

# Classification problems in conformal geometry

## Methods of Lie theory and geometry

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# Recall question from part one

Question: Which linear differential operators on  $\mathbb{R}^n$  preserve harmonic functions? Answer on  $\mathbb{R}^3$ :—

Zeroth order  $f \mapsto \text{constant} \times f$

1

First order

$$\nabla_1 = \partial/\partial x_1 \quad \nabla_2 = \partial/\partial x_2 \quad \nabla_3 = \partial/\partial x_3$$

3

$$x_1 \nabla_2 - x_2 \nabla_1 \quad \&c.$$

3

$$x_1 \nabla_1 + x_2 \nabla_2 + x_3 \nabla_3 \quad \boxed{+1/2}$$

1

$$(x_1^2 - x_2^2 - x_3^2) \nabla_1 + 2x_1 x_2 \nabla_2 + 2x_1 x_3 \nabla_3 + x_1$$

3

&c.

=====

Dimensions .....

**10**

$$[\mathcal{D}_1, \mathcal{D}_2] \equiv \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1$$

Lie Algebra  $\cong \mathfrak{so}(4, 1) = \boxed{\text{conformal algebra}} \leftarrow \text{NB!}$

# Second order

Boyer-Kalnins-Miller (1976)

Extras:  $\propto$  Laplacian ( $f \mapsto h\Delta f$  for any smooth  $h$ )  
plus a 35-dim<sup>ℓ</sup> family of new ones!

$$\{\mathcal{D}_1, \mathcal{D}_2\} \equiv \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2\mathcal{D}_1$$

$$\odot^2 \mathfrak{so}(4, 1) = ? \quad \dim = 10 \times 11/2 = 55$$

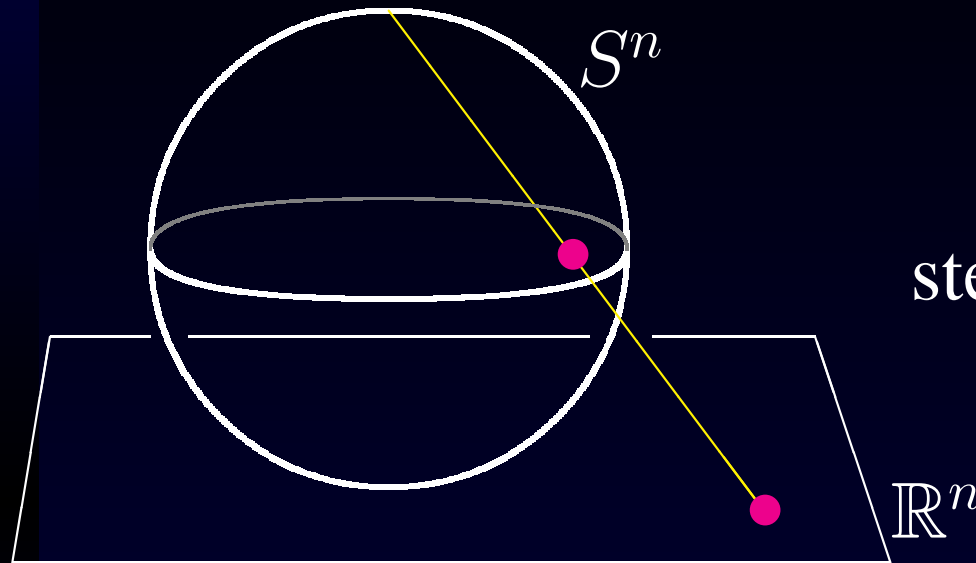
$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \odot \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \circ \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \circ \oplus \mathbb{R} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

$$55 = 35 + 14 + 1 + 5$$

Separation of variables (Bôcher, Bateman, ...).

Third order...?

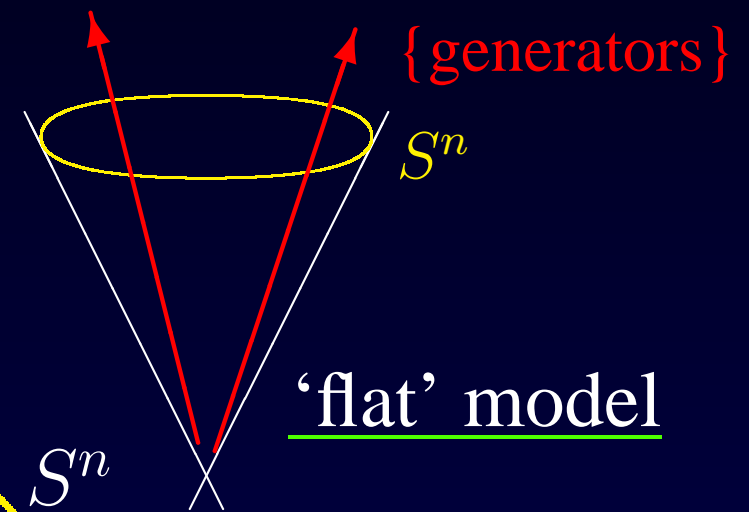
# Recall flat conformal geometry



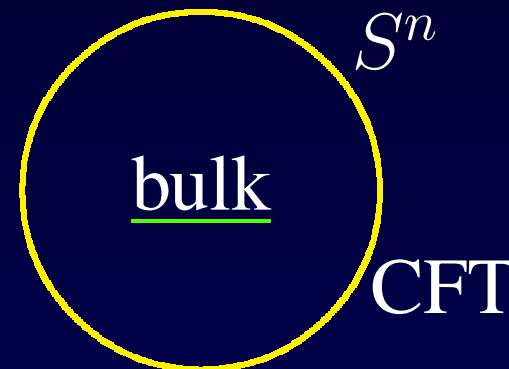
stereographic projection

Action of  $SO(n + 1, 1)$  on  $S^n$   
by conformal transformations

Beltrami model of hyperbolic space



'flat' model



# Conformal Laplacian Dirac 1935

$$r \equiv x_1^2 + \cdots + x_n^2 + x_{n+1}^2 - x_{n+2}^2$$

$$\tilde{\Delta} \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_{n+1}^2} - \frac{\partial^2}{\partial x_{n+2}^2}$$

$f$  on null cone  $\subset \mathbb{R}^{n+2}$  homogeneous of degree  $w \rightsquigarrow$

- ambiently extend to  $\tilde{f}$  of degree  $w$
- freedom  $\tilde{f} \mapsto \tilde{f} + rg$  for  $g$  of degree  $w - 2$
- calculate:  $\tilde{\Delta}(rg) = r\tilde{\Delta}g + 2(n + 2w - 2)g$

$w = 1 - n/2 \Rightarrow f \mapsto (\tilde{\Delta}\tilde{f})|_{r=0}$  is invariantly defined.

On  $\mathbb{R}^n$  it's  $\Delta$

On  $S^n$  it's  $\Delta - \frac{n-2}{4(n-1)}R$

AdS/CFT

Fefferman-Graham 'ambient' metric

# Symmetries of $\Delta$

$\mathcal{D}$  a symmetry  $\iff \Delta\mathcal{D} = \delta\Delta$  for some  $\delta$ .

trivial example:  $\mathcal{D} = \mathcal{P}\Delta$  for any  $\mathcal{P}$

equivalence:  $\mathcal{D}_1 \equiv \mathcal{D}_2 \iff \mathcal{D}_1 - \mathcal{D}_2 = \mathcal{P}\Delta$

$\mathbb{R}^n \rightsquigarrow \mathcal{A}_n \equiv$  algebra of symmetries

under composition  
up to equivalence

Write  $\mathcal{D} = \underline{V^{bc\dots d}} \nabla_b \nabla_c \dots \nabla_d +$  lower order terms

symbol

normalise w.l.g. to be **trace-free**

# Theorems

→  $\mathcal{D}$  a symmetry  $\Rightarrow$  trace-free part of  $\nabla^{(a} V^{bc\dots d)} = 0$

→ Easy 😊

→ On  $\mathbb{R}^n$ , such a conformal Killing tensor  $V^{bc\dots d} \rightsquigarrow \mathcal{D}_V$

→ Not So Easy 😞

$\mathcal{D}_V$  is a canonically associated symmetry of the form

$$\mathcal{D}_V = V^{bc\dots d} \nabla_b \nabla_c \cdots \nabla_d + \text{lower order terms.}$$

- E.g. First order

$$\mathcal{D}_V f = V^a \nabla_a f + \frac{n-2}{2n} (\nabla_a V^a) f$$

- E.g. Second order

$$\mathcal{D}_V f = V^{ab} \nabla_a \nabla_b f + \frac{n}{n+2} (\nabla_a V^{ab}) \nabla_b f + \frac{n(n-2)}{4(n+2)(n+1)} (\nabla_a \nabla_b V^{ab}) f$$

# Ingredients of proof

- We can solve the conformal Killing tensor equation

$$\nabla^{(a} V^{bc\dots d)} = g^{(ab} \lambda^{c\dots d)}$$

on  $\mathbb{R}^n$  by prolongation (use Lie algebra cohomology):–

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|} \hline & & & \dots & & & \\ \hline & & & \dots & & & \\ \hline \end{array}}_{\circ} \quad \text{w.r.t. } \mathfrak{so}(n+1, 1).$$

# of columns = # of indices on  $V^{bc\dots d}$

$$\begin{aligned} \text{E.g. } V^b &= s^b + m^{bc} x_c + \lambda x^b + r^c x_c x^b - \frac{1}{2} x^c x_c r^b \\ &= \text{translation} + \text{rotation} + \text{dilation} + \text{inversion}. \end{aligned}$$

- Use 'ambient methods' to construct  $\mathcal{D}_V$ .

# Algebra structure

As a vector space

$$\mathcal{A}_n = \bigoplus_{s=0}^{\infty} \underbrace{\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & \cdots & & & & \\ \hline & & & \cdots & & & & \\ \hline \end{array}}_s \circ$$

Question: What about the algebra structure?

Theorem

$$\mathcal{A}_n = \frac{\otimes \mathfrak{so}(n+1, 1)}{\left( \underbrace{X \otimes Y - X \odot Y}_{\text{Cartan}} - \frac{1}{2} \underbrace{[X, Y]}_{\text{Lie}} + \frac{n-2}{4n(n+1)} \underbrace{\langle X, Y \rangle}_{\text{Killing}} \right)}$$

Equivalently,

$$\mathcal{A}_n = \mathfrak{U}(\mathfrak{so}(n+1, 1)) / \underline{\text{Joseph Ideal.}}$$

# Proof of algebra structure

Calculate by ambient means that

$$\mathcal{D}_X \mathcal{D}_Y = \mathcal{D}_{X \odot Y} + \frac{1}{2} \mathcal{D}_{[X, Y]} - \frac{n-2}{4n(n+1)} \mathcal{D}_{\langle X, Y \rangle}$$

and use properties of Cartan product (due to Kostant).

Remark: simple Lie algebra  $\mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{C}) \Rightarrow$

$$\dim \frac{\mathfrak{g} \otimes \mathfrak{g}}{(X \otimes Y - X \odot Y - \frac{1}{2}[X, Y] - \lambda \langle X, Y \rangle)} = \infty$$

for precisely one value of  $\lambda$  (Braverman and Joseph)

$$\rightsquigarrow \text{graded algebra } \bigoplus_{s=0}^{\infty} \odot^s \mathfrak{g}.$$

# Conformal Laplacian cont'd

Know:  $\Delta - \frac{n-2}{4(n-1)}R$  is invariant under  $SO(n+1, 1)$ .

Better:  $\Delta - \frac{n-2}{4(n-1)}R$  is 'conformally invariant'.

Meaning:

$$g_{ab} \rightsquigarrow \hat{g}_{ab} = \Omega^2 g_{ab} \quad \nabla_a \rightsquigarrow \hat{\nabla}_a \quad R_{abcd} \rightsquigarrow \hat{R}_{abcd} \quad R \rightsquigarrow \hat{R}$$

EG:  $\hat{R} = \Omega^{-2} (R - (n-1)(2\nabla^a \Upsilon_a + (n-2)\Upsilon^a \Upsilon_a))$

where  $\Upsilon_a \equiv \nabla_a \log \Omega$

$$\left( \hat{\Delta} - \frac{n-2}{4(n-1)} \hat{R} \right) \Omega^{1-\frac{n}{2}} f = \Omega^{-1-\frac{n}{2}} \left( \Delta - \frac{n-2}{4(n-1)} R \right) f$$

$$L \equiv \Delta - \frac{n-2}{4(n-1)} R \quad \hat{L} \hat{f} = \widehat{L} f \quad \hat{f} = \Omega^{1-\frac{n}{2}} f$$

$L$  invariantly defined between suitable line bundles.

# Operators on the three-sphere

A complete list of SO(4, 1)-invariant linear differential operators between irreducible tensor bundles

- Standard (with suitable conformal weights)

$$\odot_{\circ}^b \Lambda^1 \xrightarrow{\nabla^{a+1}} \odot_{\circ}^{a+b+1} \Lambda^1 \xrightarrow{\nabla^{2b+1}} \odot_{\circ}^{a+b+1} \Lambda^1 \xrightarrow{\nabla^{a+1}} \odot_{\circ}^b \Lambda^1$$

for  $a, b \in \mathbb{Z}_{\geq 0}$  ( $a = b = 0 \rightsquigarrow$  de Rham complex)

- Non-standard

$$\odot_{\circ}^b \Lambda^1[a + 2b] \xrightarrow{\nabla^{2a+2b+3}} \odot_{\circ}^b \Lambda^1[-a - 3]$$

for  $a + 1/2, b \in \mathbb{Z}_{\geq 0}$  ( $a = -1/2, b = 0 \rightsquigarrow$  Laplacian)

Proof is by algebra (Lie theory and Verma modules)

Theorem All these operators have conformally invariant 'curved analogues.'

# Conformal-to-Einstein operator

Let  $P_{ab} \equiv \frac{1}{n-2} \left( R_{ab} - \frac{1}{2(n-1)} R g_{ab} \right)$ . Then

$$\sigma \xrightarrow{D} \text{trace-free part of } (\nabla_a \nabla_b \sigma + P_{ab} \sigma)$$

is conformally invariant when  $\hat{\sigma} = \Omega \sigma$  ( $a = 1, b = 0$ )

Geometric meaning where  $\sigma \neq 0$  (LeBrun 1985)

$$D\sigma = 0 \iff \sigma^{-2} g_{ab} \text{ is an Einstein metric}$$

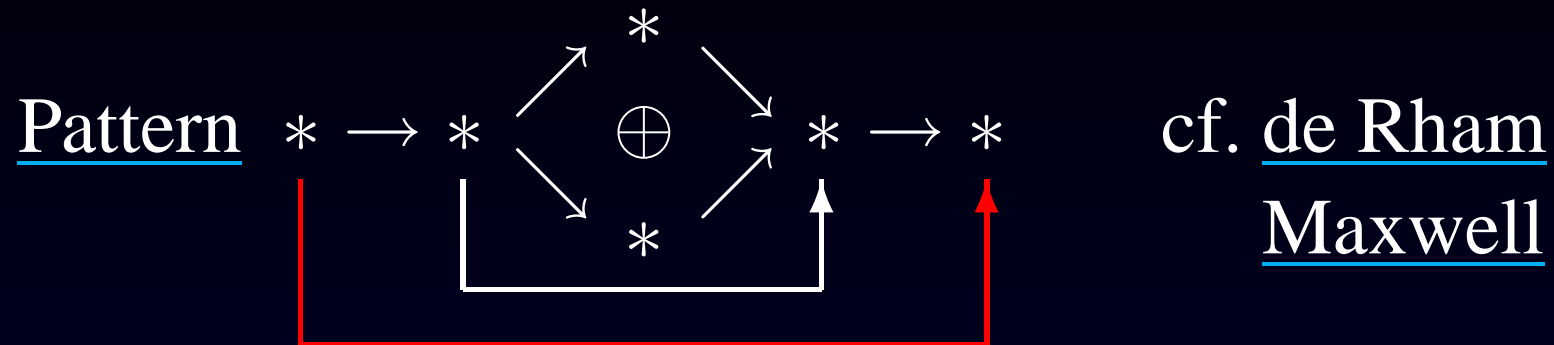
Prolong

$$\begin{aligned} \nabla_a \sigma - \mu_a &= 0 \\ D\sigma = 0 \iff \nabla_a \mu_b + P_{ab} \sigma + g_{ab} \rho &= 0 \\ \nabla_a \rho - P_a^b \mu_b &= 0 \end{aligned}$$

Cartan  
connection

$\rightsquigarrow$  Curved translation principle

# Beware the four-sphere



Theorem Most of these operators have conformally invariant 'curved analogues.'

Standard ✓ Non-standard ☹️

$\Delta$	✓	Bateman, Yamabe, et alia
$\Delta^2$	✓	Paneitz, Riegert, Eastwood & Singer
$\Delta^3$	✗	Graham
$\Delta^{\geq 4}$	✗	Gover & Hirachi
general	☹️	cf. Eastwood & Slovák

# References

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THANK YOU

THE END