SLOW STEADY FLOWS OF VISCOELASTIC LIQUIDS WITH CONSTITUTIVE EQUATIONS OF MAXWELL OR JEFFREYS TYPE

Michael Renardy

We discuss the existence of steady flows of certain viscoelastic liquids under the influence of a small body force. An existence proof is obtained by showing the convergence of an iteration scheme.

Existence theorems for steady flows of Newtonian fluids at low Reynolds number are well known (see [2]), but so far there have been no results for viscoelastic liquids, even in the case of slow flows perturbing rest. Although formal expansions in small parameters have been derived and used extensively [6], the justification of such expansions leads to difficult problems in singular perturbation theory, which have only been partially solved [4].

Here we give a summary of recent work [5], where an existence result for slow flows of a certain class of viscoelastic fluids was obtained. In these fluids the Cauchy stress tensor is related to the velocity gradient by a system of differential equations. A typical example is the "rubberlike liquid" [1], [3]:

$$\begin{gathered}\underline{\underline{T}} = \eta_0[\nabla\underline{u} + (\nabla\underline{u})^T] + \sum_{k=1}^{N} \underline{\underline{T}}_k \\ \frac{\partial \underline{\underline{T}}_k}{\partial t} + (\underline{u} \cdot \nabla)\underline{\underline{T}}_k - (\nabla\underline{u})\underline{\underline{T}}_k - \underline{\underline{T}}_k(\nabla\underline{u})^T + \lambda_k \underline{\underline{T}}_k \\ = \eta_k\lambda_k[(\nabla\underline{u}) + (\nabla\underline{u})^T] \;.\end{gathered} \tag{1}$$

Here $\nabla\underline{u}$ denotes the velocity gradient with components $(\nabla\underline{u})_{ij} = \frac{\partial u_i}{\partial x_j}$, and η_k, λ_k are positive constants (η_0 may be zero). There are many models in the literature which differ by replacing the terms $(\nabla\underline{u})\underline{\underline{T}}_k$ and $\underline{\underline{T}}_k(\nabla\underline{u})^T$ with other nonlinear combinations. Our method can also be applied to those models.

The fluid is assumed to fill a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. It satisfies no-slip conditions on the wall and is driven by a small, time-independent body force $\underline{f}$. For steady flows, we have the following equation of motion

$$\left.\begin{aligned} \rho(\underline{u}\cdot\nabla)\underline{u} - \operatorname{div}\underline{\underline{T}} + \nabla p - \underline{f} &= \underline{0} \\ \operatorname{div}\underline{u} &= 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\underline{u} = \underline{0} \qquad \text{on } \partial\Omega . \tag{2}$$

It would appear natural to replace the no-slip condition by a small prescribed velocity on the boundary. However, our proofs will only go through if this prescribed velocity is tangential to $\partial\Omega$. In fact, it is clear that inflow boundaries will require extra boundary condition and, in contrast to the Newtonian case, (2) is not well-posed with velocity boundary conditions alone.

In [5], we show that if $\|\underline{f}\|_{H^{s+1}(\Omega)}$ $(s \geq 1)$ is sufficiently small, then (1), (2) has a solution $\underline{u} \in H^{s+2}(\Omega)$, $p \in H^{s+1}(\Omega)$, $\underline{\underline{T}}_k \in H^{s+1}(\Omega)$. The proof is obtained by an iteration method, which alternates between solving "Stokes type" problems and hyperbolic problem whose characteristics are the streamlines of the flow (at this point, it is essential that streamlines do not cross $\partial\Omega$). Convergence of the iteration scheme is proved by showing first that iterates stay bounded in a certain norm and then using this information to show that the iteration is contracting in a weaker norm. We refer to [5] for details and only outline here how to set up the iteration method for the simplest case $N = 1$, $\eta_0 = 0$. We apply the divergence operator to (1) and substitute $\operatorname{div}\underline{\underline{T}}$ from (2), leading to a "Stokes-like" problem for $\underline{u}$ with a "modified pressure" $q = (\underline{u} \cdot \nabla)p + \lambda p$. The iteration used is as follows:

$$\underline{u}_0 = \underline{0},\ p^o = q^o = 0,\ \underline{\underline{T}}^o = \underline{\underline{0}} \tag{3a}$$

$$\left.\begin{aligned}
&\sum_{i,j} \underline{\underline{T}}^n_{ij} \frac{\partial^2 \underline{u}^{n+1}}{\partial x_i \partial x_j} + \eta\lambda\Delta\underline{u}^{n+1} - \rho(\underline{u}^n \cdot \nabla)(\underline{u}^n \cdot \nabla)\underline{u}^{n+1} \\
&\qquad - \nabla q^{n+1} = -[\nabla\underline{u}^n + (\nabla\underline{u}^n)^T]\nabla p^n - \underline{u}^n \cdot \nabla\underline{f} \\
&\qquad\qquad + (\nabla\underline{u}^n)\underline{f} - \lambda\underline{f} - \rho(\nabla\underline{u}^n)(\underline{u}^n \cdot \nabla)\underline{u}^n + \lambda\rho(\underline{u}^n \cdot \nabla)\underline{u}^n \\
&\operatorname{div} \underline{u}^{n+1} = 0,\ \underline{u}^{n+1} = \underline{0} \text{ on } \partial\Omega,\ \iiint_\Omega q^{n+1} = 0
\end{aligned}\right\} \tag{3b}$$

$$\left.\begin{aligned}
&(\underline{u}^{n+1} \cdot \nabla)p^{n+1} + \lambda p^{n+1} = q^{n+1} \\
&(\underline{u}^{n+1} \cdot \nabla)\underline{\underline{T}}^{n+1} - (\nabla\underline{u}^{n+1})\underline{\underline{T}}^{n+1} - \underline{\underline{T}}^{n+1}(\nabla\underline{u}^{n+1})^T \\
&\qquad + \lambda\underline{\underline{T}}^{n+1} = \eta\lambda[\nabla\underline{u}^{n+1} + (\nabla\underline{u}^{n+1})^T] .
\end{aligned}\right\} \tag{3c}$$

We note that iteration schemes of this or a similar nature are actually used to obtain numerical solutions.

<u>References</u>

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Mathematics Research Center and Department of Mathematics,
University of Wisconsin-Madison,
Madison, WI 53705
U.S.A.