LEAST GRADIENT PROBLEMS

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0. INTRODUCTION

Suppose $\Omega \subseteq \mathbb{R}^n, n \geq 2$, is a given domain with Lipschitz boundary $\partial\Omega$ and ϕ is a given continuous function defined on $\partial\Omega$. Consider the problem of minimizing

$$\int_\Omega |\nabla u|\, dx$$

amongst all functions u defined on Ω such that $u = \phi$ on $\partial\Omega$.

Even if $\partial\Omega$ and ϕ are smooth, solutions to this problem are normally only Lipschitz continuous. If ϕ is not smooth then typically the solution will be no more regular than ϕ even in the interior of Ω. If ϕ is only continuous we cannot expect the solution to be differentiable. The natural class of functions in which to consider the above problem is $\mathrm{BV}(\Omega)$, the set of functions of Bounded Variation. (For the precise definition of $\mathrm{BV}(\Omega)$ and some properties see [G].)

If $n = 1$ then a little thought soon provides all the solutions (either all decreasing or all increasing functions satisfying the boundary values). The uniqueness and regularity results we discuss later in this report do not hold and so we will assume always that $n \geq 2$.

Solutions to the above problem are known as functions of *Least Gradient* and were used by Bombieri, De Giorgi and Giusti [BDG] to prove some interesting results about minimal surfaces. We first describe the connection between functions of least gradient and minimal surfaces.

Definition: If E is a measurable set in $\mathbb{R}^n$ we define the *Perimeter* of E by

$$P(E) = \sup\left\{\int_E \operatorname{div} g\, dx : g = (g_1, g_2, \ldots, g_n),\ |g(x)| \leq 1,\ g_i \in C_0^1(\mathbb{R}^n)\right\}.$$

It is easily seen that if E has a smooth boundary then $P(E)$ equals the (n-1)-dimensional measure of $\partial\Omega$. (See [G] for this result and many other properties of the Perimeter.)

The result providing the link we are looking for is the *co-area formula.*

THEOREM. ([FR],[G,p6]) *Suppose* $u \in \mathrm{BV}(\mathbb{R}^n)$ *and for each* $t \in \mathbb{R}$ *let*

$$A_t = \{x \in \mathbb{R}^n \,:\, u(x) \geq t\}.$$

Then

$$\int_{\mathbb{R}^n} |\nabla u| \, dx = \int_{-\infty}^{\infty} P(A_t) \, dt.$$

From this formula we might expect that if u has least gradient then each A_t should minimize perimeter and this is exactly what happens.

THEOREM. ([BDG]) *Suppose* u *is a function of least gradient in* Ω *then for each* $t \in \mathbb{R}$, $A_t = \{x \in \Omega \,:\, u(x) \geq t\}$ *is a set of least perimeter.*
That is, if E *is any subset of* Ω *such that* E *and* A_t *agree outside some compact subset of* Ω *then* $P(A_t) \leq P(E)$.

By constructing functions of least gradient, Bombieri, De Giorgi and Giusti were able to show that the surface defined by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2$ in $\mathbb{R}^8$ is a minimal surface with a singularity at the origin. It was already known that in $\mathbb{R}^n$, for $n \leq 7$, minimal surfaces cannot have singularities ([S]).

More recently Parks ([P1],[P2]) proposed a method for the numerical calculation of minimal surfaces based on the construction of functions of least gradient.

1. THE CONSTRAINED PROBLEM

We now consider a new problem with the additional constraint that $|\nabla u| \leq 1$ a.e. in Ω. This was considered by Kohn and Strang [KS] and we first, briefly, describe an application given by them.

Consider a long bar, with a given constant exterior cross-section Ω, which is subjected to a given load having no vertical component and being constant up and down the rod. These assumptions mean that the problem can be considered as a two dimensional one with stress having no vertical component. Additionally, elastic-plastic theory puts on a yield condition. If the stress exceeds a given constant (depending only on the material and which we take to be 1) then the bar yields plastically. In the steady state this has the effect of assuming that the stress never exceeds 1 in magnitude. If we assume the bar is solid then, by minimizing energy, we can find the stress in the bar. (There are many papers written on this very subject; for example see [T].) However we shall not assume the bar is solid and instead consider different (constant) cross-sections obtained by removing holes from the original cross-section Ω. Of course if we remove too much then it may no longer be possible to find a stress of magnitude smaller than 1 which still supports the given load. Our problem then is to find, if possible, a cross-section of smallest area for which there is a stress which will support the load. That is we look for the lightest weight bar which will support the load.

Rather than vary the cross-section, we instead keep this fixed as Ω and then consider all possible stresses in Ω, noting that where they are zero we may remove material. In this 2 dimensional setting stresses can always be represented by single real valued functions. A function u gives a stress $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_1 = \frac{\partial u}{\partial y}$ and $\sigma_2 = -\frac{\partial u}{\partial x}$. The condition of prescribed load becomes $u = f$ on $\partial\Omega$ and the plastic yield condition becomes $|\nabla u| \le 1$. Thus if we define $w : [0,1] \to \mathbb{R}$ by

$$w(t) = \begin{cases} 0, & \text{if } t = 0, \\ 1, & \text{if } t \neq 0. \end{cases}$$

our problem is

$$(\mathcal{P}) \qquad \text{Minimize} \qquad \left\{ \int_\Omega w(|\nabla u|)\, dx \;\middle|\; u = f \text{ on } \partial\Omega\,,\ |\nabla u| \le 1 \text{ a.e. in } \Omega \right\}.$$

Unfortunately this problem will typically not have a solution. Minimizing sequences of cross-sections have more and more holes giving increasing total area of hole but the area of the individual holes decreases to zero so that they disappear in the limit. Mathematically this phenomenon is well known and occurs because the integrand is non-convex and so the integral is not lower semi-continuous with respect to weak convergence. To overcome this difficulty we consider a new problem where w is replaced by its convexification $\widetilde{w}$ (the largest convex function smaller than w). In this case $\widetilde{w}(t) = t$ and so we consider

$$(\widetilde{\mathcal{P}}) \qquad \text{Minimize} \qquad \left\{ \int_\Omega |\nabla u|\, dx \;\middle|\; u = f \text{ on } \partial\Omega\,,\ |\nabla u| \leq 1 \text{ a.e. in } \Omega \right\}.$$

Provided the set is non-empty, $(\widetilde{\mathcal{P}})$ will have a solution, minimizing sequences for $(\mathcal{P})$ will be minimizing sequences for $(\widetilde{\mathcal{P}})$ and the minimum attained in $(\widetilde{\mathcal{P}})$ equals the infimum (perhaps not attained) in $(\mathcal{P})$.

In the particular application above, information about the solution u of $(\widetilde{\mathcal{P}})$ gives good information about functions giving near to optimal results in $(\mathcal{P})$, that is about cross-sections having almost lightest weight. In regions where $|\nabla u| = 0$ cross-sections should have a hole. In regions where $|\nabla u| = 1$ cross-sections should be solid. In regions where $0 < |\nabla u| < 1$ cross-sections should have holes with average density about $|\nabla u|$ and the material should be in the form of fibres aligned along the level sets of u.

2. CHARACTERIZATION OF SOLUTIONS

It is easy to show that, provided there is at least one function v satisfying $v = f$ on $\partial\Omega$ and $|\nabla v| \leq 1$ a.e. in Ω, there is always a solution of $(\widetilde{\mathcal{P}})$. Solutions are automatically Lipschitz continuous with a Lipschitz constant depending only on the geometry of Ω. Simple examples show that in general Lipschitz continuity is the best that can be expected. Finally, given f and Ω there is at most one solution. This uniqueness result is not trivial to prove because the integrand is only convex and not strictly convex and the proof involves a consideration of level sets (see below).

It can be seen from the above that level sets of the solution are important and we would like a characterization of them similar to that given for functions of least gradient without the constraint $|\nabla u| \leq 1$. The co-area formula still applies and so we expect that level sets should minimize perimeter in some class of sets. We need to show how to build in the gradient constraint on u when looking at the level sets of u. Kohn and Strang gave the idea for this.

If we assume that Ω is convex then

$$(1) \qquad |\nabla v| \leq 1 \text{ a.e. in } \Omega \qquad \Longleftrightarrow \qquad |v(x) - v(y)| \leq |x - y| \text{ for all } x, y \in \Omega.$$

If Ω is not convex then the same equivalence is true provided we replace $|x-y|$ by the length of the shortest path in Ω which joins x and y. With this change what follows holds for non-convex Ω but for simplicity we restrict ourselves to the convex case.

It is possible to interpret (1) in terms of the level sets of v but the interpretation is complicated and the condition on the level set A_t involves all the other level sets A_s. We prefer a condition independent of the other sets A_s. The main idea of Kohn and Strang was to attempt to satisfy (1) only when $x \in \partial\Omega$ (and $y \in \Omega$) and hope that for solutions it will then automatically hold for all $x \in \Omega$ and not just $x \in \partial\Omega$. The major part of [SWZ1] is devoted to proving this.

Suppose Ω is convex and v satisfies

(i) $v = f \qquad \text{on } \partial\Omega$,

(ii) $|v(x) - v(y)| \leq |x - y| \qquad \text{for } x \in \partial\Omega,\, y \in \Omega.$

Let $A_t = \{x \in \Omega \mid v(x) \geq t\}$. If $p \in \partial\Omega$ then

either (I) $f(p) < t$, or (II) $f(p) \geq t$.

In the case (I), if $x \in \Omega$ and $|p - x| < t - f(p)$ then

$$v(x) - f(p) = v(x) - v(p) \leq |p - x| < t - f(p).$$

That is $v(x) < t$ and so $x \notin A_t$.

In the case (II), if $x \in \Omega$ and $|p - x| \leq f(p) - t$ then

$$f(p) - v(x) = v(p) - v(x) \leq |p - x| \leq f(p) - t.$$

That is $v(x) \geq t$ and so $x \in A_t$.

Thus if we define

$$L_t = \{x \in \Omega \mid \exists p \in \partial\Omega,\, |p - x| \leq f(p) - t\},$$

$$M_t = \{x \in \Omega \mid \exists p \in \partial\Omega,\, |p - x| < t - f(p)\}$$

we have

$$v \textit{ satisfies } \text{(i)} \textit{ and } \text{(ii)} \quad \Rightarrow \quad L_t \subseteq A_t \textit{ and } A_t \cap M_t = \emptyset \textit{ for each } t.$$

(In fact the converse also holds.)

For each t consider the problem

$$\text{(2)} \qquad \text{Minimize} \qquad \{\, P(E) \;\big|\; L_t \subseteq E \text{ and } E \cap M_t = \emptyset \,\}.$$

It is possible to show that this problem always has a solution and may even have more than one. To obtain a unique set we now look at

$$\text{(3)} \qquad \text{Maximize} \qquad \{\, |E| \;\big|\; E \text{ solves (2)} \,\}.$$

This always has a unique solution which we denote by $\mathcal{E}_t$. Now define, for $x \in \Omega$

$$u^*(x) = \sup\{\, t \;\big|\; x \in \mathcal{E}_t \,\}.$$

THEOREM. ([SWZ1]) *u^* is the unique solution to $(\widetilde{\mathcal{P}})$.*

Thus we have characterized the level sets of the solution as the sets solving (2) and (3) above. In the proof of the Theorem we can easily show that $u^* = f$ on $\partial\Omega$ and $|u^*(x) - u^*(y)| \leq |x - y|$ for $x \in \partial\Omega$ and $y \in \Omega$. It is necessary to show this last inequality holds also for $x \in \Omega$, in which case the co-area formula implies we have the required solution.

If x and y are in Ω then there are numbers s and t such that $u^*(x) = s$ and $u^*(y) = t$. Further, possibly after shifting x and y slightly, we may assume that $x \in \partial\mathcal{E}_s$ and $y \in \partial\mathcal{E}_t$. We need to show $|s - t| \leq |x - y|$. Since $|x - y| \geq \operatorname{dist}(\partial\mathcal{E}_s, \partial\mathcal{E}_t)$ it is sufficient to show

$$\operatorname{dist}(\partial\mathcal{E}_s, \partial\mathcal{E}_t) \geq |s - t| \quad \text{for all } s, t.$$

This now is a result about minimal surfaces and can be proved using techniques from minimal surface theory.

3. THE UNCONSTRAINED PROBLEM

We now return to the unconstrained problem of finding functions of least gradient. If we admit generalised solutions in $\mathrm{BV}(\Omega)$ as is done for the minimal surface equation

([G,§14]) then it is easy to prove the existence of solutions. Such solutions need not satisfy the boundary values, need not be unique and need not be any more regular than being in $\mathrm{BV}(\Omega)$. We would like conditions which ensure that we can find a unique continuous solution satisfying the boundary conditions.

Parks ([P1],[P2]) showed that if Ω is strictly convex and the boundary data ϕ satisfies a Bounded Slope Condition then there is a Lipschitz continuous function of least gradient with boundary values ϕ. Further, under some additional technical conditions which are normally satisfied, the solution is unique. His method was to approximate $\int |\nabla v|\,dx$ by $\int \sqrt{\varepsilon^2 + |\nabla v|^2}\,dx$, and then, using the Bounded Slope Condition, obtain uniform gradient estimates independent of ε.

The idea in [SWZ2] is to construct the required solution via its level sets as was done in the constrained problem. The process is even simpler now since there is no necessity for the sets L_t and M_t. However, instead, we must try to ensure the solution satisfies the boundary values.

Given ϕ continuous on $\partial\Omega$ construct a new function g, continuous on $\mathbb{R}^n$, such that $g = \phi$ on $\partial\Omega$ and then set

$$G_t = \{\, x \in \mathbb{R}^n \mid g(x) \geq t \,\}.$$

Now consider the problem

$$\text{(4)} \qquad \text{Minimize} \qquad \{\, P(E) \mid E - \Omega = G_t - \Omega \,\}.$$

There may be more than one solution and so let $\mathcal{E}_t$ be the solution to

$$\text{(5)} \qquad \text{Maximize} \qquad \{\, |E| \mid E \text{ solves (4)} \,\}$$

and define

$$u(x) = \sup \{\, t \mid x \in \mathcal{E}_t \,\}.$$

We want to know that u is continuous and satisfies the boundary data. If this is the case then the co-area formula says u is the required solution. However it is not hard to see that this will not always happen. For example, take Ω to be the square $[0,1] \times [0,1]$ in $\mathbb{R}^2$ and ϕ to be zero on three sides and non-negative (but not identically zero) on the bottom. Suppose also, for simplicity, that ϕ has a "bell-shaped" curve along the bottom (actually any curve will do). For each $t > 0$ the portion of $\mathcal{E}_t$ inside $\overline{\Omega}$ will be a subinterval of the bottom side, but if $t \leq 0$, $\mathcal{E}_t$ will contain the whole of Ω. Consequently the function u satisfies the boundary data but is zero inside Ω and so is not continuous up to $\partial\Omega$. To avoid the difficulty presented in this example we need a condition that will ensure $\partial\mathcal{E}_t$ does not lie along $\partial\Omega$. The precise condition is given by the next Theorem.

THEOREM. ([SWZ2]) *If $\partial\Omega$ is C^2 and has strictly positive mean curvature in a dense subset of $\partial\Omega$, then, for any continuous function ϕ defined on $\partial\Omega$, there is a unique continuous function of least gradient in Ω having boundary values ϕ.*
Further, if the condition on $\partial\Omega$ fails there is smooth boundary data, arbitrarily small in any C^k norm, for which there is no continuous solution.

If $\partial\Omega$ is not C^2 but only Lipschitz continuous then we can also give geometric conditions on $\partial\Omega$ which are necessary and sufficient for existence of continuous solutions. (Of course these conditions coincide with the one above when $\partial\Omega$ is C^2.)

The final question we consider in this report is the one of regularity of the solution u. In $\mathbb{R}^2$ the level sets of functions of least gradient are straight lines. If we take any continuous function of $(x,y) \in \mathbb{R}^2$ which is independent of x then its level sets will be straight (horizontal) lines and so it will be a function of least gradient on any set. Hence, generally, solutions can be no more regular in the interior than they are on the boundary. Indeed they may be even worse.

Suppose Ω is the circle radius 1 and centre $(0,1)$ in $\mathbb{R}^2$. Take boundary values ϕ on $\partial\Omega$ so that $\phi(x,y) = |x|$ for $y < 1$ and 1 otherwise. Then the least gradient solution will be $u(x,y) = \sqrt{y}$ for $y < 1$ and 1 otherwise. Thus even though the boundary data is Lipschitz continuous the solution itself is only Hölder continuous with exponent $\frac{1}{2}$.

However this example illustrates the worst that can happen.

THEOREM. ([SWZ2]) *Suppose $\partial\Omega$ has strictly positive mean curvature (with respect to the inner unit normal) and $\phi \in C^{0,\alpha}(\partial\Omega)$ for some α, $0 < \alpha \leq 1$. Then $u \in C^{0,\alpha/2}(\overline{\Omega})$.*

Examples as above show that these results are best possible. Other results allow the mean curvature of $\partial\Omega$ to be 0 at some points but grow like a power away from these points and again optimal regularity is given. If we assume $\phi \in C^{1,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$, and strictly positive mean curvature then $u \in C^{0,(1+\alpha)/2}(\overline{\Omega})$. For any regularity of ϕ higher than $C^{1,1}(\partial\Omega)$ we can only expect to obtain the $C^{0,1}(\overline{\Omega})$ regularity of u given by the last result.

Finally we mention that in some subsets of Ω the regularity of the solution can be improved beyond that mentioned above.

(i) [PZ] If $2 \leq n \leq 8$, $\partial\Omega$ is C^{n-1}, ϕ is C^{n-1} and u is a Lipschitz continuous function of least gradient, then u is C^{n-3} on an open dense set of Ω.
In fact Parks and Ziemer prove that if $|\nabla u(x_0)| \neq 0$ and $u(x_0) = t$ then u is C^{n-3} and $|\nabla u(x)| \neq 0$ in a neighbourhood of the component of $\partial\mathcal{E}_t = \{x \mid u(x) = t\}$ which contains x_0.

(ii) If $n = 2$ the level sets of u are straight lines and so interior regularity may be easily inferred from boundary regularity. Thus, in the case that $\phi \in C^{0,\alpha}(\partial\Omega)$ we have from above that $u \in C^{0,\alpha/2}(\overline{\Omega})$. However in the example given u fails to be in $C^{0,\alpha}$ only on approach to $\partial\Omega$. If we consider points x and y in the interior of Ω and consider the level sets of u which pass through x and y then because Ω is convex these level sets (straight lines) must meet $\partial\Omega$ at a positive angle which only depends on the distance of x and y from $\partial\Omega$. Thus the behaviour of u at x must be proportional to the behaviour of u at the boundary.
If $n = 2$, Ω is convex and $\phi \in C^{0,\alpha}(\partial\Omega)$ then $u \in C^{0,\alpha}(\Omega) \bigcap C^{0,\alpha/2}(\overline{\Omega})$.
For $n > 2$ the same argument cannot be applied as level sets, while minimal sufaces, need not be straight lines or planes and even if they come from the interior of Ω they may still come into $\partial\Omega$ in a tangential way.

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