HYPERGROUPS AND HARMONIC ANALYSIS

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0. INTRODUCTION

The modern approach to harmonic analysis on a Lie group treats the representations of the group as the central objects of study, while characters are treated as important but auxiliary objects associated to representations. This is in direct contrast to the modern approach to harmonic analysis on a finite group, which treats the determination and study of the characters of the group as the primary problem, and considers representations as important but auxiliary objects associated to characters.

For finite groups, the reason for this is three fold; 1) the determination of the irreducible characters is a vastly simpler problem than the determination of the irreducible representations 2) almost all of the standard problems of harmonic analysis may be answered solely by means of the character theory and 3) historically, the theory of characters has preceded that of the theory of representations.

This suggests the following interesting question – might it be possible to develop harmonic analysis on a Lie group as essentially a theory of characters, and thereby finesse the present difficulties and technicalities in modern representation theory? [This is not an entirely new idea – in fact Harish Chandra's pivotal work on the existence of discrete series for non-compact semi-simple groups (see for example Varadarajan [8])

takes this view.]

In this paper we show how such an approach may be initiated, and applied to various classes of groups. The first problem is of course – how does one define a character of a group if one does not know what a representation is? Our answer to this question is a variant of the one Frobenius would have given – the characters of a finite group, say, are exactly those functions on the set of conjugacy classes of the group which respect the natural algebraic structure of this set. This algebraic structure has a probabilistic nature and is called an abelian hypergroup (see [3], [5], [7]). We therefore replace the problem of non-commutative harmonic analysis on the group G with the problem of abelian harmonic analysis on the hypergroup of conjugacy classes, which we call the class hypergroup of G. This allows us to sketch a straight forward algorithm for the construction of the character table of any given finite group.

It also provides a general framework for the study of harmonic analysis on an arbitrary group.

When G is a Lie group, we show that one may expect an intimate relationship between the class hypergroup of G and the hypergroup of adjoint orbits. This provides an explanation for the general effectiveness of Kirillov theory (see Kirillov [6]) in harmonic analysis since, formally at least, the dual object of the hypergroup of adjoint orbits is the hypergroup of co-adjoint orbits, although we also see that in general Kirillov theory will provide at best a 'linear approximation' to the unitary dual of G on account of the global difference between the class hypergroup and the hypergroup of adjoint orbits.

We therefore propose a program for the determination and study of the unitary dual of a Lie group G which both incorporates and extends Kirillov theory.

1. DEFINITIONS AND BASIC STRATEGY

A finite abelian hypergroup (F.A.H.) is a set $\mathcal{C} = \{C_0, C_1, \ldots, C_n\}$ together with an associative abelian algebra structure on $\mathbb{R}\mathcal{C}$

$$(1.1) \qquad C_i C_j = \sum_k n_{ij}^k C_k$$

and an involution $*: \mathcal{C} \to \mathcal{C}$ such that

1) $n_{ij}^k \geq 0$

2) $\sum_k n_{ij}^k = 1$

3) C_0 is the identity

4) $n_{ij}^0 > 0$ if and only if $C_i^* = C_j$.

Such an object has a pleasing physical interpretation as a collection of interacting particles with (1.1) defining the transition probabilities. C_0 acts like a photon in that it is absorbed in any collision and C_i^* represents the anti-particle of C_i- the only particle for which collision with C_i has a non-zero probability of creating a photon. Particle interactions are independent of their order in time and space, and are symmetrical with respect to replacing particles with anti-particles.

We define the **mass** $m(C_i)$ of C_i to be $(n_{ij}^0)^{-1}$, where $C_i^* = C_j$, and the **total mass** of $\mathcal{C}$ to be $m(\mathcal{C}) = \sum_k m(C_k)$. A **character** of $\mathcal{C}$ is a function $X : \mathcal{C} \to \mathbb{C}$ satisfying

$$(1.2) \qquad X(C_i)X(C_j) = \sum_k n_{ij}^k X(C_k)$$

and

$$(1.3) \qquad X(C_i^*) = \overline{X(C_i)}.$$

$\mathcal{C}^\wedge$ will denote the set of characters of $\mathcal{C}$.

The characters of $\mathcal{C}$ do not necessarily form a F.A.H. under pointwise multiplication but the only axiom that may fail is 1) and this allows us also to define the notions of mass and characters for $\mathcal{C}^\wedge$. One then has the following (see Wildberger [10])

THEOREM 1.1. *i)* $|\mathcal{C}| = |\mathcal{C}^\wedge|$

ii) $(\mathcal{C}^\wedge)^\wedge \simeq \mathcal{C}$

iii) $m(\mathcal{C}) = m(\mathcal{C}^\wedge)$. ▪

There are two important F.A.H.'s associated to any finite group G.

The first consists of the set of conjugacy class under convolution. More specifically, we will identify a conjugacy class C_i with the uniform probability distribution on it (regarded as an element in the group algebra). The involution * sends a conjugacy class C_i to the class of inverse elements of C_i, and the multiplication is given by convolution. We call this F.A.H. the **class hypergroup** of G, and denote it by $\mathcal{C}(G)$.

The second is perhaps more familiar and consists of the representations under tensor products. More specifically, let χ_i and χ_j be two irreducible characters and

$$\chi_i \chi_j = \sum_k M_{ij}^k \chi_k$$

the decomposition of $\chi_i \chi_j$ into irreducibles. If $d_\ell = \chi_\ell(e)$ is the dimension of the corresponding representation and we set $X_\ell = \chi_\ell / d_\ell$ then

$$X_i X_j = \sum_k m_{ij}^k X_k$$

where $m_{ij}^k = M_{ij}^k d_k / d_i d_j$. Under this multiplication, and involution $X_i^* = \overline{X_i}$, the set $\{X_i\}$ forms a F.A.H. which we call the **representation hypergroup** of G and denote by $\mathcal{C}(G^\wedge)$.

The main fact about these two F.A.H.s associated to G is the following.

THEOREM 1.2. $\mathcal{C}(G)^\wedge \simeq \mathcal{C}(G^\wedge)$. ▪

This theorem is a reformulation into our language of a classical property of the irreducible characters of a finite group. Theorem 1.2 states that the characters of G are exactly those functions on conjugacy classes which preserve their natural algebraic structure. Representations are **not** needed in order to define and study characters – an observation which Frobenius would have considered obvious since he (and other pioneers of group theory) studied characters before representations had been defined.

This motivates us to suggest a new approach to the definition and study of $G^\wedge$, where G is an arbitrary group (although we are particularly interested in the case G a real Lie group.) This program involves the following steps.

1.   Determine $\mathcal{C}(G)$, the abelian hypergroup of (invariant) probability distributions on the conjugacy classes of G.

2.   Determine $\mathcal{C}(G)^\wedge$, the dual hypergroup of $\mathcal{C}(G)$.

3.   Establish a version of Theorem 1.2 to determine $\mathcal{C}(G^\wedge)$.

There are of course a number of serious obstacles to be overcome before this program can be applied to large families of groups. We will investigate some of these problems and possible ways to overcome them in this article.

We must first however discuss a third family of F.A.H.s, which occurs in a number of different contexts.

Let H be a finite abelian group and G a subgroup of $\text{Aut}(H)$. Then the orbits of G on H (identified as above with the corresponding probability distributions in the group algebra) form a F.A.H. which we denote by $\mathcal{C}(G; H)$. To determine the characters of $\mathcal{C}(G; H)$, consider the dual action of G on $H^\wedge$, the dual group of H, and form the hypergroup $\mathcal{C}(G; H^\wedge)$ of orbits.

THEOREM 1.3. $\mathcal{C}(G; H)^\wedge \simeq \mathcal{C}(G; H^\wedge)$.

To see this, consider an orbit $\mathcal{U} \in \mathcal{C}(G; H^\wedge)$. We can associate to $\mathcal{U}$ a function $X_\mathcal{U}$ on $\mathcal{C}(G; H)$ by

$$(1.4) \qquad X_\mathcal{U}(\mathcal{O}) = \frac{1}{|\mathcal{U}||\mathcal{O}|} \sum_{\phi \in \mathcal{U}} \sum_{x \in \mathcal{O}} \phi(x) \qquad \mathcal{O} \in \mathcal{C}(G; H)$$

which is easily checked to be a character of mass $m(\mathcal{U}) = |\mathcal{U}|$. Since $m(\mathcal{C}(G; H^\wedge)) = m(\mathcal{C}(G; H)) = |H|$, Theorem 1 iii) shows that the characters $X_\mathcal{U}, \mathcal{U} \in \mathcal{C}(G; H^\wedge)$ exhaust $\mathcal{C}(G; H)^\wedge$.

Let us note that any representation of a finite group G on a finite-dimensional vector space V over a finite field yields such a F.A.H.

§2. FINITE GROUPS

Given the multiplication table of a finite group G, it is a routine matter to compute (or to write a program to compute) the structure equations of $\mathcal{C}(G)$. For C_i, let adC_i denote the operator on $\mathbb{C}\mathcal{C}$ given by multiplication by C_i. The eigenvalues of this operator are the character values $X(C_i)$, $X \in \mathcal{C}^\wedge$. The determination of the characters of $\mathcal{C}(G)$ (and thus of G) thus reduces to the simultaneous diagonalization of the set $\{ad\ C_i\}$ of commuting operators.

Let us illustrate this procedure in the simplest possible case, that of $G = S_3$. If C_1 denotes the conjugacy class of transpositions and C_2 the class of order 3 cycles, then it is easy to calculate the equations of $\mathcal{C}(G) = \{C_0, C_1, C_2\}$ to be

$$C_1^2 = \frac{1}{3}C_0 + \frac{2}{3}C_2$$

$$C_1 C_2 = C_1$$

$$C_2^2 = \frac{1}{2}C_0 + \frac{1}{2}C_2$$

Then with respect to the basis $\{C_0, C_1, C_2\}$,

$$ad\ C_1 = \begin{vmatrix} 0 & \frac{1}{3} & 0 \\ 1 & 0 & 1 \\ 0 & \frac{2}{3} & 0 \end{vmatrix} \qquad ad\ C_2 = \begin{vmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} \end{vmatrix}$$

By simultaneously diagonalizing these operators (or in this case by a simple examination of the structural equations) one finds the characters to have values

	X_0	X_1	X_2
C_0	1	1	1
C_1	1	-1	0
C_2	1	1	$-\frac{1}{2}$

This is the **hypergroup character table** and differs from the group character table only by some normalizations which make it more symmetrical. To complete the analysis we must describe the structure of $\mathcal{C}(G^\wedge)$ as a hypergroup. This is given by

$$X_1^2 = X_0$$

$$X_1 X_2 = X_2$$

$$X_2^2 = \frac{1}{4}X_0 + \frac{1}{4}X_1 + \frac{1}{2}X_2.$$

In practice, however, this method has the difficulty that for many interesting finite groups for which one wishes to calculate a character table, the multiplication table is far from 'known', so even the determination of the group's conjugacy classes can be a difficult problem.

Let us also mention at this point an interesting question – what exactly are the class hypergroup structure constants for the symmetric group S_n?

3. COMPACT GROUPS

Let G be a compact connected Lie group. Each conjugacy class $C \subseteq G$ is closed and carries a unique G-invariant probability measure. It follows that the set $\mathcal{C}(G)$ of conjugacy classes has the structure of a (continuous) abelian hypergroup under group convolution. Similarly $\mathcal{C}(G^\wedge)$ forms an infinite discrete abelian hypergroup and Theorem 1.2 holds, so that

$$\mathcal{C}(G)^\wedge \simeq \mathcal{C}(G^\wedge).$$

Let us see how the above general considerations allow us to recover the usual description of $G^\wedge$ in terms of highest weights (or integral co-adjoint orbits in the geometric quantization picture). The key is to compare $\mathcal{C}(G)$ with the hypergroup of adjoint orbits $\mathcal{O} \subseteq \mathfrak{g}$, which we denote $\mathcal{C}(\mathfrak{g})$. The hypergroup $\mathcal{C}(\mathfrak{g})$ is an infinite continuous version of the hypergroup $\mathcal{C}(G, H)$ discussed in §1, where $\mathfrak{g}$ plays the role of H. Some Euclidean harmonic analysis on $\mathfrak{g}$ shows that the analog of Theorem 1.3 holds, namely that

$$(3.1) \qquad\qquad \mathcal{C}(\mathfrak{g})^\wedge \simeq \mathcal{C}(\mathfrak{g}^*),$$

where $\mathcal{C}(\mathfrak{g}^*)$ denotes the hypergroup of co-adjoint orbits $\mathcal{U} \subseteq \mathfrak{g}^*$. Furthermore the correspondence is given by the natural analog of (1.4), namely for each $\mathcal{U} \subseteq \mathfrak{g}^*$ and $\mathcal{O} \subseteq \mathfrak{g}$

$$X_\mathcal{U}(\mathcal{O}) = \int_\mathcal{U} \int_\mathcal{O} e^{i\phi(x)} d\mu_\mathcal{O}(x) d\mu_\mathcal{U}(\phi)$$

with $d\mu_\mathcal{O}$ and $d\mu_\mathcal{U}$ the G-invariant probability measures on $\mathcal{O}$ and $\mathcal{U}$ respectively. By G-invariance, this can be rewritten as

$$X_\mathcal{U}(\mathcal{O}) = \int_\mathcal{U} e^{i\phi(x_0)} d\mu_\mathcal{U}(\phi)$$

where x_0 is any point in $\mathcal{O}$, which we recognize as a Kirillov-type character formula.

One expects $\mathcal{C}(\mathfrak{g})$ and $\mathcal{C}(G)$ to be closely related, at least on a neighborhood of $0 \in \mathfrak{g}$ on which $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Somewhat surprisingly, the true relationship

between $\mathcal{C}(\mathfrak{g})$ and $\mathcal{C}(G)$ is only revealed when the entire exponential map is taken into consideration.

This has been described in recent work by Dooley and Wildberger [1] and we here describe the main result. Let j be an analytic square root of the determinant of the exponential map - this is a G-invariant function on $\mathfrak{g}$. For $f \in C^\infty(G)$, let $\tilde{f}(X) = f(\exp X)$ denote its lift to $\mathfrak{g}$, and for a compactly supported distribution μ on $\mathfrak{g}$, let $\Phi(\mu)$ be the distribution on G given by $\Phi(\mu)(f) = \mu(j\tilde{f}) \ \forall f \in C^\infty(G)$.

THEOREM 3.1. *(Dooley, Wildberger [1]) Let $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathfrak{g}$ be adjoint orbits with G-invariant probability measures $d\mu_1$ and $d\mu_2$ respectively. Then*

$$\Phi(d\mu_1) * \Phi(d\mu_2) = \Phi(d\mu_1 * d\mu_2),$$

where the convolution on the left is on G and the convolution on the right is on $\mathfrak{g}$.

Theorem 3.1 shows that $\Phi : \mathcal{C}(\mathfrak{g}) \to \mathcal{C}(G)$ is a hypergroup homomorphism, which is clearly surjective since $\exp$ is surjective. Now suppose that $X \in \mathcal{C}(G)^\wedge$, so that X is by Theorem 1.2 a (normalized) irreducible character of G. Then $X \circ \Phi$ is a hypergroup character of $\mathcal{C}(\mathfrak{g})$ which by (3.1) must necessarily be of the form $X_{\mathcal{U}}$ for some co-adjoint orbit $\mathcal{U} \subseteq \mathfrak{g}^*$. Conversely a character $X_{\mathcal{U}}$ of $\mathcal{C}(\mathfrak{g})$ will factor through Φ to give a character of $\mathcal{C}(G)$ precisely when $\mathcal{U}$ satisfies the obvious integrality condition with respect to the exponential map. It follows that $\mathcal{C}(G)^\wedge$ is in 1–1 correspondence with the integral co-adjoint orbits. We have therefore recovered the classical classification of $G^\wedge$ without any resort to the specific structure of G (maximal tori, root spaces etc).

In fact we have done more, we have also established the Kirillov character formula in global form for the character X.

Theorem 3.1 shows how to transfer problems concerning central convolution on G to central convolution on $\mathfrak{g}$. The explicit structure of the hypergroup $\mathcal{C}(\mathfrak{g})$ has been

recently determined in Dooley, Repka and Wildberger [2].

We shall shortly see that in some sense Theorem 3.1 holds **formally** for any Lie group, so there is at least some hope that this entire line of reasoning has an analog even for non-compact groups.

4. NILPOTENT GROUPS

If G is a connected, simply-connected nilpotent Lie group, then the exponential map is a diffeomorphism from $\mathfrak{g}$ to G. This suggests there should be a strong connection between $\mathcal{C}(G)$ and $\mathcal{C}(\mathfrak{g})$, if these objects really exist; in fact Kirillov theory leads us to predict that $\mathcal{C}(G) \simeq \mathcal{C}(\mathfrak{g})$. The typical conjugacy class is non-compact however, so while it does carry a G-invariant measure, such a measure will not generally be a probability measure. Furthermore the convolution of two such measures may easily not exist, at least in the usual sense. Nevertheless we have the following result.

THEOREM 4.1. *(Wildberger [9]) Let $\mathcal{O}_i \subseteq \mathfrak{g}$, $i = 1, 2, 3$ be adjoint orbits and $C_i \subseteq G$ the corresponding conjugacy class $\exp \mathcal{O}_i = C_i$. Then $\mathcal{O}_3 \subseteq \mathcal{O}_1 + \mathcal{O}_2$ if and only if $C_3 \subseteq C_1 C_2$.*

Since the proof is pertinent here, we recall the main idea. Working in the free Lie algebra generated by X and Y (and the corresponding algebra of formal power series) and letting $X \cdot Y = [X, Y]$, $X * Y = \ln(\exp X \exp Y)$ and $X^Y = \exp(Y) \cdot X$, one can show that there exists formal power series $A(X, Y)$ and $B(X, Y)$ such that

$$X * Y = X^{A(X,Y)} + Y^{B(X,Y)}.$$

Furthermore the assignment $Z = X^{A(X,Y)}$, $W = Y^{B(X,Y)}$ is invertible in the sense that one can write X and Y as similar formal power series in Z and W. These considerations prove that $\exp(\mathcal{O}_X + \mathcal{O}_Y) = \exp \mathcal{O}_X \exp \mathcal{O}_Y$ but they actually prove more, namely that

if we define $A : \mathcal{O}_X \times \mathcal{O}_Y \to \mathfrak{g}$ by $A(X', Y') = X' + Y'$ and $M : \mathcal{O}_X \times \mathcal{O}_Y \to \mathfrak{g}$ by $M(X', Y') = X' * Y'$, then we can find a bijection $\eta : \mathcal{O}_X \times \mathcal{O}_Y \to \mathcal{O}_X \times \mathcal{O}_Y$ such that $M = A \circ \eta$.

This is nothing but a **formal** proof of our conjecture $\mathcal{C}(G) \simeq \mathcal{C}(\mathfrak{g})$. If G was a finite nilpotent group with a Lie algebra $\mathfrak{g}$ and an exponential map $\exp : \mathfrak{g} \to G$ which was both a bijection and satisfied the Baker-Campbell-Hausdorff formula, then the above argument would allow us immediately to deduce that $\mathcal{C}(G) \simeq \mathcal{C}(\mathfrak{g})$. Are there such groups? Yes there are – if G is a finite $(p-1)$ step nilpotent p-group (for some prime p) then Howe [4] has shown that there is an abelian group $\mathfrak{g}$ with the structure of a $(p-1)$ step nilpotent Lie algebra and a bijection $\exp : \mathfrak{g} \to G$ satisfying the Campbell-Hausdorff formula. We have thus proved

THEOREM 4.2. *If G is as above, then $\mathcal{C}(G) \simeq \mathcal{C}(\mathfrak{g})$.* ∎

From this we may deduce the following result of Howe [4].

COROLLARY 4.3. *If G is as above, then $\mathcal{C}(G^\wedge) \simeq \mathcal{C}(\mathfrak{g}^\wedge)$ so that the characters of G are in 1:1 correspondence with the co-adjoint orbits of G in $\mathfrak{g}^\wedge$. In fact the character corresponding to any $\mathcal{U} \subseteq \mathfrak{g}^\wedge$ is the push down under $\exp$ of the Fourier transform of the unique G-invariant probability distribution on $\mathcal{U}$.* ∎

Note that our proof of this result has largely ignored the specific structure of the group G.

Let us now turn to the case of G a real nilpotent Lie group, where the above reasoning is only a hopeful guide. Can we make any sense out of $\mathcal{C}(G)$ or $\mathcal{C}(\mathfrak{g})$ and use them to obtain the Kirillov theory for G?

Consider the case of G the Heisenberg group with Lie algebra $\mathfrak{g}$ with basis $\{X, Y, Z\}$ satisfying the canonical commutation relations $X \cdot Y = Z$. The adjoint orbits can be

described as follows:

1) For $(x, y) \neq (0, 0)$, $\mathcal{O}_{(x,y)} = \{(x, y, z) \mid z \in \mathbf{R}\}$

2) For any z, $\mathcal{O}_z = \{(0, 0, z)\}$.

The orbits $\mathcal{O}_{(x,y)}$, carry a G-invariant measure which is just Lebesgue measure dz. One way of defining a convolution structure on these orbits is to use the space $\mathcal{F}$ of continuous almost-periodic functions on $\mathfrak{g}$. Note that $\mathcal{F}$ is closed under Euclidean translations and that any $f \in \mathcal{F}$ when restricted to $\mathcal{O}_{(x,y)}$ is G-almost-periodic in the sense of Von Neumann (that is, the set of its translates under G is conditionally compact in the sup norm.)

For any orbit $\mathcal{O}_{(x,y)}$, let $M_{(x,y)} : \mathcal{F} \to \mathbf{C}$ be the G-invariant mean associated to $\mathcal{O}$; because G acts on $\mathcal{O}_{(x,y)}$ by translations we have

$$M_{(x,y)}(f) = \lim_{N \to \infty} \frac{1}{2N} \int_{a-N}^{a+N} f(x, y, z) dz \qquad \forall f \in \mathcal{F}$$

the integral existing independent of $a \in \mathbf{R}$. Let us use this formula to also define $M_{(0,0)}$. Furthermore,

$$M_z(f) = f(0, 0, z),$$

and we note that

(4.1)
$$M_{(0,0)}(f) = \lim_{N \to \infty} \frac{1}{2N} \int_{a-N}^{a+N} M_z(f) dz \qquad \forall f \in \mathcal{F}$$

independently of $a \in \mathbf{R}$.

We may now convolve two means M and M' by the formula

$$M * M'(f) = M(x)(M'(y)f(x + y)) \qquad \forall f \in \mathcal{F}$$

where $M(x)$ means to take the mean M in the variable x etc. We obtain the following structural equations

1) $M_{(x,y)} * M_{(x',y')} = M_{(x+x', y+y')}$

2) $M_{(x,y)} * M_z = M_{(x,y)}$

3) $M_z * M_{z'} = M_{z+z'}$.

Some remarks ought to be made about this structure, which we call $\mathcal{C}(\mathfrak{g})$. First of all

1) in the case $(x', y') = -(x, y)$ yields $M_{(0,0)}$ on the right side which is to be interpreted

as a mean of the M_z's by (4.1). The structure is abelian and associative, and has a

topology since the set of orbits $\{\mathcal{O}\}$ inherits the quotient topology from $\mathfrak{g}$.

One may consider the corresponding structure on the conjugacy classes of G; a

moments thought shows that this structure, $\mathcal{C}(G)$, is identical to $\mathcal{C}(\mathfrak{g})$, more specifically

$\exp : \mathcal{C}(\mathfrak{g}) \to \mathcal{C}(G)$ is an isomorphism. Thus to determine $\mathcal{C}(G)^\wedge$, we look for the

characters of $\mathcal{C}(\mathfrak{g})$. In analogy with the theory of hypergroups, a character will mean a

continuous, bounded function $X : \mathcal{C}(\mathfrak{g}) \to \mathbb{C}$ which respects the structural equations.

Now 2) shows that either $X(M_{(x,y)}) = 0$ or $X(M_z) = 1$. In the former case 3) shows

that $X(M_z) = e^{i\lambda z}$ for some $\lambda \in \mathbb{R}$, with $\lambda = 0$ excluded since

$$X(M_{(0,0)}) = \lim_{N \to \infty} \frac{1}{2N} \int_{-N}^{N} X(M_z) dz$$

must be 0. In the latter case 1) shows that $X(M_{(x,y)}) = e^{i(ax+by)}$ for some $(a, b) \in \mathbb{R}^2$.

We get then the following description of $\mathcal{C}(\mathfrak{g})^\wedge$.

THEOREM 4.4. $\mathcal{C}(\mathfrak{g})^\wedge = \{X_\lambda \mid \lambda \in \mathbb{R}, \ \lambda \neq 0\} \cup \{X_{(a,b)} \mid a, b \in \mathbb{R}\}$

where
$$X_\lambda : M_{(x,y)} \mapsto 0$$

$$: M_z \mapsto e^{i\lambda z}$$

and
$$X_{a,b} : M_{(x,y)} \mapsto e^{i(ax+by)}$$

$$: M_z \mapsto 1.$$

The reader will recognize the usual description of the irreducible characters of the Heisenberg group, and will hopefully appreciate the fact that we have finessed entirely the Stone-Von Neumann theorem on which this description usually depends.

It seems an interesting challenge to try to extend this approach to more general nilpotent groups.

5. FURTHER DIRECTIONS

The main problem in extending our considerations to more general groups, such as non-compact semi-simple groups, is the difficulty in defining an analog of a G-invariant probability measure on a general conjugacy class. What is really needed is a more general theory of generalized functions or distributions on a Lie group.

If one is willing to abandon probability measures and simply work with the usual G-invariant measures on conjugacy classes, the question of the existence of convolutions of two such measures becomes a rather subtle problem involving the relative geometry of the classes. Let us show, in a simple but instructive example, how such convolutions may be studied and contribute to the determination of the characters of a group.

Let $G = SL(2, \mathbf{R})$ and consider the non-zero adjoint orbits in $\mathfrak{g} = sl(2, \mathbf{R})$. These are of three types; there are two nilpotent cones (whose union with the origin forms the 'light cone'), hyperboloids of one sheet outside the light cone and the individual sheets of the hyperboloids of two sheets inside the light cone.

Let $\mathfrak{t} = \{z_t = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \big| t \in \mathbf{R}\}$ with $\mathfrak{t}_+ = \{z_t \mid t > 0\}$. Let $\mathcal{O}_t$ denote the G-orbit of z_t. Then $C = \bigcup_{t>0} \mathcal{O}_t$ is an open cone so the convolution of any two orbital measures in C is a well-defined measure. Thus C forms a natural 'semi-hypergroup' of $\mathcal{C}(\mathfrak{g})$. The easiest way to calculate the structure constants in C is perhaps to use the following fact.

LEMMA 5.1. *Let μ_λ denote a G-invariant measure on $\mathcal{O}_\lambda, \lambda > 0$ and $p : \mathfrak{g} \to \mathfrak{t}$ the orthogonal projection with respect to the Killing form. Then the push down $p^*(\mu_\lambda)$ is a multiple of Lebesgue measure on $[\lambda, \infty)$, where we identify $\mathfrak{t}$ with $\mathbf{R}$ using the variable t.* ∎

This is a direct analog of the familiar (but remarkable) fact that the surface area on a sphere between two parallel planes depends only on the distance between the planes and is proportional to it.

We may thus normalize our measures μ_λ so that $p^*(\mu_\lambda) = dt$ on $[\lambda, \infty)$. From the fact that convolution and projection commute, it follows that for $\lambda, \lambda' > 0$,

$$\mu_\lambda * \mu_{\lambda'} = \int_{\lambda+\lambda'}^\infty \mu_{\lambda''} d\lambda''.$$

This algebraic structure on C is abelian and associative, and essentially controls the possibilities for the values of the characters of G on the elliptic set.

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