RIESZ CAPACITY AND THE APPROXIMATION OF SOBOLEV FUNCTIONS BY SMOOTH FUNCTIONS

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This work, done jointly with William P. Ziemer [2], is a generalisation of the following result, proved by F.C. Liu in 1977 [1].

If Ω is a strongly Lipschitz domain in R^n, $f \in W^{\ell,p}(\Omega)$ (where $1 \leq p < \infty$) and $\varepsilon > 0$ is arbitrary, then there exists a C^ℓ function g on Ω such that

(i) the set $\{x ; x \in \Omega$ and $f(x) \neq g(x)\}$ has Lebesgue measure $< \varepsilon$

and (ii) $|f-g|_{\ell,p} < \varepsilon$.

The Michael-Ziemer generalisation gives a C^m function g on Ω (with $m \leq \ell$) , the approximation in (i) is with respect to capacity and the approximation in (ii) is with respect to the norm in $W^{m,p}$. Moreover, Ω is an arbitrary open subset of R^n .

Riesz capacity can be defined in the following way. Let $1 \leq p < \infty$ and let k be a real number, such that $k > 0$ and $kp < n$. Let $f \in L^p(R^n)$ and suppose $f \geq 0$. The Riesz potential $I_k f$ of f is the function defined on R^n by

$$I_k f(x) = \frac{1}{\gamma(k)} \int_{R^n} |x-y|^{k-n} f(y)\, dy ,$$

where $\gamma(k)$ is a positive constant whose value is not important in the present context. For each subset E of R^n , the Riesz capacity $R_{k,p}(E)$ is defined by

$$R_{k,p}(E) = \inf(|f|_p)^p ,$$

where the infimum is taken over all non-negative $f \in L^p(R^n)$, such that

$$I_k f(x) \geq 1$$

for all $x \in E$. We define

$$R_{0,p} = m^* ,$$

where m^* denotes outer Lebesgue measure.

$R_{k,p}$ is not additive, hence it is not a measure; however it is countably sub-additive. Let H^α denote Hausdorff measure of dimension α . If $kp < n$, then

$$H^{n-kp}(E) = 0 \implies R_{k,p}(E) = 0$$

and

$$R_{k,p}(E) = 0 \implies H^{n-kp+\varepsilon}(E) = 0$$

for every $\varepsilon > 0$.

There is a difficulty in generalising the Liu theorem. A function $f \in W^{\ell,p}(\Omega)$ may be undefined on a set of measure zero and this set could have positive capacity. So we establish the following result on approximate limits.

<u>THEOREM</u> *Let* $\ell p < n$, Ω *be an open set of* R^n *and* $f \in W^{\ell,p}(\Omega)$. *Then there exists a subset* E *of* Ω , *such that*

$$R_{\ell,p}(E) = 0$$

and

$$\lim_{\delta\to 0+} \frac{1}{m(B_\delta(0))} \int_{|y-x|<\delta} f(y)\,dy$$

exists for all $x \in \Omega \sim E$.

Thus, each Sobolev function can be represented by a function which is approximately continuous except for a set of zero capacity. The Michael-Ziemer approximation theorem can now be stated.

THEOREM *Let* $1 \le p < \infty$, *let* ℓ, m *be positive integers with* $1 \le m \le \ell$ *and* $(\ell-m)p < n$ *and let* Ω *be an open set of* R^n . *Let* $f \in W^{\ell,p}(\Omega)$, *and be approximately continuous at every point of* Ω , *except for a set* E *with* $R_{\ell-m,p}(E) = 0$. *Let* $\varepsilon > 0$. *Then there exists a* C^m *function* g *on* Ω , *such that*

(a) *the set* $F = \{x \ ; \ x \in \Omega$ *and* $f(x) \ne g(x)\}$ *has* $R_{\ell-m,p}(F) < \varepsilon$ *and*

(b) $|f-g|_{m,p} < \varepsilon$.

Since $R_{0,p}$ is Lebesgue outer measure, Liu's theorem is the special case with $m = \ell$.

REFERENCES

[1] Fon-Che Liu, *A Lusin Type Property of Sobolev Functions*, Indiana University Mathematics Journal, 26 (1977), 645-651.

[2] J.H. Michael and William P. Ziemer, *A Lusin Type Approximation of Sobolev Functions by Smooth Functions*, to appear.

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