AN INVITATION TO THE ANTI-PERIODIC PROBLEM

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1. INTRODUCTION

Let u be a vector-valued function defined on $\mathbb{R}$. We say u is τ-*anti-periodic* for a fixed $\tau>0$ if

$$u(t+\tau)=-u(t), \qquad t\in\mathbb{R}.$$

This property seems to have been first studied in [13]. On the other hand, we call u τ-*periodic* if $u(t+\tau)=u(t)$ holds for each $t\in\mathbb{R}$. By definition, τ-anti-periodic functions are 2τ-periodic.

In this note, we shall explain results on the anti-periodic problem of nonlinear evolution equations with odd subdifferential operator terms which are defined in real Hilbert spaces. For this, we also relate the definition and properties of subdifferential operator, and some results on the periodic problem.

The author hopes that this note will interest the reader in the anti-periodic problem.

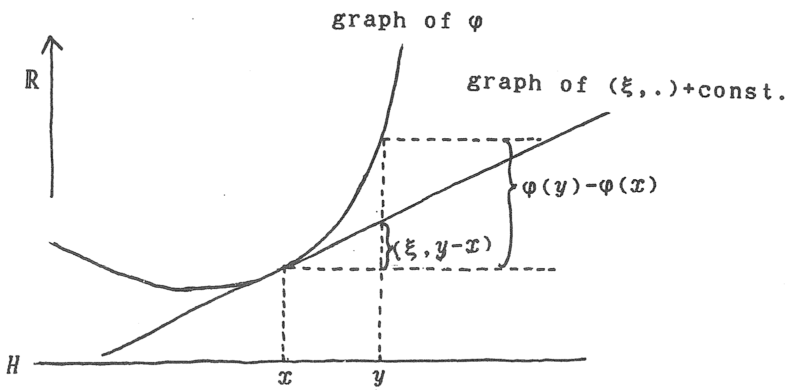
2. SUBDIFFERENTIAL OPERATOR (PRELIMINARY)

Let H be a real Hilbert space with innerproduct $(.,.)$ and norm $\|\ \|$. The subdifferential is a (possibly multivalued) operator defined as below: Let $\varphi: H\to(-\infty,+\infty]$ be a proper lower semi-continuous (l.s.c.) convex functional. The *effective domain* of

φ is the set $\{x\in H;\ \varphi(x)<+\infty\}$ and denoted by $\mathfrak{D}(\varphi)$. The *subdifferential* $\partial\varphi$ of φ is defined by

$$\partial\varphi(x)=\{\xi\in H;\ (\xi,y-x)\le\varphi(y)-\varphi(x)\quad \textit{holds for all}\quad y\in\mathfrak{D}(\varphi)\ \},$$
$$\mathfrak{D}(\partial\varphi)=\{x\in\mathfrak{D}(\varphi);\quad \textit{the set}\quad \partial\varphi(x)\quad \textit{is nonempty}\ \}.$$

By definition, the relation $\xi\in\partial\varphi(x)$ for $x\in\mathfrak{D}(\partial\varphi)$ is illustrated by the following figure;



The following properties are known ([2], [3],[11]):

(I) If φ is differentiable at $x\in\mathfrak{D}(\partial\varphi)$, then $\partial\varphi(x)$ is a singleton set and $\partial\varphi(x)=$ grad $\varphi(x)$. On the other hand, if $x\in\mathfrak{D}(\partial\varphi)$ and φ is sharp at x then $\partial\varphi(x)$ is multivalued.

(II) $\overline{\mathfrak{D}(\partial\varphi)} = \overline{\mathfrak{D}(\varphi)}$.

(III) $\partial\varphi$ is *monotone*, i.e., for all x, $y\in\mathfrak{D}(\partial\varphi)$ the estimate

(2.1) $$(x^*-y^*,\ x-y)\ge 0\ ,\qquad x^*\in\partial\varphi(x)\ ,\ y^*\in\partial\varphi(y)$$

holds. This property is obtained by convexity of φ. Moreover, since φ is proper and l.s.c., it is known that $\partial\varphi$ is *maximal monotone*. Hence, by applying the nonlinear semigroup theorem started by Kōmura [11], $\partial\varphi$ generates a (nonlinear) semigroup $\{S(t);t\geq 0\}$ defined on $\overline{D(\partial\varphi)}$ $(=\overline{D(\varphi)} \subset H)$. In other words, for each $u_0\in\overline{D(\partial\varphi)}$, the function $S(t)u_0$, $t\in[0,+\infty)$, is the unique solution in $W^{1,1}_{loc}(0,\infty;H)$ to the initial-value problem

$$(2.2)\qquad \begin{cases} \dfrac{d}{dt}u(t) + \partial\varphi(u(t)) \ni 0, & t>0, \\ u(0)=u_0 . \end{cases}$$

Here the uniqueness of solutions of (2.2) is obtained by monotonicity (2.1), which is equivalent to nonexpansion of $\{S(t);t\geq 0\}$.

(IV) The semigroup $\{S(t);t\geq 0\}$ generated by $\partial\varphi$ has a *smoothing effect property* in the sense that for each $u_0\in\overline{D(\partial\varphi)}$ one has $S(t)u_0\in D(\partial\varphi)$ for $t>0$. No other operator is known to have this property in the nonlinear case.

(V) The solution of (2.2) is the projection to H of the steepest descent on the graph $G(\varphi)\subset H\times\mathbb{R}$. Moreover

$$(2.3)\qquad \frac{d}{dt}\varphi(u(t)) = -\left\|\frac{d}{dt}u(t)\right\|^2, \quad \text{a.e. } t>0.$$

This property is easily understood in the case where φ is smooth.

We give examples in the real Hilbert space $L^2(\Omega)$ with Ω a domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$.

Example 2.1. Put

$$\varphi_1(u)=\begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx , & u\in H^1(\Omega) \ (\equiv \mathfrak{D}(\varphi_1)) , \\ +\infty , & \text{otherwise} . \end{cases}$$

Then φ_1 is proper l.s.c. convex functional on $L^2(\Omega)$ and the subdifferential $\partial\varphi_1$ is as below;

$$\partial\varphi_1(u) = -\Delta u , \quad \mathfrak{D}(\partial\varphi_1) = \{u\in H^2(\Omega);\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0 \} .$$

Example 2.2. Put

$$\varphi_2(u)=\begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx , & u\in H^1_0(\Omega) \ (\equiv \mathfrak{D}(\varphi_2)) , \\ +\infty , & \text{otherwise} . \end{cases}$$

Then

$$\partial\varphi_2(u) = -\Delta u , \quad \mathfrak{D}(\partial\varphi_2) = H^1_0(\Omega) \cap H^2(\Omega) .$$

Example 2.3. Let $p\geq 2$. Put

$$\varphi_3(u)=\begin{cases} \frac{1}{p}\int_\Omega \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^p dx , & u \in \mathfrak{D}(\varphi_3) \ \text{(defined suitably)} \\ +\infty , & \text{otherwise} . \end{cases}$$

Then

$$\partial\varphi_3(u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) ,$$

$\mathfrak{D}(\partial\varphi_3)$ is defined depending on $\mathfrak{D}(\varphi_3)$.

Example 2.4. $\partial\varphi_4(u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, \nabla u) + b(u)$,

where $\{a_i\}$ and b satisfy some conditions (see [14]).

Example 2.5. Let C be a closed convex subset of a Hilbert space H. Put

$$I_C(u) = \begin{cases} 0 , & u \in C , \\ +\infty , & \text{otherwise} . \end{cases}$$

Then

$$\partial I_C(u) = \begin{cases} \{0\} , & u \in \operatorname{Int} C , \\ \{\xi \in H;\ (\xi, v-u) \geq 0 \ \text{ for each } \ v \in C\} , & u \in \overline{C} \setminus \operatorname{Int} C , \end{cases}$$

$$\mathfrak{D}(\partial I_C) = C .$$

Example 2.5 is useful if we consider parabolic differential equations on t-dependent domains, obstacle problems or free boundary problems (e.g. [9], [16], [19]).

3. AN EVOLUTION EQUATION WITH FORCING TERM

In this section we explain results on the periodic and anti-periodic problem of the following parabolic evolution equation defined in a real Hilbert space H;

$$(3.1) \qquad \frac{d}{dt}u(t) + \partial\varphi(u(t)) \ni f(t)$$

with φ a proper l.s.c. convex functional on H, $\partial\varphi$ the subdifferential of φ and

(3.2) $f \in L^2_{loc}(\mathbb{R};H)$ and f is τ-periodic.

Here the regularity of f in (3.2) is assumed only in order to get the existence of a solution to the initial-valued problem (3.1) with any initial-value $u_0 \in \overline{D(\partial\varphi)}$.

3.1. ON THE PERIODIC PROBLEM The following results are known;

(A) (**sufficient condition**) Suppose that $\partial\varphi$ is *coercive*, i.e.,

(3.3) $$\lim_{R\to\infty} \inf_{\|x\|\geq R} \frac{(\partial\varphi(x),x)}{\|x\|} = +\infty ,$$

or eqivalently

$$\Re(\partial\varphi) = H .$$

Then there is a τ-periodic solution to (3.1).

(B) (**sufficient condition**, Haraux [6]) Suppose

(3.4) $$\frac{1}{\tau}\int_0^\tau f(t)dt \in \operatorname{Int} \Re(\partial\varphi) .$$

Then there is a τ-periodic solution to (3.1).

(C) ([6]) In the case where $\partial\varphi$ is linear, hence, by definition, $\partial\varphi$ is a nonnegative self-adjoint operator, the relation

$$\frac{1}{\tau}\int_0^\tau f(t)dt \in \Re(\partial\varphi)$$

holds **if** and **only if** (3.1) has a τ-periodic solution.

(D) (**necessary condition**, [6]) Suppose that there is a τ-periodic solution to (3.1). Then

(3.5) $$\frac{1}{\tau}\int_0^\tau f(t)dt \in \overline{\Re(\partial\varphi)} .$$

In fact, if there is a τ-periodic solution u to (3.1), then integrating (3.1) over $[0,\tau]$ and next dividing by τ one has

$$\frac{1}{\tau}\int_0^\tau f(t)dt \in \frac{1}{\tau}\int_0^\tau \partial\varphi(u(t))dt \subset \overline{\mathrm{conv}\ \Re(\partial\varphi)} = \overline{\Re(\partial\varphi)} \ .$$

Here we noted that $\overline{\Re(\partial\varphi)}$ is convex in H.

(E) (uniqueness) If $\partial\varphi$ is strictly monotone, or equivalently, φ is strictly convex, then the number of periodic solutions to (3.1) is one or less. In fact, for any two solutions u and v, one has

$$\frac{d}{dt}\|u(t)-v(t)\|^2 = 2(-\partial\varphi(u(t))+\partial\varphi(v(t)),\ u(t)-v(t)) < 0 \qquad \text{a.e.}\ t$$

(F) (Baillon and Haraux [2]) Let u and v be *periodic* solutions to (3.1). Then

$$u(t)-v(t) = \text{const.} \in H \ , \qquad t\in\mathbb{R} \ .$$

3.2. REMARKS Before relating results on the anti-periodic problem (3.1), we give some remarks about Examples 2.1-2.4: Suppose that Ω is *unbounded* in $\mathbb{R}^n$. Then, for any $\partial\varphi$ of Examples 2.1-2.3 and Example 2.4 with $b=0$, one has

(3.6) Int $\Re(\partial\varphi)$ is empty,

(3.7) The level set $C(\lambda)\equiv\{u\in H;\ \varphi(u)\le\lambda\}$ is *not* compact for any $\lambda > \min\varphi$, or equivalently, the semigroup $\{S(t)\}$ generated by $\partial\varphi$ is *not* completely continuous.

On the other hand, if Ω is *bounded*, then the properties which fail in

(3.6) and (3.7) are satisfied for many $\partial\varphi$ in Examples 2.1-2.4.

By (3.6), if Ω is *unbounded*, we can not use condition (3.4) to get the existence of periodic solution (3.1).

Property (3.7) concerns the asymptotic strong convergence of $S(t)x$ for $x\in\overline{\mathfrak{D}(\partial\varphi)}$ in the following fact; If $C(\lambda)$ is compact for $\lambda > \min \varphi$, then

(3.8) $S(t)x$ converges strongly to a minimum point of φ as $t\to+\infty$.

The following condition (3.9) is also known to be sufficient for the convergence (3.8) ([5], [18]);

(3.9) $\exists\varepsilon>0;\ \varphi(-\varepsilon x) \le \varphi(x)$ holds for $x\in\mathfrak{D}(\varphi)$.

In particular, the case of $\varepsilon=1$ in (3.9) is the *evenness* of φ, or equivalently, the *oddness* of $\partial\varphi$. Clearly, all $\partial\varphi$ of Examples 2.1-2.4 are able to satisfy (3.9) independently of whether or not Ω is bounded or unbounded.

3.3. THE ANTI-PERIODIC PROBLEM Our results are the following ([13]);

THEOREM 3.1. *Suppose*

(3.10) $\partial\varphi$ *is odd, or equivalently,* φ *is even,*

(3.11) f *is* $\frac{\tau}{2}$ *-anti-periodic;* $f(t+\frac{\tau}{2}) = -f(t)$, $t\in\mathbb{R}$.

Then equation (3.1) *has a unique* $\frac{\tau}{2}$ *-anti-periodic solution.*

COROLLARY 3.2. *Under* (3.10) *and* (3.11), *equation* (3.1) *has a*

τ-periodic solution.

Assumptions (3.10) and (3.11) together yield the anti-periodicity condition

$$\{\partial\varphi + f(t+\frac{\tau}{2})\}(-x) = -\{\partial\varphi + f(t)\}(x), \qquad x\in\mathfrak{D}(\partial\varphi).$$

Hence it seems to be reasonable to assume (3.10) and (3.11) in the anti-periodic problem.

Now we shall view the conditions (3.10) and (3.11) from Corollary 3.2.

First we verify the necessary condition (3.5) in (D) under (3.10) and (3.11). In fact, by (3.10)

$$(3.12) \qquad 0 \in \partial\varphi(0) \subset \Re(\partial\varphi) \ .$$

On the other hand, (3.11) yields

$$(3.13) \qquad \frac{1}{\tau}\int_0^\tau f(t)dt = 0 \ .$$

Hence (3.5) holds.

Relation (3.12) holds under the generalized evenness condition (3.9), since 0 is a minimum point of φ. Therefore one might expect to generalize (3.10) into (3.9) in Corollary 3.2. But we have;

PROPOSITION 3.3. *Let* $\dim H = +\infty$. *Then there is a proper l.s.c.*

convex functional φ *on* H *and* $f\in L^2_{loc}(\mathbb{R};H)$ *such that* (3.9) *and* (3.11) *hold and there is no periodic solution to* (3.1).

We also see that condition (3.11) can not be generalized into (3.13) in Corollary 3.2. In fact we have;

PROPOSITION 3.4. *Let* $\dim H = +\infty$. *Then there is a proper l.s.c. convex functional* φ *and a* τ*-periodic function* $f\in L^2_{loc}(\mathbb{R};H)$ *such that* (3.10) *and* (3.13) *hold and there is no periodic solution to* (3.1).

By these propositions, the anti-periodic problem seems to be reasonable in our situation.

4. FURTHER RESULTS ON THE ANTI-PERIODIC PROBLEM

In this section we see the existence of anti-periodic solutions to differential equations in a non-monotone framework ([8]) or t-dependent unbounded monotone framework ([15], [16], [17]), though the uniqueness of anti-periodic solutions and relation to periodic problem are also stated in [8].

4.1. NON-MONOTONE PARABOLIC EQUATIONS (Haraux [8]) We consider the differential system in $\mathbb{R}^n$ of the form

$$(4.1) \qquad u'(t) + \partial G(u(t)) = f(t) \ ,$$

with $G\in W^{2,\infty}_{loc}(\mathbb{R}^n)$, $\partial G:\mathbb{R}^n\to\mathbb{R}^n$ the gradient of G and $f\in L^2_{loc}(\mathbb{R};\mathbb{R}^n)$. Here u' denotes $(d/dt)u$ and we put $H=\mathbb{R}^n$. We assume

(4.2) (anti-periodicity) $f(t+\frac{\tau}{2}) = -f(t)$, $t\in\mathbb{R}$,

(4.3) (oddness) $\partial G(-z) = -\partial G(z)$, $z\in\mathbb{R}^n$.

We also consider the evolution equation

(4.4) $$u'(t) + \partial\varphi(u(t)) - \lambda u(t) \ni f(t)$$

defined in a real Hilbert space H, where $\lambda>0$, $\partial\varphi$ is the subdifferential of a proper l.s.c. convex functional $\varphi:H \to \mathbb{R}\cup\{+\infty\}$ and $f\in L^2_{loc}(\mathbb{R};H)$. We assume (4.2) and

(4.5) (oddness) $\partial\varphi(-z) = -\partial\varphi(z)$, $z\in H$.

The following results are obtained;

THEOREM 4.1. (i) *If* $u\in W^{1,2}_{loc}(\mathbb{R};H)$ *is a* $\frac{\tau}{2}$ *-anti-periodic solution to* (4.1) (or (4.4)), *then*

(4.6) $$\|u'\|_{L^2(0,\tau/2;H)} \le \|f\|_{L^2(0,\tau/2;H)} ,$$

(4.7) $$\|u\|_{L^\infty(0,\tau/2;H)} \le \sqrt{\tau/2}\ \|f\|_{L^2(0,\tau/2;H)} .$$

(ii) *There is a* $\frac{\tau}{2}$ *-anti-periodic solution* $u\in W^{1,2}_{loc}(\mathbb{R};H)$ *to* (4.1) (or (4.4)). *Here, in the case of* (4.4), *we assume that*

(4.8) *For each* $c>0$ *the set* $E_c=\{z\in D(\varphi);\ \|z\|\le c,\ \varphi(z)\le c\}$ *is compact.*

Proof of (i) Let u be a $\frac{\tau}{2}$ -anti-periodic solution to (4.1). Then multiplying (4.1) by u' and integrating over $(0, \frac{\tau}{2})$ we have

$$\int_0^{\tau/2} \|u'\|^2 dt + G(u(\tau/2)) - G(u(0)) = \int_0^{\tau/2} (f(t), u'(t))dt .$$

Since $u(\tau/2) = -u(0)$ and G is even, we deduce

$$\int_0^{\tau/2} \|u'\|^2 dt = \int_0^{\tau/2} (f, u')dt \leq \{\int_0^{\tau/2} \|f\|^2 dt\}^{1/2} \{\int_0^{\tau/2} \|u'\|^2 dt\}^{1/2}$$

or (4.6).

In case of considering (4.4), put

$$G(z) = \varphi(z) - \frac{\lambda}{2} \|z\|^2, \qquad z \in \mathbb{D}(\varphi).$$

Then, by (2.3), we get (4.6) in the same way.

To verify (4.7), let $t \in (0, \tau/2)$. Then by (4.6)

$$\begin{aligned}\|u(t)\| &= \|u(t) - \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} u(s)ds\| \leq \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} \|u(t) - u(s)\| ds \\ &\leq \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} \sqrt{\tau/2}\, \|f\|_{L^2(0,\tau/2;H)} ds \leq \sqrt{\tau/2}\, \|f\|_{L^2(0,\tau/2;H)}.\end{aligned}$$

Hence (4.7) holds.

Outline of Proof of (ii) For any constant $c > 0$, we can find an auxiliary potential $G_1 \in W^{1,\infty}(\mathbb{R}^n)$ to (4.1) satisfying

(4.9) $\quad G_1(z) = G(z)$ if $\|z\| \leq c$, (4.10) For any $z_0 \in H$ there is a solution $v(.,z_0) \in W^{1,2}_{loc}(0,\infty;H)$ to

$$v' + \partial G_1(v) = f(t),\ t \geq 0, \qquad v(0)=z_0\ ,$$

(4.11) There is a constant $P>0$ such that $\|v(\tau/2;z_0)\| \leq P$ holds whenever $\|z_0\| \leq P$.

By Brouwer's fixed point theorem, the map $T:\{\|z\| \leq P\} \to \{\|z\| \leq P\}$ defined by $Tz = -v(\tau/2;z)$ has a fixed point z_1 in the closed ball $\{\|z\| \leq P\}$, or equivalently, $v(.,z_1)$ is a $\tau/2$-anti-periodic solution to $v' + \partial G_1(v) = f(t)$.

Since $v(.,z_1)$ satisfies (4.7), putting $c \geq \sqrt{\tau/2}\ \|f\|_{L^2(0,\tau/2;H)}$ and noting (4.9), we get the existence of $\tau/2$-anti-periodic solution to (4.1)

In case of considering (4.4), using Schauder's fixed point theorem, we get the existence in a similar way.

COROLLARY 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, g an odd nondecreasing (continuous) function on $\mathbb{R}$ and $\lambda \geq 0$. For each $f \in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfying $f(t+\tau/2,\ .) = -f(t,.)$* a.e.$t \in \mathbb{R}$, *there is a solution $u \in L^\infty(\mathbb{R};H^1_0(\Omega)) \cap W^{1,2}_{loc}(\mathbb{R};L^2(\Omega))$ to*

(4.12) $u_t - \Delta u + g(u) - \lambda u = f(t,x), \quad (t,x) \in \mathbb{R}\times\Omega,$

(4.13) $u(t+\tau/2,\ .) = -u(t,.), \quad t \in \mathbb{R}.$

Remarks 4.3. (i) Corollary 4.2 is of interest when g is sublinear at infinity. Because, if g is superlinear, then the existence of a τ-periodic solution is obtained under τ-periodicity of f. Moreover the existence of $\tau/2$-anti-periodic solution is obtained by applying Schauder's fixed point theorem directly under the $\tau/2$-anti-periodicity of f.

(ii) In some cases of "bad" nonlinearities producing blow-up phenomona, we can also show the existence of anti-periodic solutions. (See [8; Theorem 2.4 and Corollary 2.6].)

4.2. NON-MONOTONE HYPERBOLIC EQUATIONS ([8]) We consider the differential system

$$(4.14) \qquad u''(t) + B(u'(t)) + \partial G(u(t)) = f(t)$$

in $\mathbb{R}^n$, where $\partial G:\mathbb{R}^n \to \mathbb{R}^n$ is the gradient of a function $G \in W^{2,\infty}_{loc}(\mathbb{R}^n)$, $B \in W^{1,\infty}_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ is a monotone operator on $\mathbb{R}^n$ and $f \in L^2_{loc}(\mathbb{R};\mathbb{R}^n)$. We assume

(4.15) B and G are odd,

(4.16) $(Bz,z) \geq \alpha\|z\|^2 - c$, $\quad z \in \mathbb{R}^n$

for some constants $\alpha>0$, $c \geq 0$; We also consider the nonlinear wave equation

$$(4.17) \qquad u_{tt} - \Delta u + g(u) + \beta(u_t) \ni f(t,x)$$

defined in $L^2(\Omega)$ with Ω a *bounded* domain of $\mathbb{R}^n$, $\beta:\mathbb{R}\to\mathbb{R}$ a maximal monotone operator and $f \in L^2_{loc}(\mathbb{R};L^2(\Omega))$. We assume (4.15) and (4.16) in respect of β and g and some conditon on g (see [8]).

The following result is obtained:

THEOREM 4.4. *For any* $\tau/2$*-anti-periodic* $f\in L^2_{loc}(\mathbb{R};H)$, *there exists a* $\tau/2$*-anti-periodic solution* u *to* (4.14) (*or* $\tau/2$*-anti-periodic weak solution* $u\in C(\mathbb{R};H^1_0(\Omega))\cap C^1(\mathbb{R};L^2(\Omega))$ *to* (4.17)).

Remark 4.5. Put $g=0$ and $\beta(v)=c\|v\|^{p-1}v$, $p>1$, $c>0$, in (4.17). Then, in case $n\geq 3$ and $p>(n+2)/(n-2)$, the existence of τ-periodic solutions in the natural class $C(\mathbb{R};H^1_0(\Omega))\cap C^1(\mathbb{R};L^2(\Omega))$ seems to be unknown for general τ-periodic forcing terms $f\in L^2(\mathbb{R};L^2(\Omega))$. (See [8;section 4] and its references.)

4.3. PARABOLIC EQUATIONS WITH T-DEPENDENT UNBOUNDED MONOTONE TERMS

In [15] and [16] (see also [17]), we obtain the existence of anti-periodic solutions to the parabolic evolution equation

$$\frac{d}{dt}u(t) + \partial\varphi(u(t)) + F(t)u(t) \ni 0$$

defined in a real Hilbert space, where $\partial\varphi$ is an *odd* subdifferential operator and $\{F(t);t\in\mathbb{R}\}$ is a family of *monotone* operators satisfying the anti-periodicity condition

$$\begin{aligned} &\mathfrak{D}(F(t+\tau/2)) = -\mathfrak{D}(F(t)), \quad t\in\mathbb{R}\ , \\ &F(t+\tau/2)(-z) = -F(t)z, \quad z\in\mathfrak{D}(F(t)),\ t\in\mathbb{R} \end{aligned}$$

and some conditions (see [15], [16]). We do not assume any compactness of φ or $F(t)$ in the sense of the strong topology of H.

Remark 4.6. The essential reason for assuming the monotonicity of $F(t)$, $t\in\mathbb{R}$, is to apply Browder's fixed point theorem, in which compactness in the strong topology is not assumed, but nonexpansion of the solutions is needed. On the other hand, in Theorem 4.1, the compactness (4.8) is assumed only for applying Schauder's fixed point theorem.

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