TIGHT FRAMES AND ROTATIONS: SHARP BOUNDS ON EIGENVALUES OF THE LAPLACIAN

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Abstract. Isoperimetric estimates stretch back for thousands of years in geometry, and for more than a hundred years in harmonic analysis and mathematical physics. We will touch on some of these highlights before describing recent progress that uses rotational symmetry to prove sharp upper bounds on sums of eigenvalues of the Laplacian. For example, we prove in 2 dimensions that the scale-normalized eigenvalue sum

$$(\lambda_1 + \cdots + \lambda_n)\frac{A^3}{I}$$

(where A denotes area and I is moment of inertia about the centroid) is maximized among triangles by the equilateral triangle, for each $n \geq 1$. This theorem, which is due to the author and B. A. Siudeja, generalizes a result of Pólya for the fundamental tone.

Numerous related problems will be discussed, such as the inverse spectral and spectral gap problems for triangular domains.

1. Introduction

Goals. Extremal domains frequently possess rotational symmetry. Indeed, the symmetry can help to establish the desired extremal property. This expository paper reports on recent such work identifying rotationally symmetric extremals for sums of eigenvalues of the Laplacian.

A prototypical result says that the scale-normalized eigenvalue sum

$$(\lambda_1 + \cdots + \lambda_n)\frac{A^3}{I}$$

(where A denotes area and I is moment of inertia about the centroid) is maximal for the equilateral triangle among all triangles, for each $n \geq 1$. This sharp result generalizes work of Pólya [41] for the fundamental tone $n = 1$. Like all the original work reported in this paper, it was proved jointly by the author with B. A. Siudeja. Precise references will be given later.

The method applies to competing domains that are linear transformations of the extremal, and so in particular our results cover all triangles and parallelograms, as they are the linear images of equilateral triangles and squares, both of which are rotationally symmetric. Further possible extremal domains are shown in Figure 1.

Notably, the method is valid for all major boundary conditions (Dirichlet, Robin and Neumann) and for eigenvalue sums of arbitrary length. It requires no explicit knowledge of the eigenvalues or eigenfunctions of the extremal

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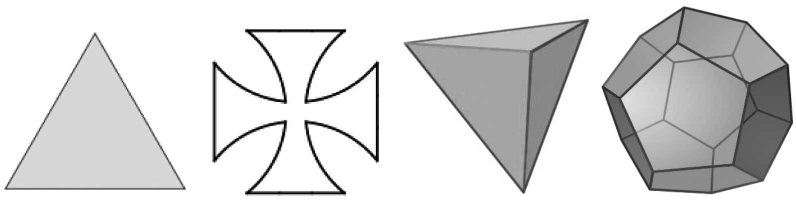

FIGURE 1. Some rotationally symmetric domains, in 2 and 3 dimensions. (Image credits for tetrahedron and dodecahedron: `http://en.wikipedia.org/`)

domain. An interesting feature of the argument is an averaging step over the rotation group of the extremal domain, which employs the tight frame (Plancherel-type) property of the rotation orbits in order to understand the effect of linear transformation on the Rayleigh quotient.

Itinerary. En route to our destination, we will enjoy a leisurely tour of the classic sights. The tour departs in the next section with separation of variables for the interval and rectangle. Then we travel onward to view ancient isoperimetric monuments and modern eigenvalue constructions, in Section 3. We state Pólya's result and our own new work in Section 4. A side trip in Section 5 reveals more about tight frames. These exceedingly useful systems behave like orthonormal bases except for their being overcomplete (linearly dependent). Returning subsequently to the main route, we encounter a curious fact about the higher dimensional case in Section 6, namely that the moment of inertia must be normalized not on the original domain where the eigenvalue was evaluated, but instead on an "inverse" domain. Our journey concludes with open problems in Section 7 and a re-cap in Section 8. Appendix A offers formulas for the eigenvalues of an equilateral triangle, and corresponding pictures of nodal patterns.

The work we describe lies in the field of "isoperimetric inequalities in mathematical physics". Pólya and Szegő wrote the classic text in the field [43]. Beginners might prefer to consult the brief online encyclopedia articles by Benguria [8], before proceeding to the survey papers by Ashbaugh, Benguria and Linde [4, 5, 9], and the appealing modern monographs by Bandle [6], Henrot [18] and Kesavan [22], which contain a wealth of results and open problems.

Thanks. This paper presents an expanded version of my talk at the AMSI International Conference on Harmonic Analysis and Applications, held at Macquarie University in February 2011. The organizers of the meeting deserve our sincere thanks for fostering a stimulating atmosphere of research interaction.

2. Eigenvalues of the Laplacian

What is harmonic analysis? Wikipedia says it is "the branch of mathematics that studies the representation of functions or signals as the superposition of basic waves." This quote ignores the group-theoretic aspects of the subject, and yet it captures the historical goal of harmonic analysis to *analyze* and *synthesize* functions in terms of simpler objects.

One dimension. Harmonic analysis began with Fourier's eigenfunction expansions of solutions of the heat (or diffusion) equation. Let us begin similarly with the wave equation in 1 dimension, for a vibrating string of length L along the x-axis:

$$c^2\phi_{xx} = \phi_{tt}$$

where ϕ represents the transverse displacement of the string at position x and time t. We separate variables by $\phi(x,t) = u(x)\sin(\sqrt{\lambda}ct)$ and deduce that the spatial function u should be an eigenfunction of the second derivative operator:

$$-u_{xx} = \lambda u.$$

We speak of the eigenvalues as frequencies or tones of vibration, although more precisely it is the square root of the eigenvalue that determines the frequency or "pitch" of the string, since

$$\sqrt{\lambda}c = \text{angular frequency of vibration in the term } \sin(\sqrt{\lambda}ct)$$

.

The allowable eigenvalues very much depend on the boundary conditions imposed at the endpoints $x = 0$ and $x = L$. The simplest conditions are "fixed endpoint" or Dirichlet conditions, namely $u = 0$ at $x = 0, x = L$, which lead to infinitely many solutions

$$u_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$

with corresponding eigenvalues given explicitly by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots$$

Two dimensions. The two dimensional analogue of a string is a membrane or drumhead, vibrating transversely in the third direction. By separating variables in the wave equation $c^2\Delta\phi = \phi_{tt}$ we arrive at an eigenvalue problem for the Laplacian $\Delta = \partial_x^2 + \partial_y^2$:

$$-\Delta u = \lambda u, \qquad \text{Dirichlet BC } u = 0 \text{ on boundary } \partial\Omega,$$

where the planar region Ω describes the rest shape of the drum and the Dirichlet boundary condition means the drum is fixed at zero displacement around the edge. The eigenvalues λ_n are positive, and increase to infinity:

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty.$$

Just like in 1 dimension, $\sqrt{\lambda_n}$ is proportional to the n-th frequency of vibration of the membrane.

Unfortunately, and in stark contrast to the 1-dimensional situation, the eigenvalues of the Laplacian are *not* given by an explicit formula. Only on a few special domains can the eigenvalues and eigenfunctions be computed, most notably by separation of variables for *disks* (using polar coordinates and Bessel functions) and for *rectangles* (using rectangular coordinates and sine functions). A method of reflection and extension to a lattice succeeds for *equilateral triangles* (using barycentric coordinates and trigonometric functions). See Appendix A for the resulting formulas.

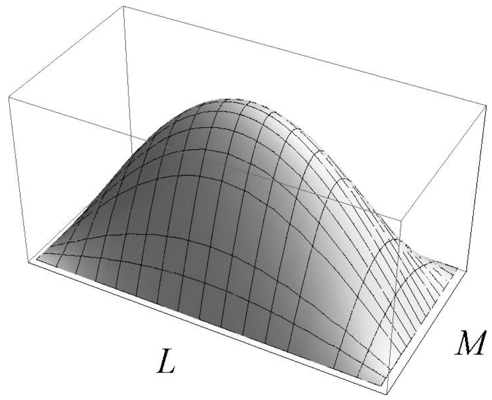

FIGURE 2. The first eigenfunction of a rectangle of sides L and M.

For example, the rectangle $[0, L] \times [0, M]$ has first eigenfunction (or "fundamental mode")

$$u_1 = \sin\Big(\frac{\pi x}{L}\Big) \sin\Big(\frac{\pi y}{M}\Big),$$

as plotted in Figure 2. That eigenfunction arises from a product of 1-dimensional eigenfunctions in the x and y directions, and it satisfies the eigenfunction equation $-\Delta u_1 = \lambda_1 u_1$ with first eigenvalue (or "fundamental tone")

$$\lambda_1 = \pi^2\Big(\frac{1}{L^2} + \frac{1}{M^2}\Big).$$

For our later purposes, the most important feature of this eigenvalue formula is its reciprocal dependence on the squares of the side lengths:

$$\lambda_1 \propto \frac{1}{(\text{length scale})^2}.$$

On dimensional grounds this scaling relation must hold for all domains, in fact, and not just for rectangles, because the dimensions of the eigenvalue λ must match those of the second order differential operator $-\Delta$, which are $(\text{length})^{-2}$.

Symmetry can help in computing eigenvalues of some other special domains. For example, the spectrum can be computed using symmetry considerations for a half-disk, and for a 45-45-90° triangle (which is half of a square), and for a 30-60-90° triangle (which is half of an equilateral). Nonetheless, the catalog of domains for which the eigenvalues of the Laplacian can be calculated remains depressingly small.

Numerical methods can provide highly accurate approximations to finitely many of the eigenvalues, and user-friendly software in 2 dimensions (such as the PDE Toolbox in Matlab) makes the job almost painless. But numerical results by themselves provide little insight into the dependence of the spectrum on the shape of the domain. To gain such insight we seek theoretical results, particularly upper and lower bounds on eigenvalues.

Figure 3. Queen Dido applying the isoperimetric theorem. (*Dido purchases land for the foundation of Carthage*, engraving by Matthäus Merian the Elder, in Historische Chronica, Frankfurt a.M., 1630.)

3. Lower bounds

Since we cannot *calculate* the eigenvalues of an arbitrary domain Ω, can we at least *estimate* them? Can we obtain sharp "isoperimetric" type results that relate the eigenvalues to simpler geometric quantities? We begin with a quick survey of some famous lower bounds.

Isoperimetric motivation — Queen Dido of Carthage. The isoperimetric theorem asserts that among all domains of given area, the circle possesses the minimal perimeter. When stated as an inequality, the theorem says

$$P^2 \geq 4\pi A$$

where P is the perimeter.

Equivalently, this isoperimetric inequality says that among all domains with given perimeter, the one maximizing the area is circular. This latter form of the result was employed to good effect by Queen Dido, or so the legend goes, when she founded the city of Carthage. Permitted by the local king to found the city on only as much land as could be encompassed by an oxhide, she cut the hide into thin strips and fashioned it into a rope (Figure 3), and then placed it to encircle a suitable hilltop location, the "Byrsa" on which the ruins of Carthage stand to this day. (Or perhaps she enclosed a semicircle of land including the hill and abutting the coast, in order to provide the city with access to the sea. Either way, isoperimetric principles come into play.)

A new proof of the isoperimetric theorem appeared in 2010, due to Lawlor [32]. This amazingly short proof requires only calculus. It begins with the trivial 1-dimensional version of the theorem (that every bounded interval has two endpoints) and proceeds by induction to higher dimensions. For a delightful exposition of many other isoperimetric proofs in 2 dimensions, see

Domain	P	$\sqrt{\lambda_1}$
Disk	3.55	4.26
Square	4.00	4.44
Quarter Disk	4.03	4.55
Rectangle 2:1	4.24	4.97
Equilateral Triangle	4.56	4.77

Table 1. Evidence for the Isoperimetric and Faber–Krahn Theorems: values of the perimeter and fundamental tone for various domains of area $A = 1$.

the survey paper by Treibergs [45], and for isoperimetry on surfaces consult the work of Howards, Hutchings and Morgan [20]. Osserman's classic survey on isoperimetric inequalities deserves close study too [40].

Rayleigh's conjecture. In his book "The Theory of Sound", Lord Rayleigh wrote

> "If the area of a membrane be given, there must evidently be some form of the boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle." [44, §210]

Expressed in modern prose (which is lamentably less elegant), Rayleigh's conjecture asserts that

$$\lambda_1 A \text{ is minimal for the disk.}$$

(The product $\lambda_1 A$ is scale invariant, and so we need not specify the size of the disk.) The intuitive motivation for Rayleigh's conjecture is clear: boundary clamping raises the frequency of a drum, and so the less boundary a domain has, the lowers its fundamental tone should be. The domain with least possible boundary (for given area) is exactly the disk, by the isoperimetric theorem.

Rayleigh presented both experimental and numerical evidence for his conjecture, along with perturbational calculations for nearly-circular domains. Table 1 presents some of the numerical evidence, and highlights the close connection with the isoperimetric problem:

Raleigh's Conjecture that $\lambda_1 A$ is minimal for the disk took a suprisingly long time to prove. Faber [13] and Krahn [23, 24] independently found proofs in the mid-1920s, using what is now called symmetric decreasing rearrangement. (A clear and highly readable version of Krahn's proof using modern terminology appears in Kesavan's book [22, Sections 2.3 and 4.1].) The proofs took so long partly because the co-area formula had not been invented; indeed, Krahn's second paper developed the formula in higher dimensions.

In the 1970s, the Faber–Krahn and isoperimetric theorems were revealed to be endpoint cases of a 1-parameter family of sharp results, in work by Luttinger [35], who proved that the spectral zeta function $\sum_n (\lambda_n)^{-s}$ is maximal for the disk of the same area, for each $s > 1$. Even stronger, he showed

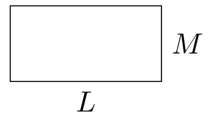

Figure 4. Rectangle.

that the trace of the heat kernel $\sum_n e^{-\lambda_n t}$ is maximal for the disk of the same area, when $t > 0$. Letting $t \to \infty$ in Luttinger's result recovers the Faber–Krahn inequality. Letting $t \to 0$ recovers the isoperimetric inequality, in view of the two-term asymptotic expansion of the trace of the heat kernel:

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{A}{4\pi t} - \frac{P}{4\sqrt{4\pi t}} + O(1) \qquad \text{as } t \to 0. \tag{3.1}$$

Pólya and Szegő pursued a different generalization of the Faber–Krahn result, by formulating a *polygonal* version of Rayleigh's conjecture. They asked [18, Ch. 3] whether

$\lambda_1 A$ is minimal for regular N-gon, among all N-gons?

They proved this conjecture for triangles and quadrilaterals ($N = 3, 4$), by means of Steiner symmetrization. For pentagons and above ($N \geq 5$), the problem remains open to this day.

4. **Upper bounds**

Upper bounds are the goal of this paper. First, let us check that the area normalization used in the Faber–Krahn lower bound is useless for upper bounds. Consider the rectangle in Figure 4, with sides of length L and M.

We have

$$\begin{aligned} \lambda_1 A &= \pi^2\left(\frac{1}{L^2} + \frac{1}{M^2}\right)LM \\ &= \pi^2\left(\frac{M}{L} + \frac{L}{M}\right) \to \infty \end{aligned}$$

as $L \to \infty$ with $M = 1$. In other words, a long thin rectangle has $\lambda_1 A$ tending to infinity. Thus we seem to need to normalize with a quantity that penalizes long thin domains. The moment of inertia

$$I = \min_{\vec{x}_0} \int_\Omega |\vec{x} - \vec{x}_0|^2 \, dA(\vec{x})$$

seems a plausible candidate. Indeed, the rectangle has moment of inertia

$$I = \int_{-L/2}^{L/2} \int_{-M/2}^{M/2} (x^2 + y^2)\, dy dx = \frac{1}{12} LM(L^2 + M^2)$$

and so

$$\begin{aligned} \lambda_1 \frac{A^3}{I} &= \pi^2\left(\frac{1}{L^2} + \frac{1}{M^2}\right)\frac{12(LM)^3}{LM(L^2+M^2)} \\ &= 12\pi^2, \end{aligned} \tag{4.1}$$

which is constant for all rectangles, regardless of the side lengths.

The main point here is to normalize the eigenvalue by dividing by the moment of intertia. The factor of A^3 simply serves to make the resulting expression $\lambda_1 A^3/I$ scale invariant. Notice one can write

$$\lambda_1 \frac{A^3}{I} = \lambda_1 A \cdot \frac{A^2}{I}$$

which expresses our quantity as the product of the Faber–Krahn expression $\lambda_1 A$ and a scale invariant geometric factor A^2/I. Since A^2/I is small when the domain is long and thin, it compensates for the largeness of $\lambda_1 A$ on such domains.

Known results. The literature contains two results that involve the moment of inertia:

Theorem 4.1.
Hersch [19, eq. (5)]*: Among parallelograms, the squares (and indeed, all rectangles) maximize* $\lambda_1 A^3/I$.
Freitas [14, Theorem 1]*: Among triangles, the equilaterals maximize* $\lambda_1 A^3/I$.

A stronger result was proved previously by Pólya.

Theorem 4.2 (Pólya [41], [42, p. 308,328])**.** *Start with an N-fold rotationally symmetric domain in the plane, where $N \geq 3$. Then among all linear images, the original domain maximizes* $\lambda_1 A^3/I$.

If the rotationally symmetric domain is an equilateral triangle centered at the origin, then its linear images yield all possible triangles (with centroid at the origin). If the original domain is a square, then the linear images consist of all parallelograms (with centroid at the origin). Thus Pólya's result encompasses those of Hersch and Freitas. In defense of the latter two authors, we note their results were formulated in terms of side lengths rather than moment of inertia, and the connection to Pólya's result is not obvious.

Interestingly, Pólya's proof does not use explicit formulas for the first eigenfunction of the extremal domain (as Hersch and Freitas do), but relies instead on rotational symmetry of the first eigenfunction of the rotationally symmetric domain. Rotational symmetry of the eigenfunction depends on two facts: (a) that the rotate of an eigenfunction is again an eigenfunction with the same eigenvalue, by rotational invariance of the Laplacian and of the domain; (b) that the first eigenfunction is unique up to constant multiples, in other words, that the first eigenvalue is simple.

Pólya's proof fails beyond the first eigenvalue, because although fact (a) holds for all eigenfunctions, fact (b) definitely does not. For example, rotating a disk eigenfunction of the form $J(r)\cos\theta$ by angle $\pi/2$ yields a different eigenfunction, of the form $J(r)\sin\theta$, which has the same eigenvalue.

Question. Can we extend Pólya's result to *sums* of eigenvalues?

Why should one care about the eigenvalue sum $(\lambda_1 + \cdots + \lambda_n)$? The first reason is physical: the sum of eigenvalues represents the energy needed to fill the lowest n quantum states under the Pauli exclusion principle. The second is mathematical: summing provides an easier route to studying the high eigenvalues, which are difficult to study individually.

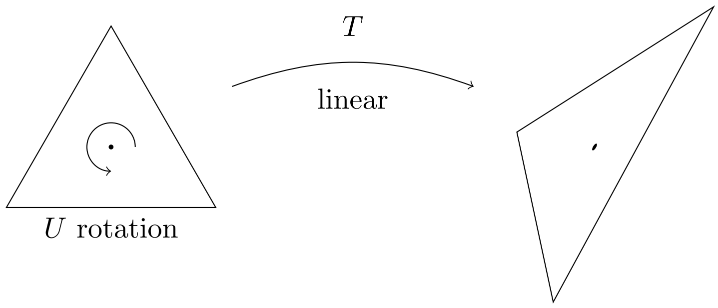

FIGURE 5. A domain with N-fold rotational symmetry (here $N = 3$), and its image under a linear transformation.

Let us explain, by recalling the variational characterization of eigenvalues. The first eigenvalue of a domain Ω is characterized by the minimum of the Rayleigh quotient:

$$\lambda_1(\Omega) = \min\Big\{ \int_\Omega |\nabla v|^2 \, dA : v = 0 \text{ on } \partial\Omega, \|v\|_2 = 1 \Big\},$$

where the boundary condition $v = 0$ means that v belongs to the Sobolev space $H_0^1(\Omega) = W_0^{1,2}(\Omega)$. The higher eigenvalues are characterized by more complicated "minimax" and "maximin" principles [6, pp. 98–99], and thus bounds are much more difficult to obtain than for the first eigenvalue. The *sum* of the first n eigenvalues behaves more like a first eigenvalue, though, in the sense that its variational characterization involves just a single minimum:

$$\begin{aligned}\big(\lambda_1 + \cdots + \lambda_n\big)(\Omega) = \min\Big\{ \int_\Omega |\nabla v_1|^2 \, dA + \cdots + \int_\Omega |\nabla v_n|^2 \, dA : \\ v_i = 0 \text{ on } \partial\Omega, \ \langle v_i, v_j \rangle_{L^2} = \delta_{i,j} \Big\}.\end{aligned}$$

Main result — upper bound. Our main result estimates how the eigenvalue sum changes under linear transformation of the domain. Later we express the result in terms of moment of inertia.

Theorem 4.3 (Laugesen & Siudeja [30]). *Suppose the plane domain D has rotational symmetry of order ≥ 3, and T is a linear transformation. (See Figure 5.) Then*

$$(\lambda_1 + \cdots + \lambda_n)|_{T(D)} \leq \frac{1}{2}\|T^{-1}\|^2 (\lambda_1 + \cdots + \lambda_n)|_D.$$

For $n = 1$, equality holds iff T = multiple of orthogonal matrix or $T(D)$ = rectangle.

Here $\|M\|^2 = \sum_{i,j} M_{ij}^2$ is the Hilbert-Schmidt norm of the matrix M.
Now we outline the proof, which involves a new

Method of Rotations and Tight Frames.

Further details and references can be found in our original paper [30].

In order to apply the variational characterization of the eigenvalue sum, we need trial functions on $\Omega = T(D)$. Begin with an orthonormal basis

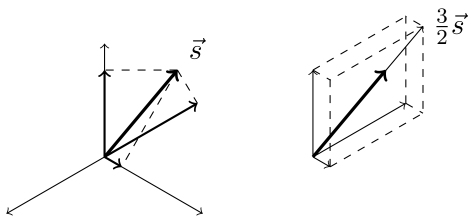

FIGURE 6. The "Mercedes–Benz" tight frame formed from the 3rd roots of unity (rotated by $\pi/2$). Projecting against these frame vectors leads to reconstruction of 3/2 times the original vector $\vec{s}$.

$\{u_i\}_{i=1}^{\infty}$ of eigenfunctions of the Laplacian on D, and write U_m for the orthogonal matrix that rotates the plane by $2\pi m/N$, where N is the order of rotational symmetry of the domain D. Then the function

$$v_i = u_i \circ U_m \circ T^{-1}$$

is defined on the image domain $T(D)$ and equals zero on the boundary. Furthermore, $\langle v_i, v_j\rangle_{L^2} = 0$ when $i \neq j$, by changing variable and using the orthogonality of u_i and u_j. Thus after multiplying each v_i by a normalizing factor, one may use the v_i as trial functions in the variational characterization of the eigenvalue sum. One deduces (after a change of variable, again) that

$$\sum_{i=1}^{n} \lambda_i\big(T(D)\big) \leq \sum_{i=1}^{n} \int_D |(\nabla u_i)U_m T^{-1}|^2 \, dA. \tag{4.2}$$

To eliminate the rotation matrix U_m from the formula, we will average over all rotations $U_m, m = 1, \ldots, N$. The key to the averaging will be a Plancherel-type identity for the rotational orbit, known as a *tight frame* identity:

$$\frac{1}{N}\sum_{m=1}^{N} |\vec{s}\,U_m M|^2 = \frac{1}{2}|\vec{s}\,|^2 \|M\|^2 \tag{4.3}$$

where $\vec{s} \in \mathbb{R}^2$ is a row vector and M is a matrix with 2 rows.

For example, if we choose $M = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ to be the unit vector in the x-direction, which has $\|M\| = 1$, then the collection $\{U_m M : m = 1, \ldots, N\}$ consists of the N-th roots of unity. The tight frame identity (4.3) says that the squares of the dot products of $\vec{s}$ with these roots of unity reproduce the norm of $\vec{s}$, up to a factor of $N/2$. Thus the roots of unity behave like an orthonormal basis!

For example, the "Mercedes–Benz" tight frame identity ($N = 3$) says that

$$\sum_{m=1}^{3} |\vec{s} \cdot U_m \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)|^2 = \frac{3}{2}|\vec{s}|^2, \qquad \vec{s} \in \mathbb{R}^2$$

as illustrated in Figure 6.

By applying the tight frame identity (4.3) with $\vec{s} = \nabla u_i$ and $M = T^{-1}$ to our estimate (4.2), we find that

$$\sum_{i=1}^{n} \lambda_i\big(T(D)\big) \le \frac{1}{2}\|T^{-1}\|^2 \sum_{i=1}^{n} \int_D |\nabla u_i|^2 \, dA.$$

Integration by parts (Green's formula) says that

$$\int_D |\nabla u_i|^2 \, dA = -\int_D u_i \Delta u_i \, dA = \lambda_i(D) \int_D u_i^2 \, dA = \lambda_i$$

because u_i is a normalized eigenfunction on D. Hence we conclude that

$$\sum_{i=1}^{n} \lambda_i\big(T(D)\big) \le \frac{1}{2}\|T^{-1}\|^2 \sum_{i=1}^{n} \lambda_i(D),$$

which proves the inequality in the theorem.

Proving the equality statement for the first eigenvalue ($n = 1$) requires quite different arguments, which can be found in [30].

Connection to moment of inertia. Our main theorem above involves the Hilbert–Schmidt norm of the matrix T^{-1}. We would like to interpret this norm geometrically, in terms of some property of the image domain $T(D)$. Since the Hilbert–Schmidt norm is a quadratic expression of the matrix entries, and is invariant under rotations, one might expect it to relate to the moment of inertia. One finds by consideration of the moment matrix of $T(D)$ and the rotational symmetry of D that [30, Lemma 5.3]

$$\frac{1}{2}\|T^{-1}\|^2 = \frac{I}{A^3}\big(T(D)\big) \Big/ \frac{I}{A^3}(D).$$

Hence we achieve our goal of extending Pólya's result to eigenvalue sums.

Corollary 4.4 (Laugesen & Siudeja [30])**.** *For each $n \ge 1$,*

$$(\lambda_1 + \cdots + \lambda_n)\frac{A^3}{I} \quad \textit{is maximal for the} \quad \begin{cases} \textit{equilateral among triangles,} \\ \textit{square among parallelograms,} \\ \textit{disk among ellipses.} \end{cases}$$

5. Aside — more about tight frames

Frame theory stretches more broadly than perhaps is suggested by the root-of-unity construction above. A *frame* in a Hilbert space H is a spanning set $\{v_m\} \subset H$ for which the two sides of the Plancherel identity are comparable:

$$A\|v\|_H^2 \le \sum_m |\langle v, v_m\rangle|^2 \le B\|v\|_H^2 \qquad \forall v \in H,$$

for some positive constants A and B known as the *frame bounds.* A *tight frame* occurs when the frame bounds are equal:

$$\sum_m |\langle v, v_m\rangle|^2 = B\|v\|_H^2 \qquad \forall v \in H.$$

Thus a tight frame satisfies the Plancherel identity.

In the previous section, the Hilbert space was $\mathbb{R}^2$ and the frame vectors were constructed from roots of unity. Other examples of tight frames include:

- $H = \mathbb{R}^3$: vertices of the tetrahedron, cube, or any Platonic solid centered at the origin,
- $H = \mathbb{R}^d$: each nontrivial orbit of an irreducible rotation group (proved by Schur's Lemma; here "irreducible" means the orbits span $\mathbb{R}^d$),
- $L^2(\mathbb{R}^d)$: wavelet frames, Gabor frames.

Frames have become indispensable in applied harmonic analysis, in recent years. They provide overdetermined (or redundant) analysis and synthesis operators, with applications to noise reduction and robustness to erasures in signal processing.

Note the tight frames in this paper consist of equal-norm vectors. For more on that fascinating special case, see the work of Benedetto and Fickus [7]. General tight frames do not impose the equal-norm restriction. All tight frames arise as (rescaled) projections of orthonormal bases in higher dimensional spaces [17, Chapter 5]. For this and many more results in the frame theory of finite and infinite dimensional spaces, one may consult the monographs of Christensen [11] and Han *et al.* [17].

6. **Higher dimensions**

Can we extend our upper bound on eigenvalue sums to higher dimensions?

The appropriate assumption on rotational symmetry, in higher dimensions, seems to be that the domain's rotation group (or full isometry group) should be *irreducible*. In this context, irreducibility means simply that the orbit under the rotation group of any nonzero vector in $\mathbb{R}^d$ must span $\mathbb{R}^d$. In 2 dimensions, irreducibility clearly holds for domains with rotational symmetry of order 3 or greater. In 3 dimensions, irreducibility is known to hold for the ball, cube and regular tetrahedron, and indeed for each of the platonic solids. In dimensions ≥ 3, the ball, hypercube and regular simplex have irreducible rotation groups, as do other domains that share the same symmetry groups.

One might guess that our main theorem would now extend straightforwardly to higher dimensions. But it does not, because the moment of inertia no longer provides a satisfactory geometric normalization. A counterexample is provided by a rectangular box squashed in one direction, having sides $1, 1, \varepsilon$, for which one calculates

$$\lambda_1 \frac{V^{7/3}}{I}\bigg|_{\text{box}} = 12\pi^2 \frac{(2+\varepsilon^{-2})\varepsilon^{4/3}}{2+\varepsilon^2},$$

which tends to ∞ as $\varepsilon \to 0$. Hence this natural (and scale invariant) normalization of the first eigenvalue by volume and moment of inertia does not yield a bounded quantity, even on the class of rectangular boxes.

The difficulty has its root in our calculation (4.1) for the rectangle, where we used that the sum of two squares can be written as the sum of reciprocal squares multiplied by a factor:

$$L^2 + M^2 = \big(\frac{1}{L^2} + \frac{1}{M^2}\big)(LM)^2.$$

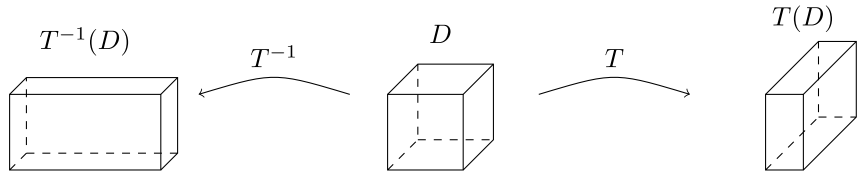

FIGURE 7. A rotationally symmetric domain D, with its forwards domain $T(D)$ and inverse domain $T^{-1}(D)$.

The corresponding formula for three squares is false:

$$L^2 + M^2 + N^2 \neq \big(\frac{1}{L^2} + \frac{1}{M^2} + \frac{1}{N^2}\big)(LMN)^2.$$

To escape the difficulty, we will normalize the moment of inertia not of the original domain but of an "inverse" domain. In the case of a rectangular box of side lengths L, M, N, the inverse box will have sides $1/L, 1/M, 1/N$, as in Figure 7. More generally, if the domain has the form $T(D)$ for some linear transformation T, then the inverse domain will have the form $T^{-1}(D)$.

The end result is that for a domain $D \subset \mathbb{R}^d$ having irreducible rotation group (*e.g.* $D =$ regular simplex, hypercube, ball), we prove:

Corollary 6.1 (Laugesen & Siudeja [31])**.**

$$(\lambda_1 + \cdots + \lambda_n)V^{2/d}\Big|_{T(D)} \cdot \frac{V^{1+2/d}}{I}\Bigg|_{T^{-1}(D)} \qquad \textit{is maximal for } T = \textit{Identity}.$$

In particular, this quantity is maximal for the regular simplex among simplexes, maximal for the hypercube among parallelepipeds, and maximal for the ball among ellipsoids.

Note the first factor $(\lambda_1 + \cdots + \lambda_n)V^{2/d}$ is a scale invariant Faber–Krahn type term on the domain $T(D)$, and the second factor $V^{1+2/d}/I$ is a purely geometric scale invariant term evaluated on the inverse domain $T^{-1}(D)$.

We know of no other eigenvalue inequality in which the geometric normalization occurs not on the original domain but on an auxiliary domain. Perhaps other such results await discovery by the reader?

7. **Open problems**

General domains. Let us consider again Pólya's result on the first eigenvalue. By substituting in the explicit eigenvalues of the equilateral triangle, square and disk, we discover that his result implies

$$\lambda_1 \frac{A^3}{I} \leq \begin{cases} 12\pi^2 & \text{for triangles, with equality for equilaterals,} \\ 12\pi^2 & \text{for parallelograms, with equality for rectangles,} \\ 2j_{0,1}^2\pi^2 & \text{for ellipses, with equality for disks,} \end{cases}$$

where $j_{0,1} \simeq 2.4$ denotes the first zero of the Bessel function J_0. Because $2j_{0,1}^2 \simeq 11.5 < 12$, the disk does *not* maximize $\lambda_1 A^3/I$ among all domains. That is something of a surprise, since the disk solves so many other extremal problems.

Conjecture 7.1. *Among convex plane domains, is*

$$\text{(degenerate sector)} < \lambda_1 \frac{A^3}{I} \leq \text{(equilateral triangle and all rectangles) ?}$$

Viewed from a different perspective, the conjecture seeks to identify the optimal constants in the comparability relation

$$\lambda_1 \asymp \frac{I}{A^3}$$

that connects the analytic quantity λ_1 to the geometric quantity I/A^3.

The convexity assumption in the conjecture serves to rule out pathological domains, such as domains with sets of measure zero removed. Such removals can drive the eigenvalue to infinity without affecting the area or moment of inertia.

For convex domains in higher dimensions, the moment of inertia must be evaluated on some kind of "inverse" domain, as we saw in Corollary 6.1. Perhaps that inverse should be the *polar dual* domain [31, Sec. 4]. For example, the polar dual of a cube is an octahedron, and vice versa. (The polar dual operation maps vertices to faces, and faces to vertices.)

Remark added in press. Recently, Siudeja and I discovered upper bounds on eigenvalue sums that hold for rather general domains (convex and starlike), with disks and balls being extremal [29]. There is no requirement that the domains be linear images of rotationally symmetric domains. The geometric normalizing factor again involves moment of inertia, as in this paper, but also a kind of "radial deformation" factor.

This recent work further explains how to extend by "majorization" from sums like $\lambda_1 + \cdots + \lambda_n$ to more general functionals such as eigenvalue products, spectral zeta functions $\sum_j \lambda_j^{-s}$, and heat traces $\sum_j e^{-\lambda_j t}$. The main results in this paper (Theorem 4.3 and Corollary 4.4) can similarly be extended to spectral zeta functions and heat traces.

Neumann eigenvalues. Our theorems hold without change for the Neumann eigenvalue sums $\mu_2 + \cdots + \mu_n$ (with boundary condition $\partial u/\partial \nu = 0$ for the eigenfunctions). We omit the first Neumann eigenvalue μ_1 from our sum, because it equals 0 for every domain.

Conjecture 7.2. *Among convex plane domains, is*

$$(\mu_2 + \cdots + \mu_n) \frac{A^3}{I} \leq \text{(disks)?}$$

The conjecture holds true for the second eigenvalue, $n = 2$, by work of Szegő and Weinberger on $\mu_2 A$. All other cases stand open.

Spectral gaps. The spectral gap $\lambda_2 - \lambda_1$ is arguably the second-most important spectral functional, after the fundamental tone λ_1. The best geometric normalizing factor for the spectral gap appears to be the diameter of the domain.

Theorem 7.3 (Andrews and Clutterbuck [1])**.**

$$(\lambda_2 - \lambda_1)(\text{diameter})^2 \geq 3\pi^2$$

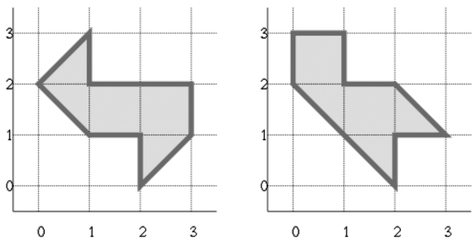

FIGURE 8. Isospectral drums found by Gordon, Webb and Wolpert. The two shapes have equal area and perimeter, but are clearly not congruent. (Image credit: `http://en.wikipedia.org/`)

for all convex domains in $\mathbb{R}^d$*. Equality holds in* 1 *dimension for the line segment, and hence in all dimensions for a degenerate rectangular box.*

This gap estimate was conjectured by van den Berg [10, formula (65)]. The proof by Andrews and Clutterbuck involves, among other innovations, a parabolic comparison principle for the modulus of concavity.

Triangles behave differently, it seems. . .

Conjecture 7.4 (Antunes–Freitas [2, Conjecture 4])**.** *Is*

$$(\lambda_2 - \lambda_1)(\textit{diameter})^2 \geq \frac{64\pi^2}{9}$$

for all triangles? Equality holds for equilaterals.

Lu and Rowlett recently provided a computer-assisted proof [33].

Note that the conjectured minimizer here (the equilateral) is *not* degenerate, in contrast to the degeneracy of the line segment in Theorem 7.3.

Incidentally, the diameter normalization also lends itself to proving bounds on *sums* of eigenvalues. The equilateral triangle has been shown to minimize $(\lambda_1 + \cdots + \lambda_n)(\text{diameter})^2$ among all triangles [27, 28]. The analogous problem remains open for general domains, with the disk proposed as the minimizer.

Can one hear the shape of a drum? Do the eigenvalues determine the domain? This inverse problem raised long ago by Kac [21] continues to stimulate research today.

The Weyl asymptotic $\lambda_j \sim 4\pi j/A$ shows that knowledge of the high eigenvalues ($j \to \infty$) determines the area A. The perimeter too can be obtained from the spectrum, as seen in the heat trace asymptotic (3.1). One might hope by such investigations to determine more and more properties of the domain in terms of the spectrum, and hence determine the shape of the domain completely, up to congruence. Gordon, Webb and Wolpert [15] showed the impossibility of this program, by finding two non-congruent polygons which generate precisely the same eigenvalues. Figure 8 shows these strikingly simple shapes. An elementary proof of the isospectrality, due to Bérard and using his technique of transplantation, can be found in the survey paper by Benguria and Linde [9].

The drums found by Gordon, Webb and Wolpert have corners at the boundary. Kac's question remains open for smoothly bounded domains. In the positive direction, one *can* hear the shape of drums with *analytic* boundary curves and a reflectional symmetry, by recent work of Zelditch [46].

Can one hear the shape of a triangular drum? The answer is Yes: the whole spectrum $\{\lambda_n\}$ determines a triangle up to congruence, as shown by Durso [12] as a consequence of results on the wave operator. D. Grieser and S. Maronna [16] have observed an alternative approach using the heat trace asymptotic for triangles, as follows:

$$(7.1)\quad \sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{A}{4\pi t} - \frac{P}{4\sqrt{4\pi t}} + \frac{1}{24}\Big(-1 + \sum_{j=1}^{3} \frac{\pi}{\theta_j}\Big) + o(1) \qquad \text{as } t \to 0,$$

where $\theta_1 \leq \theta_2 \leq \theta_3$ are the interior angles of the triangle. To indicate their idea, notice the values of A and P are known from the expansion (7.1), and so is the value of $\sum_{j=1}^{3} \theta_j^{-1}$. Also $\sum_{j=1}^{3} \theta_j$ has value π. The value of $\sum_{j=1}^{3} \cot(\theta_j/2) = \prod_{j=1}^{3} \cot(\theta_j/2)$ is known too, because it equals $P^2/4A$ by a standard formula for triangles. From the last three expressions, the three angles θ_j can be obtained uniquely (although not straightforwardly; the inversion requires insightful monotonicity arguments). The angles determine the triangle's shape, and then the area A fixes its size.

Incidentally, the asymptotic formula (7.1) has a somewhat complicated history, which is discussed by Mazzeo and Rowlett [37]. They treat domains with piecewise smooth boundary, and describe the origin of the "heat trace anomaly" in the third coefficient by means of renormalized heat invariants of auxiliary smooth domains that model the corner formation.

The solution above of the inverse spectral problem for triangles leaves one feeling slightly embarrassed at needing infinitely many pieces of spectral information (the eigenvalues used in defining the heat kernel or wave operator) in order to determine just three parameters on which the triangle depends. And so we ask:

> do the first three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ suffice to determine
> the shape of a triangle?

Numerical investigations by Antunes and Freitas [3] demonstrate convincingly that the answer is Yes. Can anyone find a proof?

8. **Conclusions**

In this expository paper we have recalled the historical origins of isoperimetric type inequalities for eigenvalues, and described modern developments giving sharp upper bounds on eigenvalue sums in terms of moment of inertia.

The method of Rotations and Tight Frames by which these bounds are proved

- works in **all dimensions**,
- handles linear images of **rotationally symmetric domains**,
- is **geometrically sharp**, meaning the rotationally symmetric domain is extremal,

- handles eigenvalue sums of **arbitrary length** (that is, any n),
- **applies universally**: to Dirichlet, Robin and Neumann boundary conditions, to Schrödinger potentials with rotational symmetry, and even to the magnetic Laplacian [26].

Further applications seem likely in future.

If this paper has whetted your appetite for more, then please consult the original papers [30, 31], which give a complete account of the proofs and a fuller guide to the literature.

Appendix A. Eigenvalues of equilateral triangles, rectangles, disks

The Dirichlet eigenfunctions of equilateral triangles were derived about 150 years ago by Lamé [25, pp. 131–135]. More recent derivations appear in the text of Mathews and Walker [36, pp. 237–239], and in a paper by McCartin [38], who treated the Neumann eigenvalues too [39].

Eigenfunctions of rectangular domains and disks are well known, of course, by separation of variables.

The Dirichlet eigenvalues of these domains are:

$$\begin{aligned} \big\{(16\pi^2/9)\big[k_1^2+k_1k_2+k_2^2\big] : k_1,k_2\geq 1\big\} &\text{ for an equilateral triangle of side 1,} \\ \big\{\pi^2\big[(k_1/l_1)^2+(k_2/l_2)^2\big] : k_1,k_2\geq 1\big\} &\text{ for a rectangle of side lengths } l_1,l_2, \\ \big\{j_{m,p}^2 : m\geq 0, p\geq 1\big\} &\text{ for the unit disk,} \end{aligned}$$

where $j_{m,p}$ is the pth zero of the Bessel function J_m.

The Neumann eigenvalues are:

$$\begin{aligned} \big\{(16\pi^2/9)\big[k_1^2+k_1k_2+k_2^2\big] : k_1,k_2\geq 0\big\} &\text{ for an equilateral triangle of side 1,} \\ \big\{\pi^2\big[(k_1/l_1)^2+(k_2/l_2)^2\big] : k_1,k_2\geq 0\big\} &\text{ for a rectangle of side lengths } l_1,l_2, \\ \big\{(j'_{m,p})^2 : m\geq 0, p\geq 1\big\} &\text{ for the unit disk,} \end{aligned}$$

where $j'_{m,p}$ is the pth zero of the Bessel derivative J'_m.

Figure 9 displays the nodal patterns of the first three Dirichlet eigenfunctions of the equilateral triangle, extended by odd reflection across the sides of the triangle.

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References

1. B. Andrews and J. Clutterbuck. *Proof of the fundamental gap conjecture.* J. Amer. Math. Soc. 24 (2011), 899–916.
2. P. Antunes and P. Freitas. *A numerical study of the spectral gap.* J. Phys. A 41 (2008), 055201, 19 pp.

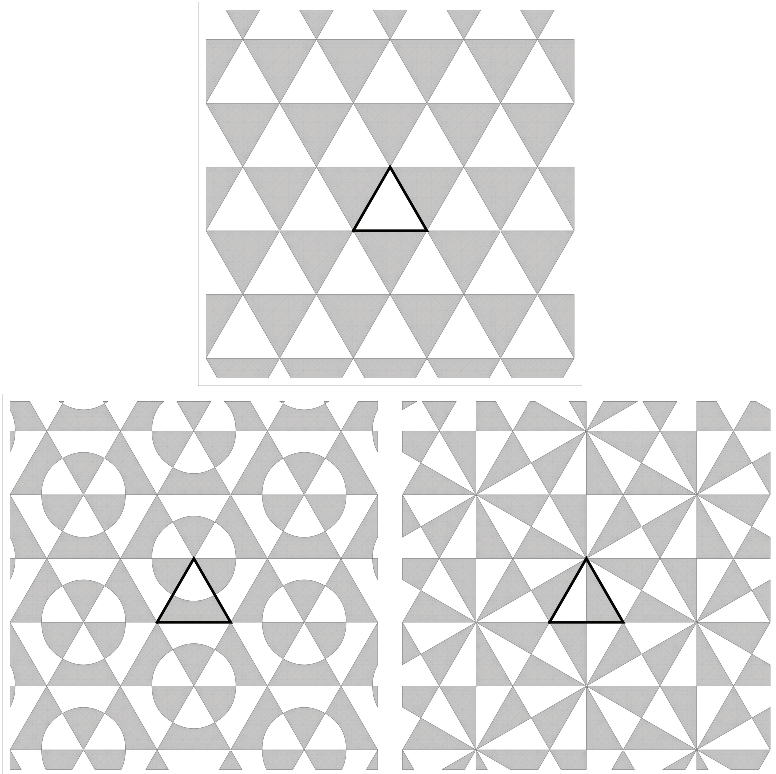

FIGURE 9. *Top:* the first nodal domain of an equilateral triangle (the central triangle), extended by odd reflection across the boundaries. The eigenfunction is positive in the light regions, and negative in the dark regions. *Bottom left:* the nodal pattern of the second Dirichlet eigenfunction of the same triangle, extended by odd reflection. *Bottom right:* The nodal pattern of the third Dirichlet eigenfunction, extended by odd reflection. *Remark.* The second and third nodal patterns are rather different, although the corresponding eigenvalues are equal. The second eigenfunction is even about the vertical bisector of the triangle, while the third eigenfunction is odd about that line.

3. P. Antunes and P. Freitas. *On the inverse spectral problem for Euclidean triangles.* Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 467 (2011), no. 2130, 1546–1562.
4. M. S. Ashbaugh. *Isoperimetric and universal inequalities for eigenvalues.* Spectral theory and geometry (Edinburgh, 1998), 95–139, London Math. Soc. Lecture Note Ser., 273, Cambridge Univ. Press, Cambridge, 1999.
5. M. S. Ashbaugh and R. D. Benguria. *Isoperimetric inequalities for eigenvalues of the Laplacian.* In: Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, 105–139, Proc. Sympos. Pure Math., 76, Part 1, Amer. Math. Soc., Providence, RI, 2007.
6. C. Bandle. Isoperimetric Inequalities and Applications. Pitman, Boston, Mass., 1979.
7. J. J. Benedetto and M. Fickus, *Finite normalized tight frames*, Adv. Comput. Math. 18 (2003), no. 2-4, 357–385.
8. R. D. Benguria. Entries on *Dirichlet eigenvalue*, *Neumann eigenvalue*, *Rayleigh–Faber–Krahn Inequality.* Encyclopaedia of Mathematics, `http://eom.springer.de/`.

9. R. D. Benguria and H. Linde. *Isoperimetric inequalities for eigenvalues of the Laplace operator.* In: Fourth Summer School in Analysis and Mathematical Physics, 1–40, Contemp. Math., 476, Amer. Math. Soc., Providence, RI, 2008.
10. M. van den Berg. *On condensation in the free-boson gas and the spectrum of the Laplacian.* J. Statist. Phys. 31 (1983), no. 3, 623–637.
11. O. Christensen. An Introduction to Frames and Riesz Bases. Birkhäuser, Boston, 2003.
12. C. Durso. *Solution of the inverse spectral problem for triangles.* Ph.D. Thesis, M.I.T., 1990.
13. G. Faber. *Beweis dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitzungsbericht der bayerischen Akademie der Wissenschaften, pp. 169–172 (1923).
14. P. Freitas. *Upper and lower bounds for the first Dirichlet eigenvalue of a triangle.* Proc. Amer. Math. Soc. 134 (2006), 2083–2089.
15. C. Gordon, D. Webb and S. Wolpert. *Isospectral plane domains and surfaces via Riemannian orbifolds.* Invent. Math. 110 (1992), 1–22.
16. D. Grieser and S. Maronna. *One can hear the shape of a triangle.* Preprint (2012). arXiv:1208.3163
17. D. Han, K. Kornelson, D. Larson and E. Weber. Frames for Undergraduates. Student Mathematical Library, 40. American Mathematical Society, Providence, RI, 2007.
18. A. Henrot. Extremum Problems for Eigenvalues of Elliptic Operators. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.
19. J. Hersch. *Contraintes rectilignes parallèles et valeurs propres de membranes vibrantes.* Z. Angew. Math. Phys. 17 (1966), 457–460.
20. H. Howards, M. Hutchings and F. Morgan. *The isoperimetric problem on surfaces.* Amer. Math. Monthly 106 (1999), no. 5, 430–439.
21. M. Kac. *Can one hear the shape of a drum?* Amer. Math. Monthly 73 (1966), no. 4, part II, 1–23.
22. S. Kesavan. Symmetrization & Applications. Series in Analysis, 3. World Scientific Publishing, Hackensack, NJ, 2006.
23. E. Krahn. *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises.* Math. Ann. 94 (1925), no. 1, 97–100.
24. E. Krahn. *Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen.* Acta Comm. Univ. Tartu (Dorpat) A, 9 (1926), 1–44. See the English translation in [34, pp. 139–175].
25. M. G. Lamé. Leçons sur la Théorie Mathématique de L'Élasticité des Corps Solides. Deuxième édition. Gauthier–Villars, Paris, 1866.
26. R. S. Laugesen, J. Liang and A. Roy. *Sums of magnetic eigenvalues are maximal on rotationally symmetric domains.* Ann. Henri Poincaré 13 (2012), 731–750.
27. R. S. Laugesen, Z. C. Pan and S. S. Son. *Neumann eigenvalue sums on triangles are (mostly) minimal for equilaterals.* Math. Inequal. Appl., 15 (2012), 381–394.
28. R. S. Laugesen and B. A. Siudeja. *Dirichlet eigenvalue sums on triangles are minimal for equilaterals.* Comm. Anal. Geom., 19 (2011), 855–885.
29. R. S. Laugesen and B. A. Siudeja. *Sharp spectral bounds on starlike domains.* Submitted (2012). `http://www.math.illinois.edu/~laugesen/starlike4.pdf`
30. R. S. Laugesen and B. A. Siudeja. *Sums of Laplace eigenvalues — rotationally symmetric maximizers in the plane.* J. Funct. Anal. 260 (2011), 1795–1823.
31. R. S. Laugesen and B. A. Siudeja. *Sums of Laplace eigenvalues — rotations and tight frames in higher dimensions.* J. Math. Phys. 52 (2011), 093703.
32. G. Lawlor. *An elementary proof of the isoperimetric inequality in all dimensions.* Preprint, `http://mathed.byu.edu/~lawlor/`.
33. Z. Lu and J. Rowlett. *The fundamental gap of simplices.* Preprint, arXiv:1109.4117.
34. Ü. Lumiste and J. Peetre, Editors. Edgar Krahn, 1894–1961. A centenary volume. IOS Press, Amsterdam; a co-publication with the Estonian Mathematical Society, Tartu, 1994.
35. J. M. Luttinger. *Generalized isoperimetric inequalities. I, II, III.* J. Mathematical Phys. 14 (1973), 586–593, 1444–1447 and 1448–1450.

36. J. Mathews and R. L. Walker. Mathematical Methods of Physics. Second edition. W. A. Benjamin, New York, 1970.
37. R. Mazzeo and J. Rowlett. *A heat trace anomaly on polygons.* Preprint, arXiv:0901.0019.
38. B. J. McCartin. *Eigenstructure of the equilateral triangle. I. The Dirichlet problem.* SIAM Rev. 45 (2003), 267–287.
39. B. J. McCartin. *Eigenstructure of the equilateral triangle. II. The Neumann problem.* Math. Probl. Eng. 8 (2002), 517–539.
40. R. Osserman. *The isoperimetric inequality.* Bull. Amer. Math. Soc. 84 (1978), no. 6, 1182–1238.
41. G. Pólya. *Sur le rôle des domaines symétriques dans le calcul de certaines grandeurs physiques.* C. R. Acad. Sci. Paris 235, (1952), 1079–1081.
42. G. Pólya and M. Schiffer. *Convexity of functionals by transplantation.* J. Analyse Math. 3 (1953–1954), 245–345. Reprinted in "George Pólya: Collected Papers," Vol. 3, pp. 290–390. MIT Press, Cambridge, Massachusetts, 1984.
43. G. Pólya and G. Szegő. Isoperimetric Inequalities in Mathematical Physics. Princeton University Press, Princeton, New Jersey, 1951.
44. J. W. Strutt (Lord Rayleigh). The Theory of Sound. Macmillan, London, 1877.
45. A. Treibergs. *Inequalities that imply the isoperimetric inequality.* `http://www.math.utah.edu/~treiberg/isoperim/isop.pdf`.
46. S. Zelditch. *Inverse spectral problem for analytic domains. II. $\mathbb{Z}_2$-symmetric domains.* Ann. of Math. (2) 170 (2009), no. 1, 205–269.

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