SQUARE FUNCTIONS IN BANACH SPACES

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1. INTRODUCTION

Suppose that $(T_t : t \in \mathbb{R}^+)$ is a bounded semigroup of operators on the Banach space X, of type C_0, with infinitesimal generator A. In the classical case, where (T_t) is the Poisson semigroup acting on $L^p(\mathbb{T})$ or on $L^p(\mathbb{R})$, the "g-function", developed by A. Zygmund and his school, is one of the important tools of Fourier analysis. In a more general setting, the functions $g_n(f)$

$$g_n(f)(x) = \left(\int_{\mathbb{R}^+} dt/t \, \left| t^n \, \partial^n/\partial t^n \, T_t f(x) \right|^2 \right)^{\frac{1}{2}}$$

were considered by E.M. Stein [1], and used to shed light on heat diffusion semigroups, again on L^p- spaces. In particular, g-functions are often used to prove pointwise convergence results and multiplier theorems; see Stein's paper [3] for a survey of their role. Roughly speaking, the finiteness of $\| g_n(f) \|$ measures degrees of "orthogonality" of the functions $t \mapsto t^n \, \partial^n/\partial t^n \, T_t f$, for different t.

In this paper, we shall present a personal approach to g-functions, and connect them to multiplier theorems which develop Stein's work [2]. We describe the multiplier results briefly before returning to the g-functions.

For φ in $(0,\pi)$, we let Γ_φ be the following open cone:

$$\Gamma_\varphi = \{ z \in \mathbb{C} : |arg(z)| < \varphi \} .$$

We say that $H^\infty(\Gamma_\varphi)$ acts on A if there is an extension of the

resolvent calculus to a Banach algebra homomorphism $m \mapsto m(A)$ of $H^{\infty}(\Gamma_{\varphi})$ into $\mathcal{L}(X)$; under very mild restrictions on (T_t) or on X, this homomorphism, if it exists, is essentially unique. Since the spectrum of the generator of a holomorphic semigroup lies in a cone $\bar{\Gamma}_{\theta}$, with $\theta < \pi/2$, it is not unnatural to look for actions of $H^{\infty}(\Gamma_{\varphi})$ on A for all φ in $(\theta, \pi/2]$; such actions would generalise the spectral mapping theorem.

2. A REFORMULATION OF G-FUNCTIONS

We assume hereafter that the semigroup (T_t) is holomorphic, so that $t^n \, \partial^n/\partial t^n \, T_t$ is a bounded operator on X, which we take to be a function lattice. Formally,

$$t^n \, \partial^n/\partial t^n \, T_t = (tA)^n \, e^{-tA}$$

$$= (2\pi)^{-1} \int_{\mathbb{R}} du \, \Gamma(n - iu) t^{iu} A^{iu} ,$$

by Mellin inversion. Of course A^{iu} need not make sense as a bounded operator on X, but in this context it is possible to justify everything afterwards. By the Plancherel-Parseval theorem, we obtain our version of $g_n(f)$:

(1) $\quad (\int_{\mathbb{R}^+} \, dt/t \, |t^n \, \partial^n/\partial t^n \, T_t f(x)|^2)^{\frac{1}{2}}$

$$= (2\pi)^{-\frac{1}{2}} \, (\int_{\mathbb{R}} \, du \, |\Gamma(n - iu) \, A^{iu} f(x)|^2)^{\frac{1}{2}} .$$

By Mellin inversion, we may also write

$$m(A) = (2\pi)^{-1} \int_{\mathbb{R}} du \, \hat{n}(u) \, A^{iu} ,$$

where $\hat{}$ denotes the Fourier transform and $n = m \circ \exp$. For f in X, set $\tilde{f}(\cdot, u) = A^{iu} f(\cdot)$. We now have the formula

(2) $\qquad (m(A) f)^{\sim}(x, u) = (m(A) \otimes I) \tilde{f}(x, u)$

$$= (2\pi)^{-1} \int_{\mathbb{R}} dv \; \hat{n}(v) \; A^{iv} \; A^{iu} f(x)$$

$$= (2\pi)^{-1} \int_{\mathbb{R}} dv \; \hat{n}(-v) \; \tilde{f}(x,u-v)$$

$$= (I \otimes \check{n}*) \; \tilde{f}(x,u) \; ;$$

here $\check{n}(v) = \hat{n}(-v)$. The point of this formula is that an operator acting in the x-variables is switched into a convolution operator in the u-variables. If one has estimates for the g-function, then questions of boundedness of operators $m(A)$ are reduced to questions of boundedness of convolution operators on weighted L^2-spaces.

More precisely, if A^* denotes the adjoint of A , then for f in X and h in X^* ,

$$|\langle f,h\rangle| = C_\alpha \; \left| \int_{\mathbb{R}} du \; \Gamma(\alpha+iu) \; \Gamma(\alpha-iu) \; \langle A^{iu}f, \; A^{*-iu}h\rangle \right|$$

$$\le C_\alpha \; \langle g_\alpha(f), \; g_\alpha(h)\rangle \; ,$$

where the g-function in X^* is also denoted g_α , and

$$C_\alpha = 2^\alpha (2\pi\Gamma(2\alpha))^{-\frac{1}{2}} \; .$$

Therefore,

$$(3) \qquad |\langle m(A)f,h\rangle| \le C_\alpha \; \langle g_\alpha(m(A)f), \; g_\alpha(h)\rangle \; .$$

Now if n is bounded and holomorphic in the strip $S_{\pi/2}$, where

$$S_\varphi = \{z \in \mathbb{C} : \; |\mathrm{Im}(z)| < \varphi\} \; ,$$

then, for all sufficiently rapidly decreasing square integrable φ ,

$$\left(\int_{\mathbb{R}} du \; \left| \Gamma(\tfrac{1}{2}+iu) \; \check{n}*\varphi(u) \right|^2 \right)^{\frac{1}{2}}$$

$$\le C \; \|n\|_\infty \; \left(\int_{\mathbb{R}} du \; \left| \Gamma(\tfrac{1}{2}+iu) \; \varphi(u) \right|^2 \right)^{\frac{1}{2}} \; ,$$

and hence, from (1) and (2), with the natural definition of $g_{\frac{1}{2}}$,

(4)
$$g_{\frac{1}{2}}(m(a)f)(x) = \left(\int_{\mathbb{R}} du \; |\Gamma(\tfrac{1}{2}+iu)(I \otimes \check{n}_*) \; \tilde{f}(x,u)|^2\right)^{\frac{1}{2}}$$

$$\leq C \; \|n\|_\infty \left(\int_{\mathbb{R}} du \; |\Gamma(\tfrac{1}{2}+iu) \; \tilde{f}(x,u)|^2\right)^{\frac{1}{2}}$$

$$= C \; \|n\|_\infty \; g_{\frac{1}{2}}(f)(x) \; .$$

Consequently, if we know that

$$\|g_{\frac{1}{2}}(f)\| \leq C \; \|f\| \qquad f \in X$$

and

$$\|g_{\frac{1}{2}}(h)\| \leq C \; \|h\| \qquad h \in X^* \; ,$$

then we may deduce from (3) that, for m in $H^\infty(\Gamma_{\pi/2})$,

$$|\langle m(A)f,h\rangle| \leq C \; \|n\|_\infty \; \|f\| \; \|h\| \; ,$$

i.e. that $m(A)$ is a bounded operator on X . (The constant C varies from line to line!)

Similarly, if $M \in L^\infty(\mathbb{R}^+)$, and

$$m(\lambda) = \lambda \int_{\mathbb{R}^+} ds \; e^{-s\lambda} \; M(s) \; ,$$

then, with the obvious notation,

$$\check{n}(u) = \int_{\mathbb{R}^+} d\lambda \; \lambda^{iu-1} \; \lambda \int_{\mathbb{R}^+} ds \; e^{-s\lambda} M(s)$$

$$= \Gamma(1+iu) \int_{\mathbb{R}^+} ds \; s^{-1-iu} \; M(s)$$

$$= \Gamma(1+iu) \; \hat{N}(u) \; .$$

Fourier analysis proves the inequality

$$\left(\int_{\mathbb{R}} du \; |\Gamma(1+iu) \; [\Gamma(1+i\cdot) \; \hat{N}(\cdot) * \varphi](u)|^2\right)^{\frac{1}{2}}$$

$$\leq \|M\|_\infty \left(\int_{\mathbb{R}} du \; |\Gamma(2+iu) \; \varphi(u)|^2\right)^{\frac{1}{2}}$$

for suitable φ , and hence

(5)
$$g_1(m(A)f)(x) \leq \|M\|_\infty \; g_2(f)(x) \; ,$$

and estimates for g_1 in X^* and g_2 in X imply that

$$\|m(A)f\| \leq C \|M\|_\infty \|f\|$$

(cf. Stein [2], pp.62-63).

Comparison of these two results indicate that g_α-estimates for higher indices α yield stronger multiplier theorems. In this vein, we have the following theorem.

THEOREM 1. *Suppose that* X *is an* L^p-*space, with* p *in* $(1,\infty)$. *Then* $H^\infty(\Gamma_\varphi)$ *acts on* A *for all* φ *in* $(\theta, \pi/2)$ *if and only if*

$$\|g_n(f)\| \leq C_\varphi \sec(\varphi)^n \Gamma(n) \|f\| \qquad f \in X$$

uniformly in n $(n = 1,2,3,\ldots)$ *, and also*

$$\|g_n(h)\| \leq C_\varphi \sec(\varphi)^n \Gamma(n) \|h\| \qquad h \in X^* ,$$

for all φ *in* $(\theta, \pi/2)$.

The condition that X be an L^p-space can be relaxed; what we really need is the possibility of randomisation in X and in X^* . I thought originally that conditions on the operator norm $\|A^{iu}\|$ of A^{iu} , such as

$$\|A^{iu}\| \leq C_\varphi \exp(\varphi|u|) \qquad u \in \mathbb{R},$$

for all φ in $(\theta, \pi/2)$, may have been equivalent to those of the theorem (and indeed they are if X is a Hilbert space), but a counterexample shows this is not so in general. However, in an arbitrary Banach space, we have the following result.

THEOREM 2. *If* $H^\infty(\Gamma_\varphi)$ *acts on* A *for some* φ *, and if, for all* φ *in* $(\theta, \pi/2)$

$$\|A^{iu}\| \leq C_\varphi \exp(\varphi|u|) \qquad u \in \mathbb{R},$$

then $H^\infty(\Gamma_\varphi)$ *acts on* A *for all* φ *in* $(\theta, \pi/2)$.

After this digression, we return to our g-functions. Since these can control multipliers pointwise (as in (4) and (5) above), they can control pointwise convergence. For instance, if different m_j's in (4) have their $H^\infty(\Gamma_{\pi/2})$-norms bounded by C, then for all j,

$$g_{\frac{1}{2}}(m_j(A)f)(x) \le g_{\frac{1}{2}}(f)(x) \ ,$$

and so we have a maximal function estimate

$$\sup_j \ g_{\frac{1}{2}}(m_j(A)f)(x) \le C \ g_{\frac{1}{2}}(f)(x) \ .$$

To control $m_j(A)f$, rather than $g_{\frac{1}{2}}(m_j(A)f)$, two methods are available, one classical, and the other given some prominence by the author [1]. First, from (2),

$$(m_j(A)f)(x) = (2\pi)^{-1} \int_{\mathbb{R}} dv \ \hat{n}_j(v) \ \tilde{f}(x,v) \ ,$$

so

$$|(m_j(A)f)(x)|$$

$$\le (2\pi)^{-1} (\int_{\mathbb{R}} dv \ |\hat{n}_j(v) \ \Gamma(\alpha+iv)^{-1}|^2)^{\frac{1}{2}} (\int_{\mathbb{R}} dv \ |\tilde{f}(x,v) \ \Gamma(\alpha+iv)|^2)^{\frac{1}{2}}$$

$$= (2\pi)^{-1} (\int_{\mathbb{R}} dv \ |\hat{n}_j(v) \ \Gamma(\alpha+iv)^{-1}|^2)^{\frac{1}{2}} g_\alpha(f)(v) \ ;$$

estimates on the integrals on the last line can be used to control the maximal function. Alternatively, one can work as follows:

$$|(m_j(A)f)(x)|$$

$$\le (2\pi)^{-1} (\sup_v |\Gamma(\alpha+iv)^{-1} \ \hat{n}_j(v)|)(\int_{\mathbb{R}} dv \ |\Gamma(\alpha+iv) \ \tilde{f}(x,v)|) \ ,$$

and use estimates on the suprema to control the maximal function. Since, in a lattice

$$\|\int_{\mathbb{R}} dv \ |\Gamma(\alpha+iv) \ \tilde{f}(.,v)|\| \le \int_{\mathbb{R}} dv \ |\Gamma(\alpha+iv)| \ \|\tilde{f}(\cdot,v)\|$$

$$\le (\int_{\mathbb{R}} dv \ |\Gamma(\alpha+iv)| \ \||A^{iv}\||) \ \|f\| \ ,$$

estimates for $\|A^{iv}\|$ instead of g-function estimates can be used. Although the second technique seems easier than the first, and is applicable when X is a lattice which does not admit randomisation, there are also times when the former is more appropriate.

We conclude with an observation on orthogonal series. When studying the pointwise convergence of these, one sometimes considers their Abel summability, and this puts one naturally in a semigroup context. More precisely, if $\sum_{n \in \mathbb{N}} a_n \varphi_n(\cdot)$ is an orthogonal series, with (a_n) in $\ell^2(\mathbb{N})$ and (φ_n) an orthonormal basis in some Lebesgue space $L^2(M)$, then the Abel means may be written

$$\sum_{n \in \mathbb{N}} a_n \exp(-tn) \varphi_n(x) \qquad t \in \mathbb{R}^+ .$$

The techniques discussed above suggest looking at

$$\sum_{n \in \mathbb{N}} a_n n^{iu} \varphi_n(x) .$$

The following theorem can be proved by starting with the Carleson-Fefferman pointwise convergence theorem and applying randomisation arguments, and actually gives the deep result back by an easy trick.

THEOREM 3. *In any orthogonal system, given a non-negative increasing sequence* (b_n), *then the set of* u *in* $\mathbb{R}$ *for which*

$$\sum_{n \in \mathbb{N}} a_n b_n^{iu} \varphi_n(x)$$

fails to converge pointwise almost everywhere is of null Lebesgue measure.

Proof. Let

$$S_{N,u}(x) = \sum_{n=0}^{N} a_n b_n^{iu} \varphi_n(x) ,$$

and set

$$S_u^{\#}(x) = \sup_N |S_{N,u}(x)| .$$

It suffices to prove $\|S_u^{\#}\|_2$ finite for almost all u. Now

$$\left(\int_{\mathbb{R}} du \ (1+u^2)^{-2} \ \| S_u^\# \|_2^2 \right)^{\frac{1}{2}}$$

$$= \left(\int_X dx \int_{\mathbb{R}} du \ (1+u^2)^{-2} \ [\sup_N |\Sigma_{n=0}^N \ a_n \varphi_n(x) b_n^{iu}|]^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\int_X dx \ c^2 \int_{\mathbb{R}} du \ (1+u^2)^{-1} \ |\Sigma_{n \in \mathbb{N}} \ a_n \varphi_n(x) b_n^{iu}|^2 \right)^{\frac{1}{2}}$$

$$= C \left(\int_{\mathbb{R}} du \ (1+u^2)^{-1} \int_X dx \ |\Sigma_{n \in \mathbb{N}} \ a_n \varphi_n(x) b_n^{iu}|^2 \right)^{\frac{1}{2}}$$

$$= C \ \pi^{\frac{1}{2}} \ (\Sigma_{n \in \mathbb{N}} \ |a_n|^2)^{\frac{1}{2}} \ .$$

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