A LANCZOS TYPE ALGORITHM FOR INVERSE STURM-LIOUVILLE PROBLEMS

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Abstract

The Lanczos algorithms can be used to find a symmetric tridiagonal matrix from its eigenvalues and the first components of its normalized eigenvectors. The direct application of this method to discretized Strum-Liouville problems is useless since the finite difference eigenvalues behave quite differently asymptotically than the eigenvalues of the continuous Strum-Liouville problem. We suggest a multiplicative asymptotic correction for the discrete equation. The corrected equations can still be solved, at least approximately, by an algorithm similar to the Lanczos algorithm. Numerical experiments show that this approach leads to results of modest accuracy.

1 Introduction

We consider Sturm–Liouville–problems in the standard form

$$\begin{aligned} -y'' + q(x)y &= \lambda y \,, \\ y(0) = y(\pi) &= 0 \ . \end{aligned} \tag{1.1}$$

Let $\lambda_1 < \lambda_2 < \dots$ be the eigenvalues and $y^1, y^2, \dots$ the normalized eigenfunctions. We consider the inverse problem

$$\text{Given} \quad \lambda_k, (y^k)'(0) \,, \quad k = 1, 2, \dots, \quad \text{find } q \tag{1.2}$$

See [3] for this and similar problems.

A method for solving (1.2) which suggests itself is to discretize (1.1). Let $x_\ell = h\ell$, $\ell = 0, \dots, n$, $h = \pi/n$. We approximate (1.1) by

$$\begin{aligned} \frac{-y_{\ell+1} + 2y_\ell - y_{\ell-1}}{h^2} + q_\ell y_\ell &= \lambda y_\ell \,, \qquad \ell = 1, \dots, n-1 \\ y_0 = y_n &= 0 \end{aligned} \tag{1.3}$$

where $q_\ell = q(x_\ell)$ and y_ℓ approximates $y(x_\ell)$. The derivative at $x = 0$ is approximated by y_1/h. Thus the discrete inverse problem reads: Given $n-1$ eigenvalues $\lambda_1, \dots, \lambda_{n-1}$ and the first components $y_1^1, \dots, y_1^{n-1}$ of the normalized eigenvectors of (1.3), find $q_1, \dots, q_{n-1}$. This is an inverse problem for a tridiagonal matrix which can easily be solved by the Lanczos algorithm, see [3], [4], [7].

Unfortunately, such an approach is quite useless. The reason is that the asymptotics of (1.1), (1.3) are quite different. This means that only the first few eigenvalues of (1.1) are close to those of (1.3). See [1] for a survey on finite difference methods for Sturm–Liouville–problems.

One of the remedies is to correct the high eigenvalues of (1.3) by introducing additional terms. The method of asymptotic corrections ([8], see also [1] and the references therein) consists in an additive correction of (1.3) which is derived from the well-known asymptotics of (1.1).

In the present note we suggest a multiplicative correction of (1.3) simply in the following way. For k large the solutions of (1.1) are close to $\sin kx$, see [5], [6]. Thus it seems reasonable to replace $-y''$ by a finite difference

expression which is accurate for the functions $1, x, \sin kx, \cos kx$. It is easily seen that

$$\frac{-y(x_{\ell+1}) + 2y(x_\ell) - y(x_{\ell-1})}{(h \operatorname{sinc}\frac{hk}{2})^2} \quad , \qquad \operatorname{sinc} = \frac{\sin x}{x} \tag{1.4}$$

is such an expression. Similarly, $y'(0)$ is given accurately by

$$\frac{y(x_1)}{h \operatorname{sinc}(hk)}$$

for $y = \sin kx$. Thus we replace (1.3) by

$$\begin{gathered} -y^k_{\ell+1} + 2y^k_\ell - y^k_{\ell-1} + q_\ell c_k y^k_\ell = \lambda_k c_k y^k_\ell \ , \qquad \ell = 1, \ldots, n-1 \ , \\ y^k_0 = y^k_n = 0 \end{gathered} \tag{1.5}$$

where

$$c_k = \left(h \operatorname{sinc} \frac{hk}{2} \right)^2 .$$

Here, the dependence of y, λ on k has been made explicit. Note that for hk small, $c_k \sim h^2$, hence (1.5) is close to (1.3) for these k.

We consider the problem of finding $q_1, \ldots, q_{n-1}$ from $\lambda_1 \ldots, \lambda_{n-1}$ and $y^k_1/(h \operatorname{sinc} hk)$ where y^k are the eigenvectors of (1.5) normalized such that

$$h \sum_{\ell=1}^{n-1} (y^k_\ell)^2 = 1 \ .$$

In the next section we show that this can be done, at least approximately, by a Lanczos type method.

2 A Lanczos type method

Introducing the $(n-1)\times(n-1)$-matrices

$$T = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & & \\ & & & -1 & 2 \end{pmatrix}, \qquad U = \sqrt{h} \begin{pmatrix} y_1^1 & \cdots & y_1^{n-1} \\ \vdots & & \vdots \\ y_{n-1}^1 & & y_{n-1}^{n-1} \end{pmatrix},$$

$$Q = \operatorname{diag}(q_\ell), \quad C = \operatorname{diag}(c_k), \quad M = \operatorname{diag}(c_k \lambda_k),$$

(1.5) can be rewritten as

$$TU + QUC = UM, \tag{2.1}$$

and U is approximately unitary. Thus we arrive at the problem of computing a unitary matrix U and a diagonal matrix Q from (2.1), the first row of U and the matrices T, C, M being known. Note that the elements in the first row of U are determined by

$$y_1^k = (y^k)'(0)\, h \operatorname{sinc}(hk).$$

This can be done by an algorithm similar to the Lanczos algorithm (see [4]) in the following way. Write

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & & \ddots & \beta_{n-2} & \\ & & \beta_{n-2} & \alpha_{n-1} & \end{pmatrix}, \qquad U = \begin{pmatrix} u_1 \\ \vdots \\ u_{n-1} \end{pmatrix}.$$

Then, (2.1) reads

$$\begin{aligned} &\alpha_1 u_1 + \beta_1 u_2 + q_1 u_1 C = u_1 M, \\ &\beta_{\ell-1} u_{\ell-1} + \alpha_\ell u_\ell + \ell_\ell u_{\ell+1} + q_\ell u_\ell C = u_\ell M, \quad \ell = 2, \ldots, n-1 \\ &\beta_{n-2} u_{n-2} + \alpha_{n-1} u_{n-1} + q_{n-1} u_{n-1} C = u_{n-1} M. \end{aligned} \tag{2.2}$$

Multiplying the first equation with u_1 we get

$$\alpha_1(u_1, u_1) + q_1(u_1 C, u_1) = (u_1 M, u_1)$$

where we have neglected (u_2, u_1) since U is close to being orthogonal. This determines q_1. Once q_1 is known we get

$$u_2 = (u_1 M - \alpha_1 u_1 - (u_1 M, u_1))/\beta_1 \ .$$

Now assume $q_1, \ldots, q_{\ell-1}$ and $u_1, \ldots, u_\ell$ to be already determined. Then, multiplying equation ℓ of (2.2) by u_ℓ yields

$$\alpha_\ell(u_\ell, u_\ell) + q_\ell(u_\ell C, u_\ell) = (u_\ell M, u_\ell)$$

where $(u_{\ell-1}, u_\ell)$, $(u_{\ell+1}, u_\ell)$ have been neglected. This determines q_ℓ. $u_{\ell+1}$ is in turn computed from

$$u_{\ell+1} = (u_\ell M - \beta_{\ell-1} u_{\ell-1} - \alpha_\ell u_\ell - q_\ell u_\ell C)/\beta_\ell \ .$$

Proceeding in this fashion we can compute $q_1, \ldots, q_{n-1}$.

3 Numerical example

The Sturm-Liouville–problem

$$-y'' + \frac{4\nu^2 - 1}{4x^2}\, y = \lambda y \ , \quad y(a) = y(b)$$

has the solution

$$y = \sqrt{x}\ (c_1 J_\nu(\sqrt{\lambda}x) + c_2 Y_\nu(\sqrt{\lambda}x))$$

where J_ν, Y_ν are the Bessel functions of first and second kind, respectively. The eigenvalues are the roots of

$$\det \begin{pmatrix} J_\nu(\sqrt{\lambda}a) & , & Y_\nu(\sqrt{\lambda}a) \\ J_\nu(\sqrt{\lambda}b) & , & Y_\nu(\sqrt{\lambda}b) \end{pmatrix} .$$

The Lanczos type algorithm from section 2 applied to (1.5) for $n = 10$ yields for $a = 1$, $b = 4$ the following result:

ℓ	x_ℓ	$q(x_\ell)$	q_ℓ
2	1.6	1.46	1.31
4	2.2	0.77	0.85
6	2.8	0.48	0.61
8	3.4	0.32	0.51

This is not a very satisfactory accuracy. For $n = 20$, the results exhibit instabilities which make them useless.

However, for a fair evaluation of these results one has to keep in mind that an uncorrected discretization (i.e. $c_k = 1$ in (1.5)) does not even produce the correct order of magnitude of the q_ℓ. Thus, our results are not quite disencouraging.

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