SUBELLIPTIC OPERATORS AND LIE GROUPS

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Abstract. We review the theory of subelliptic operators on Lie groups. The principal themes are subcoercivity, weighting and Gaussian bounds.

1. Introduction

The book [Rob2] by the second author describes in detail the theory of strongly elliptic operators on Lie groups and gives a partial description of the theory of subelliptic operators. But since 1991 the subelliptic theory has undergone major changes and the aim of this note is to review these developments.

In order to place the subelliptic operators in perspective it is convenient to reconsider the main definitions and results for strongly elliptic operators.

Let G be a connected d-dimensional Lie group with Haar measure dg and $a_1, \ldots, a_d$ a vector space basis of the Lie algebra $\mathfrak{g}$ of G. If U is a weakly, or weakly*, continuous representation of G in a Banach space $\mathcal{X}$ then $A_i = dU(a_i)$, for $i \in \{1, \ldots, d\}$, will denote the generator of the one-parameter group $t \mapsto U(\exp(-ta_i))$. Thus $U(\exp(-ta_i)) = e^{-tA_i}$. In general the operators $A_1, \ldots, A_d$ do not commute and it is useful to introduce a multi-index notation for their products. Let $J(d) = \bigcup_{n=0}^{\infty} \{1, \ldots, d\}^n$ and for $\alpha = (i_1, \ldots, i_n)$ set $A^\alpha = A_{i_1} \circ \ldots \circ A_{i_n}$ and $|\alpha| = n$. Next define $\mathcal{X}_n(U) = \bigcap_{|\alpha| \leq n} D(A^\alpha)$ for each $n \in \mathbf{N}$ and $\mathcal{X}_\infty(U) = \bigcap_{n=1}^{\infty} \mathcal{X}_n(U)$. If U is the left regular representation in the space $L_p(G)$ then we denote these spaces by $L_{p;n}(G)$ and $L_{p;\infty}(G)$. A form of order m is defined as a function $C \colon J(d) \to \mathbf{C}$ such that $C(\alpha) = 0$ for all $|\alpha| > m$ and, in addition, $C(\alpha) \neq 0$ for an $\alpha \in J(d)$ with $|\alpha| = m$. We write $c_\alpha = C(\alpha)$ and consider the operator

$$dU(C) = \sum_{|\alpha| \leq m} c_\alpha \, A^\alpha \quad ,$$

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with domain $D(dU(C)) = \mathcal{X}_m(U)$, affiliated with the representation U. The form is called strongly elliptic if there is a $\mu > 0$ such that

$$\mathrm{Re} \sum_{|\alpha|=m} c_\alpha \, (i\xi)^\alpha \geq \mu |\xi|^m \tag{1}$$

for all $\xi \in \mathbf{R}^d$. The corresponding operator $dU(C)$ is then defined to be strongly elliptic.

The main results about strongly elliptic operators are summarized in the following theorem in which $|g|$ denotes the Riemannian distance of $g \in G$ to the identity element e of G.

Theorem 1.1. *Let C be a strongly elliptic form and U a continuous representation. Set $H = dU(C)$.*

I. *The operator H is closable and its closure generates a continuous holomorphic semigroup S on $\mathcal{X}$.*

II. *If the leading coefficients are real then S is holomorphic in the open right half plane.*

III. *If $t > 0$ then $S_t \mathcal{X} \subseteq \mathcal{X}_\infty(U)$.*

IV. *If U is unitary then $\|S_z\| \leq e^{\omega |z|}$ for some $\omega \geq 0$, uniformly for all z in a sector.*

V. *The semigroup has a smooth representation independent kernel $K_t \in L_1(G\,; dg)$, i.e.,*

$$S_t x = \int_G dg \, K_t(g) \, U(g) x$$

for all $t > 0$ and all $x \in \mathcal{X}$.

VI. *There exist $b, c > 0$ and $\omega \geq 0$ such that*

$$|K_t(g)| \leq c \, t^{-d/m} e^{\omega t} e^{-b(|g|^m t^{-1})^{1/(m-1)}}$$

for all $t > 0$ and $g \in G$. Moreover, there exist $b > 0$, $\omega \geq 0$ and for each $\alpha \in J(d)$ a $c > 0$ such that

$$|(A^\alpha K_t)(g)| \leq c \, t^{-d/m} t^{-|\alpha|/m} e^{\omega t} e^{-b(|g|^m t^{-1})^{1/(m-1)}}$$

for all $t > 0$ and $g \in G$.

VII. *If U is unitary then*

$$\mathrm{Re}(x, Hx) \geq \mu \max_{\alpha;\, |\alpha|=m/2} \|A^\alpha x\|^2 - \nu \, \|x\|^2 \tag{2}$$

for some $\mu > 0$ and $\nu \in \mathbf{R}$, uniformly for all $x \in D(H)$.

Statements I, II, III and V were first proved by Langlands in his thesis [Lan]. The Gaussian bounds for higher order operators, Statement VI, were established by Robinson [Rob1] and the quasi-contractivity, Statement IV, and the Gårding inequality, Statement VII, were proved in [BGJR]. A relatively short proof of the theorem can be found in [ElR4].

The strong point of the theorem is that it is a generic result; it is valid for all strongly elliptic operators on all Lie groups. The weak point is that it is local, e.g., the estimates on the semigroup kernel are imprecise for large values of t, the time variable. Nevertheless, the estimates of the theorem have the correct local behaviour. In particular the kernel has a singularity $t^{-d/m}$ as $t \to 0$. In 1991 local lower bounds were, however, not well understood. One striking result (see [Rob2], Section III.5) established that the kernel K is positive if, and only if, C is a real second-order form and then the kernel is bounded below by a second-order Gaussian. But the existence of local lower bounds for higher order operators had not been established.

Global estimates are, of course, sensitive to special features of the group, such as volume growth, and to the specific form of the operator. Therefore detailed global properties and asymptotic estimates require restriction to specific types of group and specific classes of operators. Much recent work has concentrated on global properties but these are not the focus of the current review. Our aim is to describe the extension of the local results to the case of subelliptic operators. A good starting point to illustrate these developments is the Laplacian $-\sum_{i=1}^{d} A_i^2$.

The Laplacian is a strongly elliptic operator which is known to be hypoelliptic in the left regular representation of the group. But a 1967 result of Hörmander for vector fields implied that hypoellipticity persists under a weaker condition on the basis $a_1, \ldots , a_d$. If $d' \leq d$ then $a_1, \ldots , a_{d'}$ is called an algebraic basis for $\mathfrak{g}$ if the $a_1, \ldots , a_{d'}$ are independent and the $a_1, \ldots , a_{d'}$ together with all their multi-commutators of some order n span $\mathfrak{g}$. The smallest number n is called the rank of the algebraic basis. Given the algebraic basis and a continuous representation U one can then define $\mathcal{X}'_n(U) = \bigcap_{\alpha \in J(d');\, |\alpha| \leq n} D(A^\alpha)$ for all $n \in \mathbf{N}$. Note that in this definition only the $A_1, \ldots , A_{d'}$ occur. Now define the sublaplacian $\Delta' = -\sum_{i=1}^{d'} A_i^2$ with domain $D(\Delta') = \mathcal{X}'_2(U)$. Then Hörmander's theorem [Hör] established that Δ' is hypoelliptic in the left regular representation of G. On the other hand, Δ' is a positive symmetric operator, so one could hope that Theorem 1.1, with some modifications, might extend to Δ'. The obvious necessary modification is the use of the modulus $|g|$. The appropriate subelliptic version is

$$|g|' = \sup\{\psi(g) - \psi(e) : \psi \in C_c^\infty(G),\ \psi \text{ real},\ \sum_{i=1}^{d'} |dL(a_i)\psi|^2 \leq 1\}$$

for all $g \in G$, where L is the left regular representation in $L_\infty(G)$. If $V(s)$ denotes the volume (Haar measure) of the ball $\{g \in G : |g|' < s\}$ then there is a $D' \in \mathbf{N}$ such that $V(s) \asymp s^{D'}$ for $s \leq 1$, i.e., there is a $c \geq 1$ such that $c^{-1} s^{D'} \leq V(s) \leq c\, s^{D'}$ for all $s \in \langle 0, 1]$.

The situation for the sublaplacian in 1990 was as follows.

Theorem 1.2. *Let Δ' be a sublaplacian associated with a continuous representation U of the Lie group G in a Banach space $\mathcal{X}$.*

I. *The operator Δ' is closable and its closure generates a continuous holomorphic semigroup S on $\mathcal{X}$.*

II. *If $t > 0$ then $S_t \mathcal{X} \subseteq \mathcal{X}_\infty(U)$.*

III. $\bigcap_{n=1}^\infty D((\overline{\Delta'})^n) = \mathcal{X}_\infty(U)$.

IV. *The semigroup has a smooth representation independent kernel $K_t \in L_1(G\,; dg)$, i.e.,*

$$S_t x = \int_G dg\, K_t(g)\, U(g) x$$

for all $x \in \mathcal{X}$.

V. *There exist $b, c, b', c' > 0$ and $\omega, \omega' \geq 0$ such that*

$$c'\, t^{-D'/2} e^{-\omega' t} e^{-b'(|g|')^2 t^{-1}} \leq K_t(g) \leq c\, t^{-D'/2} e^{\omega t} e^{-b(|g|')^2 t^{-1}}$$

for all $t > 0$ and $g \in G$. More generally, if $i \in \{1, \ldots, d'\}$ then there exist $b, c > 0$ and $\omega \geq 0$ such that

$$|(A_i K_t)(g)| \leq c\, t^{-D'/2} t^{-1/2} e^{\omega t} e^{-b(|g|')^2 t^{-1}}$$

for all $t > 0$ and $g \in G$.

Statements I–IV were proved by Jørgensen [Jør], and Statement V by Robinson [Rob2], Section IV.4. The presentation was, however, far from complete. In Statement I one would expect that the semigroup is holomorphic in the right half-plane, independent of the representation. Moreover, since the kernel K is a smooth function one could well expect Gaussian bounds for all higher order derivatives.

The most striking difference between Theorems 1.1 and 1.2 is that the first deals with m-th order operators whilst the second is only for particular real second-order operators. The principal barrier to the derivation of a subelliptic analogue of Theorem 1.1 was the absence of an appropriate definition of an m-th order subelliptic operator. Moreover, since the proof of Theorem 1.2 relied heavily on the maximum principle it was far from evident how it could be extended to higher order operators or even to second-order operators with complex coefficients. But the local aspects of these problems are now well understood and progress is being made on the global aspects.

There are two themes running through the review. The first is subcoercivity, a property phrased in terms of the Gårding inequality. The second is the notion of weights and weighted paths. Geometric aspects are computed by assigning different weights to different directions. The particular weighting is determined by such factors as the algebraic structure or the asymptotic behaviour.

2. Subcoercivity, subellipticity and weights

The first clue to the correct definition of subellipticity for higher-order operators is given by the observation that strong ellipticity is equivalent to a type of Gårding inequality, a property of subcoercivity. This follows by application of Theorem 1.1 to the left regular representation of $\mathbf{R}^d$ acting on $L_2(\mathbf{R}^d)$, i.e., the representation with generators $A_i = -\partial_i = -\partial/\partial x_i$, the partial differential operators on $\mathbf{R}^d$. Then (2) gives

$$\mathrm{Re} \sum_{\alpha;\, |\alpha|\leq m} (-1)^{|\alpha|} c_\alpha \, (\varphi, \partial^\alpha \varphi) \geq \mu \sum_{\alpha;\, |\alpha| = m/2} \|\partial^\alpha \varphi\|^2 - \nu \, \|\varphi\|^2$$

for all $\varphi \in L_{2;n}(\mathbf{R}^d)$. But replacing φ by its dilation φ_λ, defined by $\varphi_\lambda(x) = \lambda^{d/2}\varphi(\lambda x)$, and taking the limit $\lambda \to 0$ one readily deduces that

$$\mathrm{Re} \sum_{\alpha;\, |\alpha| = m} c_\alpha \, (\varphi, \partial^\alpha \varphi) \geq \mu \sum_{\alpha;\, |\alpha| = m/2} \|\partial^\alpha \varphi\|^2 \tag{3}$$

for all $\varphi \in L_{2;n}(\mathbf{R}^d)$. Then, however, a Fourier transform argument establishes that (3) implies (1). The conclusion is that the strong ellipticity condition (1) for the form C is equivalent to the strong Gårding inequality (3) on $L_2(\mathbf{R}^d)$. This suggests that subellipticity of an operator should be characterized by a Gårding inequality for the principal part, i.e., the highest order term, acting on the L_2-space over an auxiliary group, i.e., the analogue of $\mathbf{R}^d$. This motivates the following definition.

Let $a_1, \ldots, a_{d'}$ be an algebraic basis for $\mathfrak{g}$ and $C\colon J(d') \to \mathbf{C}$ a form of order m. Further let $\tilde{\mathfrak{g}} = \mathfrak{g}(d', s)$ denote the nilpotent Lie algebra with d' generators which is free of step s, i.e., the quotient of the free Lie algebra with d' generators by the ideal generated by the commutators of order at least $s+1$, and $\widetilde{G} = G(d', s)$ the connected, simply connected, Lie group with Lie algebra $\tilde{\mathfrak{g}}$. We denote the generators of $\tilde{\mathfrak{g}}$ by $\tilde{a}_1, \ldots, \tilde{a}_{d'}$ and their representatives in the left regular representation of $\widetilde{G}$ by $\tilde{A}_1, \ldots, \tilde{A}_{d'}$. Then C is defined to be *subcoercive form* *of step s* if

$$\operatorname{Re} \sum_{\alpha;\, |\alpha|=m} c_\alpha \, (\varphi, \widetilde{A}^\alpha \varphi) \geq \mu \sum_{\alpha;\, |\alpha|=m/2} \|\widetilde{A}^\alpha \varphi\|^2$$

for some $\mu > 0$, uniformly for all $\varphi \in L_{2;m}(\widetilde{G};\, d\tilde{g})$ where $d\tilde{g}$ denotes Haar measure on $\widetilde{G}$. Now for a general representation $(\mathcal{X}, G, U)$ of a Lie group G with the Lie algebra $\mathfrak{g}$ we define the m-th order operators $dU(C)$ associated with subcoercive forms of step s to be subcoercive of step s. This is a direct generalization of the previous definition of strong ellipticity in which $d' = d$, $s = 1$ since then $\widetilde{G} = \mathbf{R}^d$.

Examples of subcoercive operators of step s, for any $s \in \mathbf{N}$, are $(-1)^{m/2} \sum_{i=1}^{d'} A_i^m$ and $(\Delta')^{m/2}$. The main theorem for subcoercive operators of step s is as follows.

Theorem 2.1. *Let $C \colon J(d') \to \mathbf{C}$ be a subcoercive form of order m and step s and U a continuous representation of G. Set $H = dU(C)$. If s is larger than or equal to the rank of the algebraic basis $a_1, \ldots, a_{d'}$ in $\mathfrak{g}$ then the following are valid.*

I. *The operator H is closable and its closure generates a continuous holomorphic semigroup S on $\mathcal{X}$.*

II. *The semigroup S is holomorphic in a representation independent sector.*

III. *If $t > 0$ then $S_t \mathcal{X} \subseteq \mathcal{X}_\infty(U)$.*

IV. *If U is unitary then $\|S_z\| \leq e^{\omega |z|}$ for some $\omega \geq 0$, uniformly for all z in a sector.*

V. *The semigroup has a smooth representation independent kernel $K_t \in L_1(G\,; dg)$, i.e.,*

$$S_t x = \int_G dg \, K_t(g) \, U(g) x$$

for all $x \in \mathcal{X}$.

VI. *There exist $b, c > 0$ and $\omega \geq 0$ such that*

$$|K_t(g)| \leq c \, t^{-D'/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

for all $t > 0$ and $g \in G$. More generally, there exist $b > 0$, $\omega \geq 0$ and for each $\alpha \in J(d')$ a $c > 0$ such that

$$|(A^\alpha K_t)(g)| \leq c \, t^{-D'/m} t^{-|\alpha|/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

for all $t > 0$ and $g \in G$.

VII. *If U is unitary then*

$$\operatorname{Re}(x, Hx) \geq \mu \max_{\substack{\alpha \in J(d') \\ |\alpha| = m/2}} \|A^\alpha x\|^2 - \nu \, \|x\|^2$$

for some $\mu > 0$ and $\nu \in \mathbf{R}$, uniformly for all $x \in D(H)$.

The proof [ElR1] [ElR3] involves parametrix arguments broadly analogous to those used by Langlands in his thesis [Lan] to prove the initial statements of Theorem 1.1. Langlands realized that G could be locally approximated by $\mathbf{R}^d$, through the exponential map, and hence the kernel of the resolvent $R_\lambda(H) = (\lambda I + H)^{-1}$ of H on G could be estimated in terms of the corresponding kernel $R_\lambda(H)$ on $\mathbf{R}^d$. In fact $R_\lambda(H)$, on G, can be expressed as a series expansion in $R_\lambda(H)$, on $\mathbf{R}^d$, analogous to time-independent perturbation theory, i.e., the expansion of the resolvent of a perturbed operator in terms of the resolvent of the unperturbed operator. Subsequent arguments have expressed the semigroup kernel on G in terms of the kernel on $\mathbf{R}^d$ by an analogue of the time-dependent perturbation expansion. The proof of Theorem 2.1 is broadly similar but uses the Rothschild–Stein [RoS] extension method to approximate G by $\widetilde{G}$. Since the group $\widetilde{G}$ usually has dimension strictly larger than d the local approximation is more complex. Nevertheless the semigroup kernel on G can be expressed as a series expansion in the kernel on $\widetilde{G}$ with the extra dimensions integrated out.

It is also worth noticing that the theorem immediately implies an equivalence between properties of separate and joint differentiability. Goodman [Goo], Theorem 1.1, established that one has the following identity $\bigcap_{\alpha \in J(d)} D(A^\alpha) = \bigcap_{i=1}^{d} \bigcap_{n=1}^{\infty} D(A_i^n)$ for any representation U, if $a_1, \ldots, a_d$ is a vector space basis for $\mathfrak{g}$, i.e., the subspaces of separate and joint C^∞-vectors coincide. But as a corollary of Theorem 2.1 a similar identity is valid for an algebraic basis.

Corollary 2.2. *If $a_1, \ldots, a_{d'}$ is an algebraic basis for $\mathfrak{g}$ and U a continuous representation of G then*

$$\mathcal{X}_\infty(U) = \bigcap_{\alpha \in J(d)} D(A^\alpha) = \bigcap_{i=1}^{d'} \bigcap_{n=1}^{\infty} D(A_i^n) \quad .$$

Further developments of the subelliptic theory have centred on various notions of weighting. The idea of assigning different weights to different directions is very natural in the subelliptic setting as it underlies the associated local geometry. To illustrate this let $a_1, \ldots, a_{d'}$ be an algebraic basis and assign the directions $a_1, \ldots, a_{d'}$ weight one. Then the commutators $[a_i, a_j]$ which are not in $\operatorname{span}\{a_1, \ldots, a_{d'}\}$ are assigned weight two, new directions obtained by triple commutators $[a_i, [a_j, a_k]]$ are assigned weight three, etc. Then it follows that the local dimension D' equals the sum of the weights assigned in this manner, i.e., $D' = \sum_{i=1}^{d} w_i$. Moreover, the subelliptic distance $|\cdot|'$ can be interpreted as a minimum over paths whose lengths are calculated with the appropriate weighting of the directions (see [NSW]).

Subsequent to the proof of Theorem 2.1 it was realized that a similar result could be proved for strongly elliptic operators with fairly arbitrary weightings. Let $a_1, \ldots, a_d$ of $\mathfrak{g}$ be a full vector space basis and $w_1, \ldots, w_d$ be a corresponding set of weights, i.e., positive integers, associated with the directions $a_1, \ldots, a_d$. If $\alpha = (i_1, \ldots, i_n) \in J(d)$ is a multi-index set

$$\|\alpha\| = w_{i_1} + \ldots + w_{i_n} \quad .$$

Further set $w = \operatorname{lcm}(w_1, \ldots, w_d)$ and define a modulus $\|\cdot\|$ on $\mathbf{R}^d$ by

$$\|\xi\|^{2w} = \sum_{i=1}^{d} |\xi_i|^{2w/w_i} \quad .$$

Then a form $C\colon J(d) \to \mathbf{C}$ of order m is called a weighted strongly elliptic form if there is a $\mu > 0$ such that

$$\operatorname{Re} \sum_{\|\alpha\|=m} c_\alpha \, (i\xi)^\alpha \geq \mu \|\xi\|^m$$

for all $\xi \in \mathbf{R}^d$.

One can again define a modulus $|\cdot|_w$ on G determined by the weighted basis (see [NSW]). For all $\delta > 0$ let $C(\delta)$ be the set of all absolutely continuous functions $\gamma\colon [0,1] \to G$ which satisfy the differential equation

$$\dot{\gamma}(s) = \sum_{i=1}^{d} \gamma_i(s) \, dL(a_i)\Big|_{\gamma(s)}$$

almost everywhere with $|\gamma_i(s)| < \delta^{w_i}$ for all $i \in \{1, \ldots, d\}$ and $s \in [0,1]$. Here L is the left regular representation on $L_\infty(G)$. Then

$$|g|_w = \inf\{\delta > 0 : \exists_{\gamma \in C(\delta)} [\, \gamma(0) = g \text{ and } \gamma(1) = e \,]\}$$

for all $g \in G$, and it follows that $V(s)$, the volume of the ball $\{g \in G : |g|_w < s\}$, has the property $V(s) \asymp s^{D'}$ for $s \leq 1$ where again $D' = \sum_{i=1}^d w_i$.

If $w_i = 1$ for all $i \in \{1, \ldots, d\}$ then obviously

$$(4) \qquad [a_i, a_j] \in \operatorname{span}\{a_k : k \in \{1, \ldots, d\}, \; w_k \leq w_i + w_j - 1\}$$

for all $i, j \in \{1, \ldots, d\}$, so in the unweighted case the condition (4) is valid. It is at first somewhat surprising that Condition (4) is sufficient (and in general also necessary) to prove generator theorems [ElR2].

Theorem 2.3. *Let C be a weighted strongly elliptic form and U a continuous representation of G. Set $H = dU(C)$. If Condition* (4) *is valid then statements similar to Statements* I–VII *of Theorem* 2.1 *are valid with the new definition of D' and the replacements $|g|' \to |g|_w$, $|\alpha| \to \|\alpha\|$ and $J(d') \to J(d)$ in Statements* VI *and* VII.

A typical example of a weighted strongly elliptic operator is the operator

$$\frac{\partial^4}{\partial x^4} - \frac{\partial^6}{\partial y^6}$$

on $L_p(\mathbf{R}^2)$. If one gives the x-direction weight 3 and the y-direction weight 2 then the operator has weighted order 12. But this operator can also be viewed in a different manner due to Rockland [Roc].

Let G be a homogeneous (nilpotent) Lie group with the family of dilations $(\gamma_t)_{t>0}$, i.e., a one-parameter group of automorphisms with the action $\gamma_t(a_i) = t^{w_i} a_i$ for some basis $a_1, \ldots, a_d$ of $\mathfrak{g}$ and some positive $w_1, \ldots, w_d$. Further let $|\cdot|_w$ be a homogeneous modulus, i.e., $|\gamma_t(g)|_w = |g|_w$ for all $g \in G$. Then $C\colon J(d) \to \mathbf{C}$ is called a positive Rockland form, and the operator $dU(C)$ a positive Rockland operator, whenever $dL(C)$ is a homogeneous, hypoelliptic, symmetric, positive, operator in $L_2(G)$. Here L denotes the left regular representation. Again there exists a $D' > 0$ such that $\mathrm{Vol}\{g \in G : |g|_w < s\} \asymp s^{D'}$, for $s \leq 1$, again one can define the weighted length of a multi-index $\alpha = (i_1, \ldots, i_n)$ with the $i_j \in \{1, \ldots, d\}$ by $\|\alpha\| = w_{i_1} + \ldots + w_{i_n}$ and again there is a theorem similar to Theorem 2.3.

Theorem 2.4. *Let C be a positive Rockland form and U a continuous representation. Set $H = dU(C)$. Then statements similar to Statements* I–VII *of Theorem* 2.1 *are valid with the new definition of D' and the replacements $|g|' \to |g|_w$, $|\alpha| \to \|\alpha\|$ and $J(d') \to J(d)$. The semigroup S is always holomorphic in the open right half plane.*

Statements I–III, V and VI have been proved in [AER], Statement VII with $\nu = 0$ in [ElR5] and Statement IV is a consequence of the Gårding inequality in Statement VII. Statement VI has been proved independently in [DHZ].

The proof of Theorem 2.4 is different to the proof of Theorem 2.1 insofar it does not use parametrix arguments. The proof is based on operator estimates of Helffer and Nourrigat [HeN], supplemented by scaling arguments, and in this respect is closer to the reasoning used to establish Theorem 1.2.

The above results were all known in 1994 and gave a rather diffuse description of the theory of (sub)elliptic operators with good kernel properties. There was one common conclusion following from various different assumptions. The only significant difference was in the method of weighting. In the next section we introduce the concept of weighted subcoercive operators and explain how this concept unifies all these results.

3. Weighted subcoercive operators

Let G be a connected d-dimensional Lie group and $a_1, \ldots, a_{d'}$ an algebraic basis of the Lie algebra $\mathfrak{g}$. Assign weights $w_1, \ldots, w_{d'} \in \mathbf{N}$ to the basis and again define the weighted length of a multi-index $\alpha = (i_1, \ldots, i_n) \in J(d')$ by $\|\alpha\| = w_{i_1} + \ldots + w_{i_n}$. Let $m \in \mathbf{N}$ and suppose that $m \in 2w_i\mathbf{N}$ for all $i \in \{1, \ldots, d'\}$. Then a form $C \colon J(d') \to \mathbf{C}$ of weighted order m is called a G-weighted subcoercive form if the operator $dL(C)$ satisfies a (weak) Gårding inequality in the left regular representation L on $L_2(G)$, i.e.,

$$\mathrm{Re}(\varphi, dL(C)\varphi) \geq \mu \max_{\substack{\alpha \in J(d') \\ \|\alpha\| = m/2}} \|A^\alpha \varphi\|_2^2 - \nu\, \|\varphi\|_2^2$$

for some $\mu > 0$ and $\nu \in \mathbf{R}$, uniformly for all $\varphi \in C_c^\infty(G)$. It clearly follows from the last statement, Statement VII, in the previous theorems that all operators considered in the previous sections are associated with appropriate G-weighted subcoercive forms.

In the present situation one can associate a modulus $|\cdot|'$ on G with the weighted algebraic basis. For all $\delta > 0$ let $C(\delta)$ be the set of all absolutely continuous functions $\gamma \colon [0,1] \to G$ satisfying

$$\dot{\gamma}(s) = \sum_{i=1}^{d'} \gamma_i(s)\, dL(a_i)\Big|_{\gamma(s)}$$

almost everywhere with $|\gamma_i(s)| < \delta^{w_i}$, for all $i \in \{1, \ldots, d'\}$ and $s \in [0,1]$. Then

$$|g|' = \inf\{\delta > 0 : \exists_{\gamma \in C(\delta)} [\, \gamma(0) = g \text{ and } \gamma(1) = e\,]\}$$

for all $g \in G$. It again follows from [NSW] that there is a $D' \in \mathbf{N}$ such that the volume of the ball $\{g \in G : |g| < s\}$ has the property $V(s) \asymp s^{D'}$ for $s \leq 1$.

The main theorem is also valid for G-weighted subcoercive forms [ElR6].

Theorem 3.1. *Let C be a G-weighted subcoercive form of weighted order m and U a continuous representation of G. Set $H = dU(C)$. Then statements similar to Statements* I–VII *of Theorem* 2.1, *are valid with the new definitions of D' and $|g|'$ and the replacement $|\alpha| \to \|\alpha\|$.*

The proof of the general theorem again relies on parametrix arguments but now the local approximation to G is a group G_0 of the same dimensions which is defined by a contraction process. The idea behind the construction of G_0, due to Kashiwara and Vergne [KaV], is to define a group of maps $t \mapsto \gamma_t$, where for all $t > 0$ the map $\gamma_t \colon \mathfrak{g} \to \mathfrak{g}$ is such that $\gamma_t(a_i) = t^{w_i} a_i$. Subsequently one constructs the Lie bracket

$$[a, b]_0 = \lim_{t \to 0} \gamma_t^{-1}\Big([\gamma_t(a), \gamma_t(b)]\Big) \tag{5}$$

for $a, b \in \mathfrak{g}$. Then G_0 is the connected, simply connected, Lie group with Lie algebra $(\mathfrak{g}, [\,\cdot\,,\cdot\,]_0)$. The group G_0 is homogeneous with dilations γ. Although the limiting process (5) is called a contraction in Lie theory the group G_0 corresponds to a 'blow-up' of G in the terms of differential geometry, i.e., a local magnification with respect to the dilations γ. It is for this reason that it is particularly appropriate for parametrix approximations.

The group G_0 has several interesting features. First $G_0 = \mathbf{R}^d$ if, and only if, $d' = d$, i.e., $a_1, \ldots, a_{d'}$ is a vector space basis, and the weights satisfy (4). Secondly, C is G-weighted subcoercive if, and only if, it is G_0-weighted subcoercive. This is a far-reaching generalization of the earlier observation that an operator is strongly elliptic if, and only if, it satisfies the Gårding inequality (3). Thirdly, there is an example that shows the assumptions of Theorem 2.1 are strictly stronger than those of Theorem 3.1. These properties, among others, can be found in [ElR6].

Note that the Gaussian bounds on the kernel and its derivatives given by the theorem imply that for all $\alpha \in J(d')$ there are $c > 0$ and $\omega \geq 0$ such that $\|A^\alpha S_t\|_{\mathcal{X} \to \mathcal{X}} \leq c\, t^{-\|\alpha\|/m}\, e^{\omega t}$ for all $t > 0$. Surprisingly, a converse of the above theorem is valid.

Theorem 3.2. *Let C be a form of weighted order m and L the left regular representation of G in $L_2(G)$. Set $H = dL(C)$. Suppose the following three conditions are valid.*

- **I.** *The operator H is closable and its closure generates a continuous holomorphic semigroup S on $L_2(G)$.*
- **II.** *$\|S_z\| \leq e^{\omega|z|}$ for some $\omega \geq 0$, uniformly for all z in a sector.*
- **III.** *There exist $c > 0$ and $\omega \geq 0$ such that $\|A_i S_t\|_{2\to 2} \leq c\, t^{-w_i/m} e^{\omega t}$ for all $t > 0$ and $i \in \{1, \ldots, d'\}$.*

Then the form C is G-weighted subcoercive.

There are examples of a G-weighted subcoercive operators which do not belong to any of the classes of elliptic operators described in the earlier sections. Therefore Theorem 3.1 is not only a unification of the earlier results but also a genuine extension. A simple example of an operator which fits the present setting but is not covered by the previous results can be constructed for $G = SO(3)$, the rotations in $\mathbf{R}^3$. If a_1, a_2, a_3 is a basis of $so(3)$ satisfying $[a_1, a_2] = a_3$, $[a_2, a_3] = a_1$ and $[a_3, a_1] = a_2$ then a_1, a_2 is an algebraic basis. If one specifies weights $w_1 = 3$ and $w_2 = 2$ then the operator

$$H = A_1^4 - A_2^6 - A_1^2 A_2^3$$

has (weighted) order 12 and satisfies the Gårding inequality because a straightforward calculation gives

$$\mathrm{Re}(\varphi, H\varphi) \geq 2^{-1}(\|A_1^2\varphi\|_2^2 + \|A_2^3\varphi\|_2^2) \geq 2^{-1} \max_{\|\alpha\|=6} \|A^\alpha\varphi\|_2^2 \quad .$$

Hence H generates a holomorphic semigroup with a smooth kernel satisfying Gaussian bounds in each continuous representation of the group.

Finally each weighted subcoercive operator satisfies good regularity properties if the representation is either unitary, or a left regular representation in $L_p(G)$.

Theorem 3.3. *Let $C\colon J(d') \to \mathbf{C}$ be a G-weighted subcoercive form and U a unitary representation of G or the left regular representation in $L_p(G)$ for some $p \in \langle 1, \infty\rangle$. Set $H = dU(C)$. Then*

I. *H is closed*

II. *$\bigcap_{\|\alpha\|=nw} D(A^\alpha) = D((\lambda I + H)^{nw/m})$ for all $n \in \mathbf{N}$ and all large $\lambda > 0$, where $w = \mathrm{lcm}(w_1, \ldots, w_{d'})$.*

III. *$\bigcap_{\|\alpha\|=nw} D(A^\alpha) = \bigcap D(A_i^{nw/w_i})$.*

IV. *If $\lambda > 0$ is large enough then the operator $\lambda I + H$ has a bounded H_∞-functional calculus.*

Recently Smulders [Smu], Theorem 5.1.2, proved that the statements of the previous theorem are also valid if U is a continuous representation of G in a space $L_p(M\,;\mu)$, where $(M\,;\mu)$ is a σ-finite measure space and $p \in \langle 1, \infty\rangle$.

4. Miscellani

We conclude with three remarks. The first considers two open problems, the second comments on recent results on asymptotic behaviour and the third deals with lower bounds on the semigroup kernel.

4.1. **Two open problems.**

4.1.1. *Choice of bases.* Although Theorems 3.1 and 3.2 state that Gaussian kernel bounds are equivalent with G-weighted subcoercivity, the result is only with respect to a fixed weighted algebraic basis. If one starts with a differential operator with constant coefficients on $L_2(G)$ then it might be G-weighted subcoercive with respect to one weighted algebraic basis, but not with respect to another. For example, the operator

$$H = -\Big(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\Big)^2 + \Big(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\Big)^4$$

on $L_2(\mathbf{R}^2)$ is weighted subcoercive with respect to a suitable basis, but if one expands the brackets, then it is not evident how one would recognize the weighted algebraic basis.

In general, let H be a differential operator with constant coefficients on $L_2(G)$. Suppose the following conditions are valid.

I. The closure $\overline{H}$ generates a continuous holomorphic semigroup S.
II. $\|S_z\| \leq e^{\omega|z|}$ for some $\omega \geq 0$, uniformly for all z in a sector.
III. $S_t L_2(G) \subset L_{2;\infty}(G)$ for all $t > 0$.
IV. For all $a \in \mathfrak{g}\backslash\{0\}$ there exist $c_a, \omega_a > 0$ such that $\|dL(a)S_t\|_{2\to 2} \leq c_a\, t^{-n_a} e^{\omega_a t}$ for all $t > 0$.

Is it true that H is a weighted subcoercive operator with respect to a suitable algebraic basis and suitable weights? Is Condition II necessary?

4.1.2. *Steps.* The second problem concerns the step in Theorem 2.1. Let $d' \in \mathbf{N}$, $m \in \mathbf{N}$ and let $C\colon J(d') \to \mathbf{C}$ be a form of (unweighted) order m. If $s \in \mathbf{N}$ and C is subcoercive of step s then it is not hard to prove that C is also subcoercive of step s' for all $s' \in \mathbf{N}$ with $s' \leq s$. On the other hand, if $m = 2$ and C is subcoercive of step 2 then C is also subcoercive of step s for all $s \geq 2$ (see [ElR3], Corollary 3.8). It is an open problem whether for a general m and subcoercive form C of order m and step m the form is also subcoercive of step s for all $s \geq m$.

4.2. **Large time behaviour.** The kernel bounds in Statements VI of the previous theorems have the optimal form for small time t. But for large time the bounds contain a factor $e^{\omega t}$ which is a reflection of the semigroup property but does not usually give an accurate estimate of the asymptotic behaviour. Expressed in terms of the volume function the kernel bounds read

$$|K_t(g)| \leq c\, V(t)^{-1/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}} \tag{6}$$

for all $t > 0$ and $g \in G$. If G is a homogeneous group then the volume has a simple scaling relation. If, in addition, the operator H transforms in a simple way, e.g., if $H = dU(C)$ with C a positive Rockland form, then large t bounds can be obtained from small t bounds by scaling. In particular one may choose $\omega = 0$ and deduce that $\|K_t\|_\infty \le c\,V(t)^{-1/m}$ for all $t > 0$. The inhomogeneous operator $-d^2/dx^2 + d/dx$ on $L_2(\mathbf{R})$ shows, however, that $\omega > 0$ in general. The kernel of this latter operator is a non-centred Gaussian which can only be bounded by a multiple of a centred Gaussian at the cost of an additional exponential factor $e^{\omega t}$.

Even for homogeneous operators on a homogeneous Lie group it is not always possible to choose $\omega = 0$. An example is on the connected, simply connected, Heisenberg group. If a_1, a_2, a_3 is a basis for its Lie algebra such that $[a_1, a_2] = a_3$ then for all $\lambda \in \mathbf{R}$ the operator

$$H = -A_1^2 - A_2^2 - A_3^2 + i\lambda[A_1, A_2]$$

is strongly elliptic, but if λ is suitably large then the kernel K associated to H does not satisfy Gaussian bounds (6) with $\omega = 0$. If one would have Gaussian bounds with $\omega = 0$, then the semigroup S would be uniformly bounded in each unitary representation. But with respect to the standard infinite dimensional representation of the Heisenberg group one easily sees that this is not possible.

Despite this counterexample, there is a positive result on nilpotent Lie groups. The Heisenberg group has rank 2, whilst the above operator is subcoercive of step 1 with respect to the (algebraic) basis a_1, a_2, a_3. If the step is large enough then the following theorem is valid by [ERS1] (a different proof under weaker assumptions is given in [DERS]).

Theorem 4.1. *Let $a_1, \ldots, a_{d'}$ be an (unweighted) algebraic basis for the Lie algebra $\mathfrak{g}$ of a connected nilpotent Lie group and $C\colon J(d') \to \mathbf{C}$ a homogeneous form which is subcoercive of step r, where r is the rank of the Lie algebra $\mathfrak{g}$. Then the bounds (6) are valid with $\omega = 0$. Moreover,*

$$|(A^\alpha K_t)(g)| \le c\,V(t)^{-1/m} t^{-|\alpha|/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}} \tag{7}$$

for all $\alpha \in J(d')$, all $t > 0$ and all $g \in G$.

If H is a sublaplacian on a connected group with polynomial growth, i.e., a group such that $V(s) \asymp s^D$ for $s \ge 1$ for some $D \in \mathbf{N}$, then the bounds (6) and (7) for the kernel K and its first derivatives $A_i K$, $i \in \{1, \ldots, d'\}$, are valid with $\omega = 0$ (see [Rob2], Subsection IV.4b). The situation for higher derivatives and large t is, however, more complicated since the structure relations

$$A_i A_j K_t = A_j A_i K_t + \sum_{k=1}^{d} c_{ij}^k A_k K_t$$

show that there are algebraic obstructions to the extra derivatives always contributing an extra $t^{-1/2}$ decay. Good bounds for the second derivatives are nevertheless valid on nilpotent groups by Theorem 4.1 and on compact groups by a spectral gap argument. Basically, these are the only two cases on which second-order derivatives behave well.

Theorem 4.2. *Let $a_1, \ldots, a_{d'}$ be an algebraic basis and K the kernel associated to the corresponding sublaplacian on a connected Lie group with polynomial growth. If there is a $c > 0$ such that*

$$|(A_i A_j K_t)(g)| \le c\, V(t)^{1/m} t^{-1}$$

uniformly for all $i, j \in \{1, \ldots, d'\}$, $t > 0$ and $g \in G$ then G is the local direct product $K \times_l N$ of a compact Lie group K and a nilpotent Lie group N.

This theorem has been proved in [ERS2], together with good Gaussian bounds for all higher derivatives on groups $K \times_l N$.

Higher order operators on Lie groups with polynomial growth which are not a local direct product of a compact and a nilpotent Lie group are subject to current research. The behaviour is still not fully clear but it is evident that the asymptotics are not totally determined by geometric factors such as the volume growth. In fact there are many cases where the asymptotics involve a different form of weighting which is related to the algebraic properties of the basis but in a more complicated manner than that occurring in Section 2. Even if the m-th order operator H is defined by a basis with all weights equal to one the kernel K can behave asymptotically like a Gaussian in which different directions are given different weights; some directions have weight one and others have weight $m/2$. This is the generic situation for operators $H = (-1)^{m/2} \sum_{i=1}^{d'} A_i^m$ on groups of polynomial growth which have an abelian nilshadow [DER]. Only very special choices of the basis give kernels satisfying m-th order Gaussian bounds. Generally the kernel behaves like a convolution of an m-th order Gaussian and a second-order Gaussian over a nilpotent subgroup. The simplest example of this behaviour is given on the three-dimensional group of Euclidean motions in the plane [ElR8]. The anomalous second-order behaviour occurs for the two directions corresponding to the translations.

4.3. **Lower bounds.** The foregoing results establish criteria for the kernel K associated with a weighted subcoercive operator $H = dU(C)$ to satisfy Gaussian upper bounds. But characterization of the singularity structure requires complementary lower bounds. The kernel, however, is positive if, and only if, the operator is second-order with real coefficients (see [Rob2]). Hence in the general case one can at best hope for local lower bounds. The simplest problem is to bound

$t \mapsto \|K_t\|_\infty$ from below and this is straightforward for small t by an argument of Varopoulos [Var] (see also [ElR7]).

If $H = dL(C)$ is the operator in the left regular representation on $L_\infty(G)$ then $H\mathbb{1} = c_0\mathbb{1}$ where $\mathbb{1} \in L_\infty(G)$ is the identity function and $c_0 = c_\alpha$ if $|\alpha| = 0$. Therefore $S_t\mathbb{1} = e^{-c_0 t}\mathbb{1}$ and $\int_G K_t = e^{-c_0 t}$. Then since $\|K_t\|_\infty \geq |K_t(g)|$ for all $g \in B'_r = \{g \in G : |g|' < r\}$ one has

$$\|K_t\|_\infty \geq V(r)^{-1}\Big(e^{-\operatorname{Re} c_0 t} - \int_{G\backslash B'_r} dg\, |K_t(g)|\Big)$$

for all $r > 0$, where $V(r)$ is the volume of B'_r. Thus setting $r = \sigma t^{1/m}$ and using the Gaussian bounds (6) on the kernel one finds

$$\|K_t\|_\infty \geq c_\sigma t^{-D'/m}\Big(e^{-\operatorname{Re} c_0 t} - c\, e^{\omega t}\, e^{-2^{-1}b\sigma^{m/(m-1)}}\Big)$$

for all $t \in \langle 0, 1]$. Choosing σ sufficiently large one concludes that $\|K_t\|_\infty \geq c' t^{-D'/m}$ for all $t \in \langle 0, 1]$. Therefore $\|K_t\|_\infty \asymp t^{-D'/m}$ for $t \leq 1$.

Similar arguments under more stringent assumptions allow one to characterize the asymptotic behaviour of $t \to \|K_t\|_\infty$ or to bound $|K_t(g)|$ below in a set $\{g : |g|' \leq \kappa t^{1/m}\}$ (see [Dav] [ElR7]). A typical a result is the following.

Theorem 4.3. *Let G be a connected Lie group with polynomial growth. Next let $a_1, \ldots, a_{d'}$ be a weighted algebraic basis of $\mathfrak{g}$ and $C\colon J(d') \to \mathbf{C}$ an m-th order weighted form with $C(\alpha) = 0$ if $|\alpha| = 0$. Assume the kernel K associated with $H = dU(C)$ satisfies the Gaussian bounds*

$$|K_t(g)| \leq c\, V(t)^{-1/m} e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

for all $t > 0$ and $g \in G$.

It follows that $\|K_t\|_\infty \asymp V(t)^{-1/m}$ for all $t > 0$.

Proof. Since there is no constant term one deduces that $\int K_t = 1$ and hence the previous argument gives

$$\|K_t\|_\infty \geq V(r)^{-1}\Big(1 - c\, e^{-2^{-1}b(r^m t^{-1})^{1/(m-1)}}\Big)$$

for all $t > 0$. It is important that G has polynomial growth for this estimate to be true for all large t . Now if $r = \sigma t^{1/m}$ and one chooses σ sufficiently large then

$$\|K_t\|_\infty \geq 2^{-1} V(\sigma t^{1/m})^{-1} \geq c_\sigma V(t)^{-1/m}$$

for all $t > 0$, again because of the polynomial growth of G. Since $\|K_t\|_\infty \leq c V(t)^{-1/m}$ for all $t > 0$, by the Gaussian bounds, one has $\|K_t\|_\infty \asymp V(t)^{-1/m}$ for all $t > 0$. □

This result is applicable if G is nilpotent, all weights equal one and C is homogeneous and subcoercive of step r, with r the rank of the Lie algebra of G. A similar result would be valid for weighted operators if one assumed appropriately weighted Gaussian bounds.

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