + iA_\nu), \qquad (29)
\end{aligned}$$

i.e.

$$\tfrac{1}{2}(i\not{D})^2 \to (P_\mu - iA_\mu)^2 - \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}.$$

Here the position dependence enters through $A(X)$ and $F_{\mu\nu}(X)$ which provide the potential for the bosonic part, and the interaction with the degrees of freedom, respectively. At this point we make the further assumption that the potential can be expanded, at least locally, as [5]

$$A_\mu = \tfrac{1}{2}X^\rho F_{\rho\mu} + O(X^2)\,, \tag{30}$$

where $F_{\mu\nu}$ is $X$-independent. This is analogous to the choice of Riemannian normal coordinates in the generally covariant case, and is justified by the topological invariance of the index. (The $O(X^2)$ terms give rise to $O(1/\tau)$ corrections which vanish in the limit; see below).

With (30) in place it is clear that bosonic and fermionic degrees of freedom are decoupled, (and moreover the integrations are all Gaussian), and for the discrete approximation to the path integral it only remains to identify the fermionic coherent state variables. For the internal degrees of freedom, we introduce [5, 7] a set of creation and annihilation operators $\xi, \xi^\dagger$ transforming in the appropriate representation of the Lie algebra of the gauge group. If $\{t^a\}$ are the (antihermitian) matrix generators then the operators

$$\xi^\dagger t^a \xi \tag{31}$$

will have the correct action on the internal Fock space. The latter is of course reducible, with the original representation as the one particle space.

For the Dirac algebra, fermionic creation and annihilation operators are defined via

$$\gamma^{2r-1} = \eta^r + (\eta^r)^\dagger, \quad i\gamma^{2r} = \eta^r - (\eta^r)^\dagger, \quad r = 1, 2, \ldots, D/2$$

and the corresponding coherent state Grassmann variables $\eta, \bar\eta$ introduced (with indices suppressed). After normal ordering, the real Grassmann variables $\psi^\mu$,

$$\psi^{2r-1} = (\eta^r + \bar\eta^r)/\sqrt{2}, \quad \psi^{2r} = (\eta^r - \bar\eta^r)/i\sqrt{2}, \tag{32}$$

can be re-introduced.

Finally from (26-32) we have [7]

$$I = (2\pi\epsilon)^{-ND/2}\int \Pi d\chi \Pi d\psi \Pi d\xi d\bar{\xi} e^{-S_E}$$

$$\begin{aligned} S_E = \sum_{k=1}^{N} \{ & \tfrac{1}{2}(x_k - x_{k-1})^2/\epsilon - \tfrac{1}{2}(x_k - x_{k-1})\cdot(\tfrac{1}{2}F_k\cdot x_k + \tfrac{1}{2}F_{k-1}\cdot x_{k-1}) \\ & + \tfrac{1}{2}\psi_k\cdot(\psi_k - \psi_{k-1}) - \epsilon\psi_k\cdot F_k\cdot\psi_{k-1} + \tfrac{i}{4}(\psi_k - \psi_{k-1})\cdot\Omega\cdot(\psi_k - \psi_{k-1}) \\ & - \epsilon(\psi_k - \psi_{k-1})G_k(\psi_k - \psi_{k-1}) + \bar{\xi}_k(\xi_k - \xi_{k-1})\} \end{aligned} \tag{33}$$

where $F_k^a = F^a\bar{\xi}_k t^a \xi_{k-1}$, the normal kernel for the appropriate internal operator, and $\Omega^{\alpha\beta}$, $G_k^{\alpha\beta}$ are certain non-covariant terms thrown up by normal ordering (cf. [6]).[5]

Both the $x$ and $\psi$ parts of (33), although quadratic, share the property of being symmetric under cyclic label permutations.

Thus before integration it is necessary to transform to relative and average coordinates $(x_*, \vec{z})$, viz.

$$\begin{aligned} x_* &= (x_1 + \ldots + x_n)/\sqrt{N}, \\ z_1 &= (x_1 - x_2)/\sqrt{2}, \\ z_2 &= (x_1 + x_2 - 2x_3)/\sqrt{6}, \\ z_{N-1} &= (x_1 + \ldots + x_{N-1} - (N-1)x_N)/\sqrt{N(N-1)}. \end{aligned}$$

If $(\psi_*, \vec{\chi})$ are the analogous coordinates for the fermions, then the integrand becomes

$$\begin{aligned} S_E = & - \{z^T(\mathcal{B} + \epsilon\mathcal{B}')z + z^T\Gamma x_*\}/2\epsilon \\ & - \left\{\tfrac{1}{2}\chi^T(\mathcal{F} + \epsilon\mathcal{F}')\chi + \chi^T\Delta\psi_* - \epsilon\psi_*\sum_{k=1}^{N} F_k\psi_*/2N\right\} + \ldots, \end{aligned}$$

apart from the $\bar{\xi}\dot{\xi}$ "kinetic energy" terms.

[5] The continuum limit of (33) is an $N = \frac{1}{2}$ supersymmetric nonlinear $\sigma$ model in $D+1$ dimensions. The supersymmetry is $\delta x^\mu = i\epsilon\psi^\mu$, $\delta\psi^\mu = -\epsilon\dot{x}^\mu$. The noncovariant pieces vanish in the continuum.

We assert [7] that the matrix elements of $\Gamma$, $\Delta$ are bounded by $\epsilon/\sqrt{N}$, $\epsilon/N$ respectively; thus the $\chi_*^2$, $\psi_*^2$ terms induced by the $z, \chi$ integrations (28) will be explicitly of higher order than the $\epsilon\psi_*(\Sigma F/2N)\psi_*$ term. Moreover $\mathcal{B}'$ and $\mathcal{F}'$ are $\propto F_{\mu\nu}$ so that their trace vanishes:

$$\det(\mathcal{X}+\epsilon F) = \det(\mathcal{X})\exp\operatorname{tr}(\epsilon F\mathcal{X}^{-1}+O(\epsilon^2)) = \det(\mathcal{X})(1+O(\epsilon^2))$$

where $\mathcal{X}$ is $\mathcal{B}$ or $\mathcal{F}$. Explicit calculation shows $\det^{1/2}(\mathcal{F}/\mathcal{B}) = N^{D/2}$. Also the $\vec{z}$ integrations cancel all but one factor of $(2\pi\epsilon)^{D/2}$. Finally, since $\Pi d\eta d\bar{\eta} = i^{D/2}\Pi d\psi$ from (32)), we are left with

$$I = (2\pi\epsilon N)^{D/2}\int d\chi_* d\psi_* \Pi d\xi d\bar{\xi}\cdot\exp -S_E'\cdot i^{D/2}$$

$$S_E' = \sum_{k=1}^{N}\bar{\xi}_k(\xi_k-\xi_{k-1})+\epsilon\bar{\xi}_k(\psi^* F\psi^*)\xi_{k-1}/2N.$$

The $\xi, \bar{\xi}$ integrals may be performed directly:

$$\begin{aligned} &-\bar{\xi}\,\xi_1+\bar{\xi}_1(1+F)\xi_0-\bar{\xi}_2\xi_2+\bar{\xi}_2(1+F)\xi_1+\dots \\ &\quad\to -\bar{\xi}_2\xi_2+\bar{\xi}_2(1+F)^2\xi_0-\bar{\xi}_3\xi_3+\bar{\xi}_3(1+F)\xi_2+\dots \\ &\quad\to\dots \\ &\quad\to -\bar{\xi}_N\xi_N+\bar{\xi}_N(1+F)^N\xi_0 \end{aligned} \tag{34}$$

where we have used (28) repeatedly.

A slight generalization obtains when one adds an additional term $\alpha N_{int}$ to the supersymmetric hamiltonian. This gives

$$I(x) = \operatorname{str}(e^{-\tau H+\alpha H_{int}}) = \sum_{n=0}^{\infty}x^n I_n(x)$$

where $x = e^\alpha$ is a generating parameter for the $n$th symmetrized (or antisymmetrized) product of the one particle representation of the gauge group (where $\xi, \xi^\dagger$ are bosons (fermions), respectively). Noting

$$e^{\alpha N_{int}}|\varphi\rangle = |0\rangle + |1\rangle e^\alpha\varphi \equiv |x\varphi\rangle,$$

then the only difference to the path integral is that the $\xi_0$ in (34) becomes multiplied by $x$ (and with it the entire $(1+F)^N$ coefficient). Thus applying PBC, $\xi_N = \xi_0$ as before, the result is

$$I(x) = \left(\frac{i}{2\pi}\right)^{D/2} \int d\psi^* \det{}^{\pm 1}(1 - x \exp \tfrac{1}{2}\psi_* F \psi_*) \tag{35}$$

where we have applied a rescaling

$$(N\epsilon)^{-D/2} dx_* d\psi_* = d(x_*/\sqrt{N}) d(\psi^* \sqrt{\epsilon}).$$

If necessary [5–7], we can replace $\frac{1}{2}\psi_*{}^\mu \psi_*{}^\nu F_{\mu\nu}$ and the $\psi_*$ integral by the 2-form equivalent $\mathcal{F}$, and ordinary volume integration, to get the index in standard form, with the correct normalization (cf. (13), (14)).

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