Transference from Lipschitz graphs to periodic Lipschitz graphs

by

Tao Qian

Department of Mathematics, The New England University,
Armidale, NSW 2351 Australia
E-mail address: tao@neumann.une.edu.au

§1. Introduction

In this note we will study Fourier multiplier operators on Lipschitz surfaces. On Lipschitz curves the notion of a Fourier transform is initially introduced by R. Coifman and Y. Meyer ([CM]). The monogenic extensions of the exponential functions (see [LMcQ]) enable us to define this notion on surfaces. The paper extends a proof in [GQW] using the monogenic extensions of the Gauss-Weierstrass kernels, and hence proves that the boundedness of certain operators on infinite surfaces can be transferred to the induced operators on periodic surfaces. More general Fourier multipliers rather than the H^∞ ones are considered. For the latter the reader is referred to [McQ1]-[McQ3], [LMcS], [LMcQ] and [GQW].

The author would like to thank A. McIntosh. The discussions with him in Brisbane on the monogenic extensions of the Gauss-Weierstrass kernels $\exp(-\pi t|x|^2), t > 0$, benefit this study.

§2. Transference from γ to Γ

Denote the standard basis vectors of $\mathbf{R}^{n+1}$ by $e_0, e_1, ..., e_n$, where $e_0^2 = 1, e_i^2 = -1, i = 1, ..., n$, and $e_i e_j = -e_j e_i, 1 \leq i < j \leq n$. We then imbed $\mathbf{R}^{n+1}$ into the real Clifford algebra $\mathbf{R}^{(n+1)}$ generated by $e_0, e_1, ..., e_n$, according to which we write a typical $x \in \mathbf{R}^{n+1}$ as $x = \mathrm{x} + x_0 e_0$, where $\mathrm{x} = x_1 e_1 + \cdots + x_n e_n \in \mathbf{R}^n$. In the sequel we will identify $e_0 = 1$.

We will use the following sets: For $\mu \in (0, \frac{\pi}{2}], \mathbf{C}_{\mu,+} = \{0 \neq x = \mathrm{x} + x_L e_L \in \mathbf{R}^{n+1} \mid x_L > -|\mathrm{x}| \tan \mu\}, \mathbf{C}_{\mu,-} = -\mathbf{C}_{\mu,+}$, and $\mathbf{S}_\mu = \mathbf{C}_{\mu,+} \cap \mathbf{C}_{\mu,-}$.

Let γ be an infinite Lipschitz graph parameterized by

$$\gamma = \{\mathrm{x} + g(\mathrm{x})e_0 | \mathrm{x} \in \mathbf{R}^n, g : \mathbf{R}^n \to \mathbf{R},\ g, \nabla g \in L^\infty(\mathbf{R}^n)\}.$$

Denote by $N = \|\nabla g\|_\infty < \infty$ its Lipschitz constant. Without loss of generality, we assume $-M = \min\{g(\mathrm{x}) | \mathrm{x} \in \mathbf{R}^n\} = -\max\{g(\mathrm{x}) | \mathrm{x} \in \mathbf{R}^n\}, 0 < M < \infty$. Denote $D_l = \sum_{i=0}^n e_i \frac{\partial}{\partial x_i}$ and $D_r = \sum_{i=0}^n \frac{\partial}{\partial x_i} e_i$. For a Clifford-valued function f we define

$$D_l f = \sum_{i=0}^n e_i \frac{\partial f}{\partial x_i}, \quad D_r f = \sum_{i=0}^n \frac{\partial f}{\partial x_i} e_i.$$

A function f is said to be left-monogenic or right-monogenic, if $D_l f = 0$ or $D_r f = 0$, respectively. The good thing with monogenicity is that if f is right-monogenic and g is left-monogenic in a neibourghood of a domain D with smooth boundary, then Cauchy's theorem holds:

$$\int_{\partial D} f(x)n(x)g(x)d\sigma(x) = 0,$$

where $n(x)$ is the outer nomal on ∂D and $d\sigma(x)$ the area element on the boundary. The Cauchy integral formula also holds (see [LMcQ] for example). Let $\mathcal{A}(\gamma) = \{f | f \text{ is left} - \text{monogenically defined in } -M-\delta < x_0 < M+\delta \text{ for some } \delta > 0\}$. It is easy to prove that $\mathcal{A}(\gamma)$ is dense in $L^p(\gamma), 1 < p < \infty$ (see [CM]).

A function defined on $\mathbf{R}^n$ is said to be 2π-periodic if it is 2π-periodic in every coordinate. A Lipschitz surface is said to be 2π-periodic if its parameterization function g is 2π-periodic. We denote such a g by G. Denote $\mathbf{D}^n = [-\pi,\pi] \times \cdots \times [-\pi,\pi]$ (n-factors), and $\Gamma = \{\mathbf{x} + e_0 G(\mathbf{x}) | \mathbf{x} \in \mathbf{D}^n\}$. One can similarly define the subspace $\mathcal{A}(\Gamma)$ of $L^p(\Gamma)$ with the difference that the left-monogenic functions consisting of $\mathcal{A}(\Gamma)$ are 2π-periodic. The density of $\mathcal{A}(\Gamma)$ in $L^p(\Gamma)$ holds too.

Let $\zeta = \xi + i\eta = \sum_{j=1}^n \zeta_j, \zeta_j \in \mathbf{C}, j = 1, ..., n$. The following defined $e(x,\eta)$ is the left- and right-monogenic extension of $\exp(-i\mathbf{x}\xi) : e(x,\zeta) = e_+(x,\zeta) + e_-(x,\zeta), e_\pm(x,\zeta) = e^{i<\mathbf{x},\zeta>} e^{\mp x_0|\zeta|_{\mathbf{C}}} \chi_\pm(\zeta), \chi_\pm(\zeta) = \frac{1}{2}(1 \pm i\zeta|\eta|_{\mathbf{C}}^{-1})$, where $|\zeta|_{\mathbf{C}}^2 = \sum_{i=1}^n \zeta_i^2$, $Re(|\zeta|_{\mathbf{C}}) > 0$ (see [LMcQ]).

Let $n = \mathbf{n} + n_L e_L$ be a unit vector in $\mathbf{R}^{n+1}$. We will use the open half tubes in $\mathbf{R}^{n+1} : C_n^\pm = \{x \in \mathbf{R}^{n+1} \mid \mathbf{x} \in \mathbf{D}^m, \pm\langle x, n\rangle > 0\}$, and the real n-dimensional surface $n(\mathbf{C}^n)$ in $\mathbf{C}^n$, defined by

$$n(\mathbf{C}^n) = \{\zeta = \xi + i\eta \in \mathbf{C}^n \mid |\zeta|_{\mathbf{C}}^2 \notin (-\infty, 0] \text{ and } n_L\eta = Re(|\zeta|_{\mathbf{C}})\mathbf{n}\}.$$

See [LMcQ] for some equivalent characterizations of $n(\mathbf{C}^n)$ and the relation that $n \in \mathbf{T}_\mu(\pi)$ if and only if $n(\mathbf{C}^n) \subset \mathbf{S}_\mu(\mathbf{C}^n)$, where

$$\mathbf{S}_\mu(\mathbf{C}^n) = \{\zeta = \xi + i\eta \in \mathbf{C}^n \mid |\zeta|_{\mathbf{C}}^2 \notin (-\infty, 0] \text{ and } |\eta| < \mathrm{Re}(|\zeta|_{\mathbf{C}}) \tan\mu\}.$$

Functions $e_\pm(x,\zeta)$ satisfy the following relations:

$$\begin{aligned} |e_\pm(x,\zeta)| &= e^{-<\mathbf{x},\eta> \mp x_L Re|\zeta|_{\mathbf{C}}} |\chi_\pm(\zeta)| \\ &\leq \frac{\sec(\mu)}{\sqrt{2}} e^{\mp<\mathbf{x},\mathbf{n}> Re|\zeta|_{\mathbf{C}}/n_L}, \quad \zeta \in n(\mathbf{C}^n) \subset \mathbf{S}_\mu(\mathbf{C}^m). \end{aligned}$$

The Banach space of bounded Clifford-valued holomorphic functions defined on $\mathbf{S}_\mu(\mathbf{C}^n)$, denoted by $H^\infty(\mathbf{S}_\mu(\mathbf{C}^n))$ in the sequel, is of special interest (see [LMcQ]). Theorem 1 below will be applied to conclude the boundedness of the Fourier multiplier operators on periodic Lipschitz surfaces induced by $H^\infty(\mathbf{S}_\mu(\mathbf{C}^n))$ functions.

In virtue of the extended exponential function one can define Fourier transforms or Fourier coefficients on a given Lipschitz surface, depending on the surface type being infinite or periodic, respectively. For $f \in \mathcal{A}(\gamma)$, define

$$\hat{f}(\xi) = \int_\gamma e(-x,\xi)n(x)f(x)d\sigma(x), \quad \xi \in \mathbf{R}^n.$$

If γ is periodic, i.e. $\gamma = \Gamma$, then for $f \in \mathcal{A}(\Gamma)$, one can define the Fourier coefficients

$$\hat{f}(l) = \frac{1}{(2\pi)^n}\int_\Gamma e(-x,l)n(x)f(x)d\sigma(x), \quad l \in \mathbf{Z}^n.$$

Now we are ready to introduce Fourier multiplier operators. Let b be a bounded function from $\mathbf{R}^n \to \mathbf{R}^n$. Then for $f \in \mathcal{A}(\gamma)$ the following integral is absolutely convergent, that can be proved similarly to the series case in [Q2], and so it well definesa the operator

$$m_b f(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} b(\xi)e(x,\xi)\hat{f}(\xi)d\xi.$$

For $f \in \mathcal{A}(\gamma)$, the following series is absolutely convergent, as proved in [Q2], and so it defines the operator

$$M_b f(x) = \sum_{l\in\mathbf{Z}^n} b(l)e(x,l)\hat{f}(l).$$

Owing to the fact $\mathbf{S}_\mu(\mathbf{C}^n) \cap \mathbf{R}^n = \mathbf{R}^n \setminus \{0\}$, we make the convention that for $b \in H^\infty(\mathbf{S}_\mu(\mathbf{C}^n))$

$$M_b f(x) = \sum_{0\neq l\in\mathbf{Z}^n} b(l)e(x,l)\hat{f}(l).$$

Theorem 1. *Let the function b be continuous at every $l \in \mathbf{Z}^n$. If m_b is a bounded operator on $L^p(\gamma), 1 \le p < \infty$, then M_b is a bounded operator on $L^p(\Gamma), 1 \le p < \infty$.*

The proof given below will follow the pattern in pp 261-262 [SW] (also see [GQW]). What is new here is the calculation of the monogenic extensions of the Gauss-Weierstrass kernels $\exp(-t|\mathrm{x}|^2), t > 0$.

The monogenic extensions of the Gauss-Weierstrass kernels can be calculated as follows. First we calculate the Fourier transforms of the kernels:

$$\int_{\mathbf{R}^n} e^{-i<x,y>}\exp(-t|x|^2)dx = a_n \exp(-\frac{|y|^2}{4t}),$$

where $a_n = a_n(t) = (\frac{t}{\pi})^{-\frac{n}{2}}$. Then we calculate the inverse Fourier transforms of the result functions using the monogenic extensions of the exponential functions,

$$a_n(\frac{1}{2\pi})^n \int_{\mathbf{R}^n} e(-x,y)\exp(-\frac{|y|^2}{4t})dy = a_n(\frac{1}{2\pi})^n \int_{\mathbf{R}^n} e_+(-x,y)\exp(-\frac{|y|^2}{4t})dy +$$
$$+ a_n(\frac{1}{2\pi})^n \int_{\mathbf{R}^n} e_-(-x,y)\exp(-\frac{|y|^2}{4t})dy$$
$$= a_n(\frac{1}{2\pi})^n \int_{\mathbf{R}^n} e^{i<\mathrm{x},y>}\exp(-x_0|y|)\frac{1}{2}(1+i\frac{y}{|y|})\exp(-\frac{|y|^2}{4t})dy +$$
$$+ a_n(\frac{1}{2\pi})^n \int_{\mathbf{R}^n} e^{i<\mathrm{x},y>}\exp(x_0|y|)\frac{1}{2}(1-i\frac{y}{|y|})\exp(-\frac{|y|^2}{4t})dy.$$

Both integrals in the last sum are absolutely convergent, and the result functions are the monogenic extensions of the kernels. Denote by

$$\mathcal{H}^{\pm}(x) = (\frac{1}{2\pi})^n \int_{\mathbf{R}^n} e^{i<\mathrm{x},y>}\exp(\mp x_0|y|)\exp(-\frac{|y|^2}{4t})dy,$$

the unique harmonic extensions of $a_n^{-1}\exp(-t|x|^2)$ to $\mathbf{R}^{n+1}_{-}$, respectively. We conclude that the desired monogenic extension of $\exp(-t|\mathrm{x}|^2)$, denoted by $\mathcal{M}_t(x)$, is

$$\mathcal{M}_t(x) = a_n(\frac{1}{2}(1+iD(\mathrm{x})I(x_0))\mathcal{H}^+(x) + \frac{1}{2}(1-iD(\mathrm{x})I(x_0))\mathcal{H}^-(x)),$$

where $D(\mathrm{x}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} e_i$ and $I(x_0)$ is the indefinite integral with respect to the variable x_0 that vanishes at $x_0 = \infty$.

The following assertions are needed in following the argument in [SW].

Lemma 1. *The monogenic exponential function system* $\{e(x,l)\}, l \in \mathbf{Z}^n$, *is dense in* $L^p(\Gamma), 1 \le p < \infty$.

Lemma 2. *Let* $f \in L^1(\mathbf{D}^n)$ *and* f_p *its periodic extension to* $\mathbf{R}^n$. *Then*

$$\lim_{t\to 0} \int_{\mathbf{R}^n} f(\mathrm{x})a_n^{-1}\exp(-t|\mathrm{x}|^2)d\mathrm{x} = \int_{\mathbf{D}^n} f(\mathrm{x})d\mathrm{x}.$$

Lemma 3. *Let* $f \in L^1(\Gamma)$ *and* f_p *its periodic extension to* γ. *Then*

$$\lim_{t\to 0} a_n^{-1}(t)\int_\gamma f_p(x)n(x)\mathcal{M}_t(x)d\sigma(x) = \int_\Gamma f(x)n(x)d\sigma(x).$$

Lemma 4. *Suppose* P *and* Q *are monogenic trigonometric polynomials,* m_b *is a bounded operator in* $L^p(\gamma), 1 \le p < \infty$, *then*

$$\lim_{t\to 0} a_n^{-1}(t)\int_\gamma m_b(P\mathcal{M}_{t\alpha})(x)n(x)Q\mathcal{M}_{t\beta}(x)d\sigma(x) = \int_\Gamma (M_bP)(x)n(x)Q(x)d\sigma(x).$$

Proofs of Lemma 1, 2 are similar to Lemma 3.9 in [SW] and the corresponding Lemmas in [GQW]. Lemma 4 can be proved using Cauchy's theorem and the argument in [SW] (also see [GQW]).

We only provide a proof of Lemma 3.

Proof of Lemma 3. We first change integral variable $x \to \mathrm{x}$. Owing to Lemma 2, what we need to prove is

$$\lim_{t\to 0} a_n^{-1}(t) \int_{\mathbf{R}^n} F(\mathrm{x})(\mathcal{M}_t(x) - \mathcal{M}_t(\mathrm{x}))d\mathrm{x} = 0,$$

where $F(\mathrm{x}) = f(x)n(x)\sqrt{1 + |\nabla g(x)|^2}$.

In doing so we use the formula for $\mathcal{M}_t(x)$. Changing variable in the integral expression of $\mathcal{H}^{\pm}(x)$, we have

$$\mathcal{H}^{\pm}(x) = (\frac{\sqrt{t}}{2\pi})^n \int_{\mathbf{R}^n} e^{i<\mathrm{x},\sqrt{t}y>} \exp(\mp\sqrt{t}x_0|y|) \exp(-\frac{|y|^2}{4})dy.$$

Then $M_t(x) - M_t(\mathrm{x})$ can be expressed as a sum of certain differentials and indefinite integrals of the mentioned type of the following integral:

$$(\frac{\sqrt{t}}{2\pi})^n \int_{\mathbf{R}^n} e^{i<\mathrm{x},\sqrt{t}y>}(\exp(\mp\sqrt{t}x_0|y|) - 1) \exp(-\frac{|y|^2}{4})dy.$$

Since on the surface γ the x_0 coordinate is bounded, the last integrals with respect to different t's, their differentials and indefinite integrals of the mentioned type all tend to 0 as $t \to 0$. Using the Lebesgue dominated theorem, we conclude the desired relation.

In completing the proof we use Hölder's inequality to the left hand side, before taking the limit $t \to 0$, of the equality in Lemma 4, and use the boundedness assumption on m_b. Multiplying $a_n^{-1}(t)$ to both sides of the obtained inequality, taking the limit $t \to 0$ and using Lemma 4 and 3 to the two ends, respectively, we conclude the desired boundedness for M_b.

Corollary. *Let Γ be a 2π-periodic Lipschitz surface, N its Lipschitz constant, and $\omega = \arctan N$. Then for any function $b \in H^\infty(\mathrm{S}_\mu(\mathrm{C}^n))$, $\frac{\pi}{2} \geq \mu > \omega$, M_b is extensible to a bounded operator on $L^p(\Gamma), 1 < p < \infty$.*

References

[CM] R. Coifman and Y. Meyer, *Fourier analysis of multilinear convolutions, Calderón's theorem, and analysis on Lipschitz curves,* Lecture Notes in Mathematics, Springer-Verlag 779 (1980), 104-122.

[GQW] G. Gaudry, T. Qian and S-L. Wang, *Boundedness of singular integral operators with holomorphic kernels on star-shaped Lipschitz curves,* in preparation.

[LMcS] C. Li, A. McIntosh and S. Semmes, *Convolution singular integral operators on Lipschitz surfaces,* Journal of the American Mathematical Society 5 (1992), 455-481.

[LMcQ] C. Li, A. McIntosh and T. Qian, *Clifford algebras, Fourier transforms and singular convolution operators on Lipschitz surfaces,* to appear in Revista Mathematica Iberoamericana.

[McQ1] A. McIntosh and T. Qian, *Convolution singular integral operators on Lipschitz curves,* Lecture Notes in Mathematics, Springer-Verlag 1494 (1991), 142-162.

[McQ2] A. McIntosh and T. Qian, *A note on singular integrals with holomorphic kernels,* Approximation Theory and its Applications, No.4, vol. 6 (1990) 40-57.

[McQ3] A. McIntosh and T. Qian, *L^p Fourier multipliers along Lipschitz curves,* Trans. Amer. Math. Soc. (1992), 157-176.

[Q1] T. Qian, *Singular integrals with holomorphic kernels and H^∞-Fourier multipliers on star-shaped Lipschitz curves,* preprint.

[Q2] T. Qian, *Singular integrals with monogenic kernels on the n-torus and its Lipschitz perturbations,* preprint.

[SW] E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces,* Princeton Univ. Press, Princeton, New Jersey (1971).