ON KAKUTANI'S CRITERION AND SHIRYAEV'S THEOREM

A.H. Dooley

Kakutani's classical "dichotomy" result gives a criterion for when two product measures $\mu = \bigotimes_{i=1}^{\infty} \alpha_i$ and $\nu = \bigotimes_{i=1}^{\infty} \beta_i$ on the infinite product space $\prod_{i=1}^{\infty} \mathbb{Z}_2$ are absolutely continuous. Here, for each i, α_i and β_i are probability measures on the two-point space $\mathbb{Z}_2$. In fact, it turns out that either $\mu \prec \nu$, in the case where $\sum |\alpha_i(0) - \beta_i(0)|^2 < \infty$, or else $\mu \perp \nu$. (See [4]).

Similar results were obtained by Brown and Moran [3] and Peyrière [5] for Riesz products on the circle $\mathbb{T}$. They showed that if

$$\mu = w^*\text{-}\lim \prod_{i=1}^{k} (1 + a_i \cos(3^i t + \phi_i))dt$$

and

$$\nu = w^*\text{-}\lim \prod_{i=1}^{k} (1 + b_i \cos(3^i t + \psi_i))dt$$

with $a_i, b_i \in (-1, 1)$,

then $\mu \sim \nu \quad iff \quad \sum |a_i e^{i\phi} - b_i e^{i\psi}|^2 < \infty$, and otherwise $\mu \perp \nu$.

A far-reaching generalization of Kakutani's theorem is discussed in [7], where we consider a measurable space $(\Omega, \mathcal{C})$ equipped with a non-decreasing family $(\mathcal{C}_n)_{n \geq 0}$ of σ-algebras such that $\mathcal{C} = \vee_n \mathcal{C}_n$.

Suppose that μ and ν are two probability measures on $(\Omega, \mathcal{C})$ such that their re-

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strictions $\mu_{(n)}$ and $\nu_{(n)}$ to $\mathcal{C}_n$ satisfy $\mu_{(n)} \prec\prec \nu_{(n)}$, and denote $\frac{d\nu_{(n)}}{d\mu_{(n)}}$ by α_n. Then

$$\mu \prec\prec \nu \Longleftrightarrow \nu\{\sum_{n=1}^{\infty} E_\nu(1-\sqrt{\alpha_n} \mid \mathcal{C}_{n-1}) < \infty\} = 1$$

and

$$\nu \perp \nu \Longleftrightarrow \nu\{\sum_{n=1}^{\infty} E_\nu(1-\sqrt{\alpha_n} \mid \mathcal{C}_{n-1}) = \infty\} = 1.$$

In order to deduce the Brown-Moran-Peyrière result from Shiryaev's theorem, one needs to prove a new version of Shiryaev for decreasing σ-algebras and reversed martingales. Such a theorem is the object of this article. Since martingale convergence is much nicer for reversed martingales, the proof of the theorem is much easier than Shiryaev's result. It turns out that the theorem is applicable to a much wider class of measures than just Riesz products, and gives a dichotomy theorem for G-measures introduced by G. Brown and the author in [1]. This theorem may be applied to certain problems in wavelets where one wishes to multiply mother functions and father functions in an infinite product and to decide which products formed in this way are absolutely continuous with respect to Lebesgue measure.

For the purposes of the exposition, I have limited myself to the infinite product space $X = \prod_{i=1}^{\infty} \mathbb{Z}_2$. In general, one may allow the spaces to be any compact metric space. Details of proofs will be given in [2]. All this material is joint work with Gavin Brown.

Our reversed version of Shiryaev's theorem is proved from the following theorem, surely somewhere between "an exercise in third year analysis" and "a folklore result known to the experts".

Let $(X, \mathcal{C}, \mu)$ be an arbitrary probability space, $(\mathcal{C}^n)_{n \geq 0}$ a decreasing family of sub-σ-algebras of $\mathcal{C}$. Let $\{\beta_n\}$ be a sequence of nonnegative measurable functions on X, such that β_n is $\mathcal{C}^n$ measurable and $E_\mu(\beta_n \mid \mathcal{C}^{n+1}) = 1$. Let $y_n = E_\mu(\beta_n^{1/2} \mid \mathcal{C}^{n+1})$. The problem is to analyse $\prod_{k=1}^{\infty} \beta_k^{1/2}$ in terms of the behaviour of $\prod_{k=1}^{\infty} y_k$. Notice that the latter product exists; it is either zero if $\sum(1-y_k(x))=\infty$, or nonzero if $\sum(1-y_x(x))<\infty$. With this notation, we may state our folklore theorem.

THEOREM 1. *Let* $B=\{x\in X:\sum_{k=1}^{\infty}(1-y_k(x))<\infty\}$. *Then*

(i) $\prod_{k=1}^{\infty}\beta_n^{1/2}(x)$ *converges in* $L^2(\mu|_B)$ *as* $n\to\infty$

(ii) $\prod_{k=1}^{\infty}\beta_k^{1/2}(x)\to 0$ *in* $L^1(\mu|_{B^c})$ *as* $n\to\infty$.

The ingredients of the proof are the dominated convergence theorem, L^2 estimates of products, etc.

We may apply this theorem to see that if there exists a unique weak*-limit ν of $\prod_{k=1}^{n}\beta_k\mu$, then either $\nu(B)=1$, in which case $\nu\prec\prec\mu$, and $\frac{d\nu}{d\mu}=\prod_{k=1}^{\infty}\beta_k(x)$, or $\nu(B)=0$, in which case $\nu\perp\mu$.

This is our reversed Shiryaev theorem.

To see how the last result generalizes the Brown-Moran-Peyrière theorem on Riesz products, let us consider the class of uniquely ergodic G-measures [1]. Thus, let $(g_i)_{i\geq 0}$ be a sequence of nonnegative functions on $X=\prod_{i=1}^{\infty}\mathbb{Z}_2$ such that

(i) $g_i(x)$ is independent of $x_1,\ldots,x_{i-1}$.

(ii) $\frac{1}{2}(g_i(0,\ldots,0,x_{i+1},x_{i+2}\cdots)+g_i(0,\cdots,0,1,x_{i+1}x_{i+2}\cdots))=1$.

Then the probability measure μ is said to be G-**measure** if

$$\frac{d\mu}{d\mu^n}(x)=g_1(x)\cdots g_n(x)=G_n(x).$$

Here, μ^n is the "tail" measure

$$\frac{1}{2^n}\sum_{\gamma\in\mathbb{Z}_2^n}\mu\circ\gamma,$$

and G denotes the sequence (G_n).

It turns out that every probability measure is up to equivalence, a G-measure with continuous G_n's. An interesting class of G-measures consists of the **uniquely ergodic** ones, where we fix the sequence (G_n) and insist that there be a unique G-measure. It turns out that measures in this class are ergodic in the sense that measurable sets invariant under the group of finite coordinate changes on X are either null or conull. Further, μ is uniquely ergodic if and only if for all $f \in C(X)$, $\frac{1}{2^n} \sum_{\gamma \in \mathbb{Z}_2^n} f(\gamma x) G_n(\gamma x)$ approaches a constant as $n \to \infty$. The constant is then $\int f d\mu$, and μ is the unique weak*-limit of $g_1 \cdots g_n \cdot \nu^n$ for any probability measure ν. In [1], very general sufficient conditions are given on the sequence (g_i) for unique ergodicity to hold.

We expect to have dichotomy for uniquely ergodic G-measures, since ergodic measures are either singular or mutually absolutely continuous. Further, it is desirable to have a criterion for singularity in terms of the functions g_n; such a criterion could not exist except in the uniquely ergodic case.

Application of our reversed Shiryaev criterion yields the following criterion.

THEOREM 2. *Suppose μ is a uniquely ergodic G-measure associated to a sequence (g_i) and ν is a uniquely ergodic F-measure associated to a sequence (f_i) as above. Then $\mu \sim \nu$ iff the series $\sum_{n=1}^{\infty} (\sum_{\gamma \in \mathbb{Z}_2} (\sqrt{f_n(\gamma x)} - \sqrt{g_n(\gamma x)})^2)$ converges on a set of ν-positive measure. (In fact, in this case the series converges ν a.e.) Otherwise, $\mu \perp \nu$.*

This theorem immediately implies the theorem on Riesz products above. It also generalises a criterion found by G. Ritter [6] to the G-measure setting.

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School of Mathematics
University of New South Wales
P.O. Box 1
Kensington 2033.