A SUPPORT MAP CHARACTERIZATION OF THE OPIAL CONDITIONS

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A Banach space [dual space] X satisfies the *weak* [*weak**] *Opial condition* if whenever (x_n) converges weakly [weak*] to x_∞ and $x_0 \neq x_\infty$ we have

$$\liminf_n \|x_n - x_\infty\| < \liminf_n \|x_n - x_0\|.$$

Zdzisław Opial [1967] introduced the weak condition to expand upon results of Browder and Petryshyn [1966] concerning the weak convergence of iterates for a nonexpansive selfmapping of a closed convex subset to a fixed point. In particular he observed that a uniformly convex Banach space with a weak to weak* sequentially continuous support mapping satisfies the weak condition. A *support mapping* is a selector for the *duality map*

$$D: X \to 2^{X^*} : x \mapsto \{f \in X^*: f(x) = \|f\|^2 = \|x\|^2\}$$

Uniform convexity is not sufficient for the weak to weak* sequential continuity of the unique support mapping. Browder [1966], and independently Hayes and Sims in connection with operator numerical ranges, had observed that the uniformly convex space $L_4[0, 1]$ does not have a weak to weak ($=$ weak*) continuous support mapping, while all of the sequence spaces ℓ_p $(1 < p < \infty)$ do. Opial [1967] demonstrated that with the exception of $p = 2$ none of the spaces $L_p[0, 1]$ have weak to weak continuous support mappings. Indeed, Fixman and Rao characterize $L_p(\Omega, \Sigma, \mu)$ spaces with a weak to weak continuous support mapping as those spaces for which every element of Σ with finite positive measure contains an atom.

That uniform convexity is not necessary is shown by the example of ℓ_1 with an equivalent smooth dual norm. That the unique support mapping is weak to weak* sequentially continuous follows from the norm to weak* upper semi-continuity of a duality mapping and the fact that ℓ_1 is a Schur space.

These early results were considerably improved by Gossez and Lami Dozo [1972]. In particular they show the following.

(1) The assumption of uniform convexity is unnecessary for Opial's result: *Any Banach space [dual space] with a weak [weak*] to weak* sequentially continuous support mapping satisfies the weak [weak*] Opial condition.* Indeed, their proof is easily adapted to show that a space has the weak [weak*] Opial condition if the Duality mapping is such that *given any weak* - neighbourhood* N *of zero, if* (x_n) *converges weakly [weak*] to* x_∞ *then eventually* $D(x_n) \cap (D(x_\infty) + \ ) \neq \phi$.

(2) The weak Opial condition implies the fixed point property for non-expansive self-maps of weak-compact convex sets. We give a direct proof [Van Dulst, 1982] which also applies in the weak* case.

Proposition 1: *Let* X *be a Banach space [dual space with a weak* - sequentially compact ball*[1]*] satisfying the weak [weak*] Opial condition. If* C *is a weak [weak*] - compact convex subset of* X, *than any non-expansive mapping* $T: C \to C$ *has a fixed point.*

Proof: Choose $x_0 \in C$, then since C is closed and convex, for any n the mapping $(1 - \frac{1}{n})T + \frac{1}{n}x_0$ is a strict contraction on C which by the Banach contraction mapping principle has a unique fixed point x_n in C.

Using the boundedness of C if follows that

$$\|x_n - Tx_n\| \to 0 .$$

Passing to a *subsequence* if necessary we may also assume that (x_n) converges weak [weak*] to a point x_∞.

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[1]For example; the dual of a separable space, or more generally the dual of any smoothable space.

Then,

$$\liminf_n \|Tx_\infty - x_n\| = \liminf_n \|Tx_\infty - Tx_n\|$$
$$\leq \liminf_n \|x_\infty - x_n\|$$

contradicting the weak [weak*] Opial condition unless $Tx_\infty = x_\infty$ □

Gossez and Lami Dozo [1972] in fact proved that the weak Opial condition implies *normal structure* thereby deducing the weak version of the above result via Kirk [1965].

Whether or not the weak* Opial condition implies normal structure for weak* comapct convex sets remains an open question.

(3) Weak to weak* sequential continuity of a support mapping is not necessary for the weak Opial condition. For $1 < p < q < \infty$ the space $(\ell_p \oplus \ell_q)_2$ satisfies the weak Opial condition, but [Bruck, 1969] the unique support mapping is not weak to weak continuous.

Karlovitz [1976] explored other connections between the Opial conditions and the space's goemetry, establishing a relationship with *approximate symmetry* in the Birkhoff-James notion of orthogonality.

The purpose of this note is to provide the following characterization of the weak [weak*] Opial condition in terms of support mappings.

Theorem 2: *The Banach space [dual space] X satisfies the weak [weak*] Opial condition if and only if whenever (x_n) converges weakly [weak*] to a non-zero limit x_∞ there exists a $\delta > 0$ such that eventually* $D(x_n)x_\infty \subset [\delta, \infty)$.

Proof: ($\Rightarrow$) Assume this were not the case, then by passing to subsequences we can find (x_n) converging weakly [weak*] to x_∞ with $\|x_n\| \geq \|x_\infty\| > 0$ and $f_n \in D(x_n)$ such that $\lim_n f_n(x_\infty) \leq 0$.

But

$$\begin{aligned}\lim_n \inf \|x_n\|^2 &= \lim_n \inf \|x_n - 0\|^2 \\ &> \lim_n \inf \|x_n - x_\infty\|^2 \\ &\geq \lim_n \inf f_n(x_n - x_\infty) \\ &= \lim_n \inf (\|x_n\|^2 - f_n(x_\infty)) \\ &= \lim_n \inf \|x_n\|^2 - \lim_n f_n(x_\infty),\end{aligned}$$

whence $\lim_n f_n(x_\infty) > 0$, a contradiction.

($\Leftarrow$ a modification of the proof in Gossez and Lami Dozo [1972].)

Using the integral representation for the convex function $t \mapsto \frac{1}{2}\|x + ty\|^2$ [Roberts and Varberg, 1973, 12 Theorem A] we have

$$\tfrac{1}{2}\|x + y\|^2 = \tfrac{1}{2}\|x\|^2 + \int_0^1 g^+(x + ty;\ y)\ dt$$

where

$$g^+(u;\ y) = \underset{h \to 0+}{\text{limit}} \ \frac{\frac{1}{2}\|u+hy\|^2 - \frac{1}{2}\|u\|^2}{h}$$

To establish the weak [weak*] Opial condition it suffices to show that if y_n converges weakly [weak*] to $y_\infty \neq 0$ then

$$\lim_n \inf \tfrac{1}{2}\|y_n\|^2 > \lim_n \inf \tfrac{1}{2}\|y_n - y_\infty\|^2 .$$

Now,

$$\tfrac{1}{2}\|y_n\|^2 = \tfrac{1}{2}\|y_n - y_\infty\|^2 + \int_0^1 g^+(y_n - y_\infty + ty_\infty;\ y_\infty)\ dt$$

So

$$\begin{aligned}\lim_n \inf \tfrac{1}{2}\|y_n\|^2 \geq{} & \lim_n \inf \tfrac{1}{2}\|y_n - y_\infty\|^2 \\ & + \lim_n \inf \int_0^1 g^+(y_n - y_\infty + ty_\infty;\ y_\infty)\ dt .\end{aligned}$$

By Fatou's lemma [Halmos, 1950] it is therefore sufficient to prove for each $t \in (0, 1)$ that

$$\lim_n \inf g^+(y_n - y_\infty + ty_\infty;\ y_\infty) > 0.$$

But,

$$g^+(y_n - y_\infty + ty_\infty;\ y_\infty) = \mathrm{Max}\{f(y_\infty):\ f \in D(y_n - y_\infty + ty_\infty)\}$$

[Barbu and Precupanu, 1978, §2.1 example 2° and Proposition 2.3] and $y_n - y_\infty + ty_\infty$ converges weakly [weak*] to $ty_\infty \neq 0$, so for n sufficiently large and some $\delta > 0$ we have $f(ty_\infty) > \delta$ for all $f \in D(y_n - y_\infty + ty_\infty)$

□

Remarks:

(1) Using the weak* - neighbourhood $\{g \in X^*: g(x_\infty) > -\frac{1}{2}\|x_\infty\|^2\}$ of 0 in X^* it is easily seen that the condition of the theorem is satisfied if the Duality mapping is sequentially weak [weak*] to weak* upper semi-continuous.

(2) From the details of the proof we see that if for some selection of f_n from $D(x_n)$ we have $\lim_n \inf f_n(x_\infty) > 0$, where x_n converges weak [weak*] to $x_\infty \neq 0$, then the same is true for all selections.

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