SOME REMARKS ON CHOQUET BOUNDARY

E. Tarafdar

ABSTRACT

In this note we prove some results related to Choquet boundary and Korovkin functions without using measure theory and make some interesting remarks. Our tool is Weierstrass-Stone type of arguments.

INTRODUCTION

Let X be a compact Hansdorff space and L a linear subspace of $C(X)$, the space of real valued continuous functions on X such that L contains the constant functions. If L separates the points of X and is separable, then the Choquet boundary $\partial_L X$ of X with respect to L (for definition see section 1) is a G_δ set. This result is due to Bishop and de Leeuw [5] and a somewhat different proof of this result has been given by Edwards [7]. The proofs in [5] and [7] use measure theory as the principal tool. In this note we have given a completely different proof of this and related results without using any measure theory. Our approach also shows that there is hardly any usefulness in a theorem of Bauer [2] and Edwards ([7], Theorem 2, p.118) (see our remark 1.2).

CHOQUET BOUNDARY L-AFFINE AND KOROVKIN FUNCTIONS.

Throughout this note X will denote a compact Hausdorff space with more than one point and $C(X)$ the Banach space of all real valued continuous functions f on X with sup norm, i.e. $\|f\| = \sup_{x \in X} |f(x)|$. L will denote a linear subspace, of $C(X)$ which contains the constant functions.

DEFINITIONS. Given $f \in C(X)$ we define two real valued functions on X by

$$f^*(x) = \inf \{q(x) : f \leq q \in L\}$$

$$\text{and} \quad f_*(x) = \sup \{q(x) : f \geq q \in L\} \quad , \quad x \in X ,$$

f^* and f_* are well defined as f being continuous on the compact space is bounded and L contains the constant functions.

The following properties are obvious: for $f, g \in C(X)$,

(i) $f_* \leq f \leq f^*$; (ii) $(f+g)^* \leq f^* + g^*$; (iii) $(\lambda f)^* = \lambda f^*$ for $\lambda \geq 0$;
(iv) $(-f)^* = -f_*$ and (v) $f \leq g$ implies $f^* \leq g^*$ and $f_* \leq g_*$.

A function $f \in C(X)$ is said to be L - *affine* if $f_* = f^*$. We set $L^* = \{f \in C(X) : f_* = f^*\}$ = the set of all L - affine functions. In view of the above properties (i) to (iv) L^* is a linear subspace of $C(X)$ and contains L as subspace.

The set $\partial_L X = \{x \in X : f_*(x) = f^*(x) \text{ for } \forall f \in C(X)\}$ is called the *Choquet boundary* or the *abstract boundary* of X with respect to L.

REMARK 1.1 *By virtue of (iv) we have*

$$\partial_L X = \{x \in X : f^*(x) = f(x) \text{ for } \forall f \in C(X)\}.$$

An equivalent definition of Choquet boundary is given in terms of Radon measures. A non negative Radon measure μ on X is a positive linear functional on $C(X)$. [By Riesz representation theorem for each such μ, there is a non negative regular Borel measure μ_0 on X such that $\mu(f) = \int f d\mu_0$ for $\forall f \in C(X)$]. For each $x \in X$, the point measure (Dirac measure e_x defined by $e_x(f) = f(x)$ for $\forall f \in C(X)$ is such a measure. Let M^+ denote the set of all non negative Radon measures and for $x \in X$, $M_x(L) = \{\mu \in M^+ : \mu(q) = q(x)$ for $\forall q \in L\}$. Then the Choquet boundary $\partial_L X = \left\{x : M_x(L) = \{e_x\}\right\}$. The equivalence of these two definitions can be proved by a simple application of Hahn - Banach theorem (for proof see [1], p,638). For yet another equivalent definition see [11], p.38. We note here that an important property of Choquet boundary $\partial_L X$ is that each $q \in L$ attains its global maximum and minimum on X at points of $\partial_L X$, i.e. for each $q \in L, \|q\| = |q(y)|$ for some point $y \in \partial_L X$. This is so called maximum principle (for proof see ([2], p.96) or ([11], p.40)).

A related concept is of Korovkin functions. A net $\{F_i\}_{i\in I}$ of positive linear operators $F_i : C(X) \to C(X)$ is called L-admissible if for each $q \in L$, the net $\{F_i(q)\}_{i\in I}$ converges uniformly to q on X, i.e.

$$\lim_{i\in I} \|F_i(q) - q\| = 0.$$

A function $f \in C(X)$ is called a Korovkin function with respect to L if the net $\{F_i(f)\}_{i\in I}$ converges uniformly to f on X for each L-admissible net $\{F_i\}_{i\in I}$ of positive linear operations $F_i : C(X) \to C(X)$.

We note that a function $f \in C(X)$ is a Korovkin function with respect to L if and only if f is an L - affine function (for proof of this result see [1] and [3] and for further development of the topic see [4] [6], [8] to [10], [12] to [15] and references thereof).

For each $f \in C(X)$ we denote by W_f the set $\{x \in X : f^*(x) > f(x)\}$. Then we have

(1) $\partial_L X = \bigcap_{f\in C(X)} \{W_f\}^c$ where A^c denote the complement of the set A.

This follows from our remark 1.1.

(2) For each $f \in C(X)$, W_f is a F_σ - set.

Since f^* and hence $f^* - f$ are upper semi continuous, for each positive integer n, $F_n = \{x \in X : f^*(x) - f(x) > \frac{1}{n}\}$ is a closed set, Hence

$$W_f = \bigcup_{n=1}^{\infty} F_n \quad \text{is a } F_\sigma \text{ set.}$$

In what follows the following lemmas will be useful.

LEMMA 1.1. L^* *is a closed linear subspace of* $C(X)$.

PROOF. We recall that $L^* = \{f \in C(X) : f^* = f = f_*\}$.
Let f be a limit point of L^* and $\varepsilon > 0$ be given. Then there exists $g \in L^*$ such that $\|f-g\| < \varepsilon/2$, i.e.

$$g(x) - \varepsilon/2 < f(x) < g(x) + \varepsilon/2 \quad \text{for} \quad \forall\, x \in X .$$

Now by (i) and (v)

$$(g-\varepsilon/2)_* (x) \leq f_*(x) \leq f(x) \leq f^*(x) \leq (g+\varepsilon/2)^*(x) \text{ for } \forall\, x \in X.$$

Since L^* is a linear subspace containing the constant functions, these reduce to

$$g(x) - \varepsilon/2 \leq f_*(x) \leq f(x) \leq f^*(x) \leq g(x) + \varepsilon/2 \quad \text{for} \quad \forall\, x \in X .$$

Hence

$$f^*(x) - f_*(x) = f^*(x) - g(x) + g(x) - f_*(x) < \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ for } \forall\, x \in X$$

Since ε is arbitrary, we have $f^* = f_*$, i.e. $f \in L^*$.

LEMMA 1.2. $L^*_a = \{f \in C(X) : f_*(a) = f^*(a)\}$ *is a closed linear subspace of* $C(X)$.

The proof is exactly the same as that of the above lemma if we restrict the argument to the point a only.

LEMMA 1.3. $\hat{L}_a = \{f \in C(X) : f^*(a) = f(a)\}$ *is a closed set.*

PROOF. Let f be a limit point of $\hat{L}_a$ and $\varepsilon > 0$ be given. Then we have $g \in \hat{L}_a$ such that $\|f-g\| < \varepsilon/2$, i.e.

$$g(x) - \varepsilon/2 < f(x) < g(x) + \varepsilon/2 \quad \text{for} \quad \forall\, x \in X.$$

By (v) $\qquad f(x) \leq f^*(x) \leq (g+\varepsilon/2)^*(x).$

Hence

$$g(a) - \varepsilon/2 < f(a) \le f^*(a) \le (g+\varepsilon/2)^*(a \le g^*(a) + \varepsilon/2 \quad \text{by (ii)}$$
$$= g(a) + \varepsilon/2$$

Thus $f^*(a) - f(a) = f^*(a) - g(a) + g(a) - f(a) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

As ε is arbitrary, $f^*(a) = f(a)$, i.e. $f \in \hat{L}_a$.

<u>THEOREM 1.1.</u> *If* L *separates the points of* X *(i.e. given points* $x, y \in X$ *with* $x \neq y$, *there exists* $q \in L$ *such that* $q(x) \neq q(y)$) *and is a vector lattice with usual ordering, then* $L^* = C(X)$, *i.e.* $\partial_L X = X$ *and every function is a Korovkin function with respect to* L.

<u>PROOF.</u> By Stone-Weierstrass theorem L is dense in $C(X)$. In lemma 1.1 we have proved that L^* is a closed subspace of $C(X)$. Also L^* always contains L as a subspace. Hence it follows that $L^* = C(X)$. The rest follows from our previous observations.

The following remark is the main objective of this section.

<u>REMARK 1.2.</u> *We now consider the following result which is originally due to Bauer* [2] *and of which a new proof has been given by Edwards* ([7], *Th.2, p.118).*

If L separates points and is a vector lattice, then $\partial_L X$ is a closed set and the restriction map $f \to f/\partial_L X$ from L into $R(\partial_L X)$ is an isometric linear and lattice isomorphism onto a dense subset of $R(\partial_L X)$. Furthermore, given $\tau \in M$, the set of Radon measures, there exists a unique $\mu = \mu_\tau \in M$ satisfying

(i) $\qquad \mu(g) = \tau(g) \quad$ for $\forall\, g \in L$;

(ii) $\qquad \operatorname{supp} \mu \subseteq \partial_L X$.

where supp μ denotes the support of μ and $R(\partial_L X)$ the set of real valued continuous functions on $\partial_L X$.

The map $\tau \to \mu_\tau$ in m is linear and it maps m^+ isometrically into itself.

In view of our theorem 1.1 the choquet boundary $\partial_L X$ in the above result coincides with X and hence this result loses all its interest.

2. In this section our aim is to prove without using measure theory a result of Bishop and Leeuw in the form as stated in [7].

We first note the following facts:

(2.1) For $f_1, f_2, \ldots . f_n \in C(X)$ with $f_i^*(x) = f_i(x)$ for $i = 1,2,\ldots,n$ we have

$$(f_1 \wedge f_2 \wedge \ldots \wedge f_n)^*(x) = (f_1 \wedge f_2 \wedge \ldots \wedge f_n)(x) \;;$$

and (2.2) for $f_1, f_2, \ldots, f_n \in C(X)$ with $f_{i*}(x) = f_i(x)$ for $i = 1,2,\ldots,n$ we have

$$(f_1 \vee f_2 \vee \ldots \vee f_n)_*(x) = (f_1 \vee f_2 \vee \ldots \vee f_n)(x).$$

To see (2.1) let us denote by $A_f(x) = \{q(x) : f \le q \in L\}$. Then it is clear that $A_{f_i}(x) \subseteq A_{f_1 \wedge f_2 \wedge \ldots \wedge f_n}(x)$ for each $i = 1,2,\ldots,n$. This implies that $(f_1 \wedge f_2 \wedge \ldots \wedge f_n)^*(x) \le f_i^*(x)$ for each $i = 1,2,\ldots,n$.
Thus

$$(f_1 \wedge f_2 \wedge \ldots \wedge f_n)^* \le (f_1^* \wedge f_2^* \wedge \ldots \wedge f_n^*)(x) = (f_1 \wedge f_2 \wedge \ldots \wedge f_n)(x)$$

Now (2.1) follows from the property (i) of section 1.
The proof of (2.2) is similar.

<u>PROPOSITION 2.1</u> Assume that L separates the points of X. Then $L^* = C(X)$, i.e. $\partial_L X = X$ (also each $f \in C(X)$ is a Korovkin function with respect to L) if for each $f \in C(X)$ with $f^* = f$ we have $f_* = f$ and for each $f \in C(X)$ with $f_* = f$ we have $f^* = f$.

PROOF. By (2.1),(2.2) and hypothesis L^* is a vector lattice. Hence by Stone-Weierstrass theorem $L^* = C(X)$ as L^* is closed.

A subset A of $C(X)$ is said to be a lower (an upper) semilattice if $f \wedge g (f \vee g) \in A$ whenever $f, g \in A$.

To prove our next result we will need the following lemma.

LEMMA 2.1 *Let* $A \subset C(X)$ *be a lower semi lattice and* $B \subset A$ *satisfy the following:*

(0) $f \vee g \in A$ *whenever* $f, g \in B$; *and*

(00) *given distinct points* x *and* y *in* X *and real numbers* a *and* b, *there exists* $f \in B$ *such that* $f(x) = a$ *and* $f(y) = b$. *Then* A *is dense in* $C(X)$.

PROOF. Let $f \in C(X)$ be arbitrary. Let $\varepsilon > 0$ be given. We choose a point $x \in X$ and fix it. Then for each $y \in X$ with $y \neq x$ by (00) there exists $q_y \in B$ such that $q_y(x) = f(x)$ and $q_y(y) = f(y)$. The set $O_y = \{u \in X : q_y(u) > f(u) - \varepsilon\}$ is an open set containing x and y. Clearly $\{O_y : y \in X \text{ and } y \neq x\}$ is an open covering of X. Since X is compact, there is a finite open sub covering, $O_{y_1}, O_{y_2}, \ldots, O_{y_n}$, say. The function $q_x = q_{y_1} \vee q_{y_2} \vee \ldots \vee g_{y_n}$ is in A by condition (0) and has the property that $q_x(x) = f(x)$ and $q_x(u) > f(u) - \varepsilon$ for all $u \in X$. This way for each $x \in X$ we obtain $q_x \in A$ having the above property. Again the set $G_x = \{u \in X : q_x(u) < f(u) + \varepsilon\}$ is an open set containing x. Thus $G_x : x \in X$ is an open covering of X and has, therefore, a finite subcovering, $G_{x_1}, G_{x_2}, \ldots, G_{x_m}$, say. Since A is a lower semilattice,

$g = q_{x_1} \wedge q_{x_2} \wedge \ldots \wedge q_{x_m} \in A$ and clearly has the property that $\| f - q \| < \varepsilon$.

Similarly we can prove the following lemma.

<u>LEMMA 2.2.</u> *Let* $A \subset C(X)$ *be an upper semilattice and* $B \subset A$ *satisfy the following:*

$$(0)' \qquad f \wedge g \in A \quad \textit{whenever} \quad f, g \in B;$$

and $\quad (00)' \quad$ *same as* (00) *of lemma 2.1*

Then A *is dense in* $C(X)$.

<u>THEOREM 2.1.</u> *If* L *separates the points of* X, *then* $a \in \partial_L X$ *if and only if* $|g|^*(a) = |g(a)|$ for $\forall\, g \in L$.

<u>PROOF.</u> If $a \in \partial_L X$, then $f^*(a) = f(a)$ for $\forall\, f \in C(x)$ and hence in particular $|g|^*(a) = |g|(a)$ for $\forall\, g \in L$. So let us assume that $|g|^*(a) = |g|(a)$ for $\forall\, g \in L$. We set $A = \hat{L}_a = \{f \in C(X) : f^*(a) = f(a)\}$ which is nonempty as $A \supset L$. By (2.1) A is a lower semilattice We take $B = L$. Then by using (i), (ii) and (iii) of section 1 and hypothesis we have for $p, q \in B$,

$$(p\vee q)(a) \le (p\vee q)^*(a) = \tfrac{1}{2}(p+q+|p-q|)^*(a) \le \tfrac{1}{2}[(p+q)^* + |p-q|^*(a)]$$

$$= \tfrac{1}{2}[(p+q)(a) + |p-q|(a)] = (p\vee q)(a).$$

Thus $(p\vee q)^*(a) = (p\vee q)(a)$ whenever $p,q \in L$. Hence B satisfies the condition (0) of lemma 2.1. Finally since L separates the points of X, given $x, y \in X$ with $x \neq y$, there exists $g \in L$ such that $g(x) \neq g(y)$. The function f defined by

$$f(u) = a\,\frac{g(u) - g(y)}{g(x) - g(y)} + b\,\frac{g(u) - g(x)}{g(y) - g(x)}, \quad u \in X \text{ is in } L$$

and satisfies the condition (00) of lemma 2.1. Thus by lemma 2.1 and 1.3, $A = C(X)$. By remark 1.1 this implies that $a \in \partial_L X$.

<u>REMARK 2.1</u> *If* M *is a linear subspace of* L *such that* M *contains the constant functions and separates the points of* X, *then* $a \in \partial_L X$ *if and only if* $|g|^*(a) = |g|(a)$ *for* $\forall\, g \in M$.

To prove this we take $A = \hat{L}_a$ and $B = M$ and repeat the same argument as in the above theorem.

<u>THEOREM 2.2.</u> *If* L *is a separable subspace of* $C(X)$ *and separates the points of* X, *then there exists a function* $f \in C(X)$ *such that*

$$\partial_L X = (W_f)^c.$$

<u>PROOF.</u> Let $\{g_n : n = 1,2,\ldots\}$ be a countable dense subset of L. Let M = linear span of the set $\{1, g_1, g_2, \ldots, g_n, \ldots\}$. Then M being a linear dense subspace of L also separates the points of X. Let $\{r_n : n=1,2,\ldots\}$ be an enumeration of the rationals. We define

$$f(x) = \sum_{m,n\geq 1} \frac{1}{2^{m+n}} \frac{1}{1+\|h_{mn}\|} \quad , \quad x \in X$$

where $h_{mn}(x) = |g_m(x) - r_n|$, $m,n = 1,2,\ldots$ and $x \in X$.
Let $a \notin \partial_L X$. Then by remark 2.1 there exists $g \in m$ such that $|g|^*(a) > |g|(a)$. Thus

(2.3) $\quad |g - g(a)|^*(a) \geq (|g| - |g|(a))^*(a) = |g|^*(a) - |g|(a) > 0.$

Now since g is of the form $a_0 + a_p g_p + \ldots\ldots + a_s g_s$, $a_0, a_p, \ldots, a_s$ being real numbers, we have by (V) of section 1

$|g - g(a)|^* \leq |a_p||g_p - g_p(a)|^*(a) + |a_s||g_s - g_s(a)|^*(a)$. This together with (2.3) above implies that $\delta = |g_p - g_p(a)|^*(a) > 0$ for some p.
Now we can find a rational number r_q such that $|g_p - r_q| < \frac{1}{2}\delta$.
Hence by (v) again

$$|h_{pq}|^*(a) = |g_p - r_q|^*(a) \geq |g_p - g_p(a)|^*(a) - |g_p - r_q| > \delta - \tfrac{1}{2}\delta = \tfrac{1}{2}\delta.$$

Also by (1) $|h_{m,n}|^*(a) \geq |h_{mn}|(a)$ for $\forall\, m,n = 1,2,\ldots.$
Hence it follows that $f^*(a) > f(a)$. Hence $a \in W_f$.
Thus $(\partial_L X)^c \subset W_f$. On the other hand $W_f \cap \partial_L X = \phi$ by definition of $\partial_L X$ and hence the conclusion of the theorem follows.

<u>COROLLARY 2.1</u> *If* L *is a separable subspace of* $C(X)$ *and separates the points of* X, *then* $\partial_L X$ *is a* G_δ *set.*

PROOF By theorem 2.2, there exists a function $f \in C(X)$ such that $\partial_L X = (W_f)^c$. Now since W_f is a F_σ set (vide (2) of section 1), $\partial_L X$ is a G_δ set.

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Department of Mathematics
University of Queensland
St. Lucia, Brisbane
AUSTRALIA 4067