The Wave Equation

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This lecture introduces some of the basic properties of hyperbolic equations, illustrated by the specific example of the wave equation in $\mathbf{R}^3$. Emphasis is placed on the geometric nature of many of the constructions, and on describing different approaches to the existence problem.

1 The Wave Equation

The prototypical hyperbolic equation, and the object of study in this lecture, is the wave equation

$$\Box u := u_{tt} - \Delta u = f. \tag{1}$$

We shall consider mainly the case where u and f are defined on $\mathbf{R}^3 \times \mathbf{R}_+$, as this simple case suffices to illustrate most of the basic properties of general linear hyperbolic equations.

Solutions of (1) satisfy the *energy estimate*,

$$\frac{d}{dt}E(t) = \int_{\mathbf{R}^3} f u_t \, dx \tag{2}$$

where

$$E(t) = \frac{1}{2}\int_{\mathbf{R}^3} (u_t^2 + |Du|^2) \, dx,$$

$(Du = (D_1 u, D_2 u, D_3 u))$. In particular, if $f = 0$ then $E(t) = E(0)$ (energy conservation) and in general, we may estimate

$$\left| \int_{\mathbf{R}^3} f u_t \, dx \right| \le \sqrt{2E(t)} \|f(t)\|_{L^2}$$

and thereby control $E(t)$.

From (2) we immediately deduce *uniqueness* for the *Cauchy problem*

$$\begin{aligned} u_{tt} - \Delta u &= f \\ u(0) &= u_0 \in H^1(\mathbf{R}^3) \\ u_t(0) &= u_1 \in L^2(\mathbf{R}^3) \end{aligned} \tag{3}$$

since if $u, v \in C^1(\mathbf{R}^+, H^1(\mathbf{R}^3))$ satisfy (3) then $w = u - v$ satisfies (3) with $f = 0$ and $u_0 = u_1 = 0$, and hence $E_w(0) = 0$ and by (2), $E_w(t) = 0 \ \forall t \ge 0$.

More detailed information about (1) may be obtained by exploiting the geometry underlying the energy estimate. Multiplying (1) by u_t leads to the divergence identity

$$D_t(\tfrac{1}{2}(u_t^2 + |Du|^2)) - D_i(u_t u_i) = u_t \Box u, \tag{4}$$

$(u_t = D_t u = \partial u/\partial t, u_i = D_i u = \partial u/\partial x^i)$, which we may rewrite as

$$d(\iota_X \mu) = u_t \Box u \mu, \tag{5}$$

where $\mu = dt \wedge dx^1 \wedge dx^2 \wedge dx^3$ and X is the vector field

$$X = \tfrac{1}{2}(u_t^2 + |Du|^2)\, \partial_t - u_t u_i \, \partial_i. \tag{6}$$

Integrating (4) over a lens–shaped region

$$V = \{(x,t) : \phi(x) \le t \le \psi(x),\ x \in \Omega \subset \mathbf{R}^3\}$$

where $\phi = \psi$ on $\partial\Omega$ and Ω is bounded, leads by Stokes' theorem to

$$\int_{S(\psi)} \iota_X \mu - \int_{S(\phi)} \iota_X \mu = \int_V u_t \Box u \, dt \, dx \tag{7}$$

where $S(\psi) = \{(x,t), t = \phi(x), x \in \Omega\} = \text{graph}(\psi)$ and both $S(\phi), S(\psi)$ are oriented by $dx^1 \wedge dx^2 \wedge dx^3$. Since the tangent 3–plane to $S(\phi)$ is

$$\begin{aligned} v &= (\partial_1 + \phi_1 \partial_t) \wedge (\partial_2 + \phi_2 \partial_t) \wedge (\partial_3 + \phi_3 \partial_t), \\ &= \partial_1 \wedge \partial_2 \wedge \partial_3 + \partial_t \wedge (\phi_1\, \partial_2 \wedge \partial_3 + \phi_2\, \partial_3 \wedge \partial_1 + \phi_3\, \partial_1 \wedge \partial_2), \end{aligned}$$

where $\phi_i = \partial_i \phi$, it follows that

$$\begin{aligned} (\iota_X \mu)(v) &= X^0 - X^i \phi_i \\ &= \tfrac{1}{2}(u_t^2 + |Du|^2) + u_t u_i \phi_i \end{aligned} \tag{8}$$

and hence

$$\int_{S(\phi)} \iota_X \mu = \int_\Omega \left(\tfrac{1}{2}(u_t^2 + |Du|^2) + u_t \phi_i u_i \right)\Big|_{t=\phi(x)} dx. \tag{9}$$

Now setting $v_1 = |D\phi|^{-1} D\phi$ if $D\phi \ne 0$ we have

$$u_t^2 + |Du|^2 + 2u_t u_i \phi_i = (1 - |D\phi|) u_t^2 + (|Du|^2 - |D\phi|(v_1 \cdot Du)^2) + |D\phi|(u_t + v_1 \cdot Du)^2, \tag{10}$$

and this is positive semidefinite for any function u whenever $|D\phi| \le 1$. A surface $S = S(\phi)$ satisfying $|D\phi| < 1$ is said to be *(strictly) spacelike*; if $|D\phi| \le 1$ then $S(\phi)$ is *weakly spacelike*. From (7) and (10) we may derive the fundamental uniqueness result.

Proposition 1 *Suppose u, v satisfy $\Box u = \Box v$ in $\mathbf{R}^3 \times \mathbf{R}_+$ and*

$$\left.\begin{aligned} u(x,0) &= v(x,0) \\ u_t(x,0) &= v_t(x,0) \end{aligned}\right\} \quad \forall x \in \Omega \subset \mathbf{R}^3, \tag{11}$$

then $u(x,t) = v(x,t)$ for all $(x,t) \in D^+(\Omega)$, where $D^+(\Omega)$ is the future domain of dependence *of Ω,*

$$D^+(\Omega) = \{(x,t) : 0 \le t \le \mathit{dist}(x, \partial\Omega),\ x \in \Omega\}. \tag{12}$$

Proof : Suppose $(x_0, t_0) \in D^+(\Omega)$ and evaluate the energy integrals for the function $w = u - v$ over the region bounded by the past light cone $C^-_{x_0,t_0} = \{(x,t) : t = t_0 - |x - x_0|\}$ and the initial surface $t = 0$. Since $B_{t_0}(x_0) = \{x : |x - x_0| \leq t_0\} \subset \Omega$, (7) gives

$$\int_{B_{t_0}(x_0)} \left(w_t^2 + |Dw|^2\right) dx = \int_{C^-_{x_0,t_0} \cap \{t \geq 0\}} \left(|D_\theta w|^2 + |D_\rho w|^2\right) dx \qquad (13)$$

where

$$D_\rho w = \partial_t w + |x - x_0|^{-1}(x - x_0) \cdot Dw$$

is the radial null derivative along the light–like lines generating $C^-_{x_0,t_0}$ and $D_\theta w$ denotes the remaining non–radial spatial derivatives. The integral over $B_{t_0}(x_0)$ vanishes by (10), thus $D_\rho w = 0$ and $w = 0$ on $\Omega \times \{0\}$ and it follows that $w(x_0, t_0) = 0$. ■

This says the initial values of u outside Ω do not affect the solution inside $D^+(\Omega)$, or equivalently, the solution propagates at speed 1.

Corollary 2 *Suppose u satisfies (3) in $\mathbf{R}^3 \times \mathbf{R}_+$ with $f = 0$. If $spt(u_0) \cup spt(u_1) \subset \Omega$, then $u(x,t) = 0$ for $(x,t) \notin I^+(\Omega)$, where $I^+(\Omega)$ is the* domain of influence *of Ω,*

$$I^+(\Omega) = \{(x,t) : dist(x, \Omega) < t\}. \qquad (14)$$

The argument of Proposition 1 extends readily to show the uniqueness of the initial value problem (IVP) posed on a strictly spacelike surface $S(\phi)$,

$$\begin{aligned} u_{tt} - \Delta u &= f \\ u(x, \phi(x)) &= u_0(x) \\ u_t(x, \phi(x)) &= u_1(x), \quad \forall x \in \mathbf{R}^3, \end{aligned} \qquad (15)$$

since the energy integral over $S(\phi)$ given by (9), (10) vanishes if $u = 0$. A similar result holds then for the uniqueness in the domain of dependence over a subset of $S(\phi)$.

Note that it is essential here that $S(\phi)$ be strictly spacelike, $|D\phi| < 1$, since if $|D\phi| = 1$ on an open set Σ, then there are nontrivial data (u, u_t) along $S(\phi)$ for which the density (10) vanishes, and then the above argument would fail. If $|D\phi| = 1$, then the vanishing of (10) constrains only the derivatives of u tangential to $S(\phi)$, and does not otherwise restrict the transverse derivative u_t. This is very different from the strictly spacelike case $|D\phi| < 1$. The set $\Sigma \subset S(\phi)$ where $|D\phi| = 1$ is called a *characteristic surface*, and this remark indicates that the initial value problem posed on a characteristic surface will have a significantly different nature from the usual Cauchy IVP.

The energy identity also implies stability in the energy norm: if $\Box u = \Box v$ and u and v are initially close in energy norm, then they remain close in the energy norm. Again this argument applies not only to (3), but also to the Cauchy problem with initial data posed on any strictly spacelike surface S. A precise formulation of the stability property is left as an exercise; suffice only to note that this stability justifies the emphasis placed on the Cauchy problem, and on spacelike surfaces.

The *stress–energy tensor* of a solution of $\Box u = 0$ is

$$T_{\alpha\beta} = u_\alpha u_\beta - \tfrac{1}{2}\left(\eta^{\gamma\delta} u_\gamma u_\delta\right)\eta_{\alpha\beta} \tag{16}$$

where $u_\alpha = D_\alpha u$, $\alpha = 0, \cdots, 3, (D_0 = \partial_t)$, and $\eta_{\alpha\beta}$ is the Lorentz metric, $\eta_{\alpha\beta} = diag(-1,1,1,1)$. It follows from $\Box u = 0$ that $T_{\alpha\beta}$ satisfies the conservation law

$$\eta^{\beta\gamma} D_\gamma T_{\alpha\beta} =: D^\beta T_{\alpha\beta} = 0, \tag{17}$$

and the energy identity (4) corresponds to the special case $\alpha = 0$. Comparing with the vector field X of (6) we find that

$$\tfrac{1}{2}(u_t^2 + |Du|^2)\,\partial_t - u_t u_i\,\partial_i = T_{0\alpha}\eta^{\alpha\beta}\,\partial_\beta,$$

and the energy density (8) may be rewritten as

$$\left(\tfrac{1}{2}(u_t^2 + |Du|^2) + u_t u_i \phi_i\right)\,dx = T_{0\alpha} n^\alpha \sqrt{1 - |D\phi|^2}\,dx,$$

where $n = (n^\alpha) = (1 - |D\phi|^2)^{-1/2}(\partial_t + \phi_i \partial_i)$ is the future timelike unit normal vector to $S(\phi)$ and $(1 - |D\phi|^2)^{1/2} dx$ is the metric volume measure on $S(\phi)$ induced by the spacetime metric $\eta_{\alpha\beta}$.

It is clear that the energy identities above are associated with the vector field $\partial_t = \partial_0$. More general energy identities may be obtained by replacing ∂_t by K, where $K = K^\alpha \partial_a$ is any *conformal Killing vector* in Minkowski space $\mathbf{R}^{3,1}$ i.e.

$$D_\alpha K_\beta + D_\beta K_\alpha = \frac{1}{2}(D^\gamma K_\gamma)\eta_{\alpha\beta} \tag{18}$$

where $K_\alpha = \eta_{\alpha\beta} K^\beta$. The conformal Killing equation (18) in Minkowski space is satisfied by the following vector fields:

(a) translations

$$\partial_\alpha, \quad \alpha = 0, \cdots, 3 \tag{19}$$

(b) rotations

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i, \quad 1 \le i, j \le 3 \tag{20}$$

(c) boosts (Lorentz rotations)

$$B_i = x_i \partial_t + t\partial_i, \quad i = 1, \cdots, 3 \tag{21}$$

(d) dilation

$$S = t\partial_t + x^i \partial_i = x^\alpha \partial_\alpha, \tag{22}$$

(e) conformal translations

$$C_{(\alpha)} = \left((\eta_{\gamma\delta} x^\gamma x^\delta) \delta^\beta_\alpha - x_\alpha x^\beta \right) \partial_\beta, \quad \alpha = 0, \cdots, 3. \tag{23}$$

The rotations and boosts may be grouped into the Lorentz rotations $L_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha$, $0 \le \alpha < \beta \le 3$ (where $x_\alpha = \eta_{\alpha\gamma} x^\gamma$), which generate the Lorentz group $SO(3,1)$; the 15 vector fields $(\partial_\alpha, L_{\alpha\beta}, S, C_{(\alpha)})$ generate the Lorentz conformal group $C(3,1)$. Since $\eta^{\alpha\beta} T_{\alpha\beta} = -u_\alpha u^\alpha$, by combining (17) and (18) we have

$$D^\beta (T_{\alpha\beta} K^\alpha) = -\tfrac{1}{4} D^\alpha K_\alpha \, u_\beta u^\beta$$

for any conformal Killing vector K. Equivalently

$$d(K^\alpha T_{\alpha\beta} \partial^\beta \iota_{\partial_\beta} \mu) = -\tfrac{1}{4} D^\alpha K_\alpha \, u_\beta u^\beta$$

and we may apply Stokes' theorem to obtain generalised energy estimates. The usual energy identity (4) follows from $K = \partial_t$; if we choose $K = C_{(0)} = (r^2 - t^2)\,\partial_t + 2t(t\partial_t + x^i \partial_i) = (r^2 + t^2)\,\partial_t + 2rt\partial_r$, $r = |x| = \left(\sum_1^3 (x^i)^2 \right)^{1/2}$, then we obtain the Morawetz identity

$$\frac{d}{dt} \int_{\mathbf{R}^3} \left(\tfrac{1}{2}(r^2 + t^2)(u_t^2 + |Du|^2) + 2tx^i u_t D_i u \right) dx = \int_{\mathbf{R}^3} 2t(-u_t^2 + |Du|^2)\, dx, \tag{24}$$

which is valid for any solution of $\Box u = 0$ with sufficiently rapid decay. This may be used to obtain more detailed information about decay of a solution; another more useful trick is to exploit the commutation relations

$$\begin{aligned} [\Box, \partial_\alpha] &= 0 \\ [\Box, L_{\alpha\beta}] &= 0 \\ [\Box, S] &= 2\Box \\ [\Box, C_{(\alpha)}] &= -4(\partial_\alpha + x_\alpha \Box) \end{aligned} \tag{25}$$

to derive energy estimates for derivatives of solutions. For example, if u is a (smooth) solution of (3) with $f = 0$ then $\Box(\partial_\alpha^k u) = 0$ and it follows that the higher order energies

$$E_k(t) = \int_{\mathbf{R}^3} \sum_{|A|=k} |D_A u|^2 \, dx \tag{26}$$

where $A = (\alpha_1, \cdots, \alpha_k)$ is a multi–index of length k, are all conserved,

$$\frac{d}{dt} E_k(t) = 0 \quad \Longleftrightarrow \quad E_k(t) = E_k(0). \tag{27}$$

This gives

Proposition 3 *Suppose u satisfies (3), with $f = 0$, $u_0 \in H^{k+1}(\mathbf{R}^3)$ and $u_1 \in H^k(\mathbf{R}^3)$, $k \geq 0$, then $u(t) \in H^{k+1}(\mathbf{R}^3)$.*

More detailed information (about the t–differentiability of $u(t)$) may also be determined from the higher order energies (26), and further energy-type estimates for the angular and dilation derivatives $(L_{\alpha\beta})^k u$, $S^k u$ are possible. If $E_k(u) < \infty$ for sufficiently large k, then the Sobolev embedding may be used to infer continuity of u and its derivatives.

We now turn briefly to the problem of establishing *existence* of solutions to the Cauchy IVP.

First, we observe that it suffices to treat the case where $f = 0$, for if we let $\phi(x, t; \tau)$ denote the solution of

$$\begin{aligned} \Box\phi(x, t; \tau) &= 0, \quad t \geq \tau \\ \phi(x, \tau; \tau) &= 0 \\ \phi_t(x, \tau; \tau) &= f(x, \tau), \end{aligned}$$

then

$$u(x, t) = \int_0^t \phi(x, t; \tau)\, d\tau$$

satisfies $\Box u = f$, $u(0) = u_t(0) = 0$. This trick is known as Duhamel's principle, and reduces the existence question to the homogeneous equation $\Box u = 0$.

One method of showing existence is to exhibit an explicit formula. This technique is however limiting, in that it will not extend simply to more general hyperbolic equations. For example, taking the Fourier transform in the spatial variables only and solving the resulting ordinary differential equations gives the expression

$$\hat{u}(\xi, t) = \hat{u}_0(\xi) \cos(|\xi| t) + |\xi|^{-1} \hat{u}_1(\xi) \sin(|\xi| t). \tag{28}$$

If u_0, u_1 are sufficiently smooth (for example, in the Schwartz class $\mathcal{S}$) then this formula may be inverted to give a formula for $u(x, t)$ satisfying the wave equation. Using the energy estimates and the fact that the Schwartz class is dense in $H^k(\mathbf{R}^3), k \geq 0$, we may then use an approximating sequence to construct solutions with less smooth initial data.

Another interesting explicit formula is based on the method of *spherical means.* If we denote the average over spheres by

$$M_\psi(x,r) = \frac{1}{4\pi}\int_{S^2} \psi(x+r\omega)\,d\omega \tag{29}$$

where $\omega \in S^2$ and $d\omega$ denotes the usual surface measure over S^2, then it is straightforward to show that $v(x,r) = M_\psi(x,r)$ satisfies the *Darboux equation*

$$v_{rr} + \frac{2}{r}v_r - \Delta v = 0 \tag{30}$$

with initial conditions $v(x,0) = \psi(x)$, $v_r(x,0) = 0$. It follows that

$$u(x,t) = tM_{u_1}(x,t) + \frac{d}{dt}\left(tM_{u_0}(x,t)\right) \tag{31}$$

satisfies the Cauchy problem (3) with $f = 0$. Huygen's Principle follows directly from (31), since the right hand side depends only on the values of u_0, u_1 on the sphere $|y-x| = t$. Thus, for example, initial data with support concentrated near $x = 0$ will give a solution with support concentrated near the outgoing light cone $C^+_{0,0}$. This phenomenon persists in all odd spatial dimensions; however solutions of the wave equation in even spatial dimension exhibit "tails" to the outgoing radiation. This effect is easily observed, in ripples on a pond for example.

The operator-theoretic approach to existence starts by setting $v = u_t$ and rewriting the wave equation as the first order system

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}(0) &= \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \in H^1_0(\mathbf{R}^3) \times L^2(\mathbf{R}^3). \end{aligned} \tag{32}$$

Let B denote the right hand operator and let $U = [u,v]^t$; we may then view the equation $U_t = BU$ as an evolution equation on the Hilbert space $F = H^1_0(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$. Note that $H^1_0(\mathbf{R}^3)$ is the completion of $C^\infty_c(\mathbf{R}^3)$ by the norm

$$||f||^2_{H^1_0} = \int_{\mathbf{R}^3} |Df|^2\,dx \tag{33}$$

and is thus not equal to $H^1(\mathbf{R}^3)$, since it contains functions which are not in L^2 because they do not decay fast enough near infinity. However by the Sobolev inequality,

$f \in H_0^1(\mathbf{R}^3)$ implies $f \in L^6(\mathbf{R}^3)$. The advantage in using H_0^1 is that then the operator B becomes skew-symmetric under the norm on F, ($W = [w, z]^t$)

$$
\begin{aligned}
(W, BU)_F &= \int_{\mathbf{R}^3} (D\bar{w} \cdot Dv + \bar{z}\Delta u) \\
&= \int_{\mathbf{R}^3} (D\bar{w} \cdot Dv - D\bar{z} \cdot Du) \\
&= -(BW, U)_F. \qquad (34)
\end{aligned}
$$

Since B is densely defined in F, and since B is closed (this follows from elliptic regularity, for example), it follows from the general theory of abstract evolution equations, as described in Derek Robinson's lectures, that B generates a unitary group $\exp(tB)$ on F. Then $U(t) = \exp(tB)U_0$ satisfies the evolution equation $U_t = BU$, and thus solves the Cauchy problem for the wave equation. This approach has the advantage that it automatically gives existence for solutions with initial data satisfying only the regularity required to have bounded energy norm.

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