SOME PROBLEMS OF SPECTRAL THEORY

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As noted by several authors (eg. [7], [8]), the spectrality of operators with spectrum contained in $\mathbb{R}$ or the unit circle $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$, can often be determined by an examination of the groups $\{e^{isT};\ s \in \mathbb{R}\}$ and $\{T^n;\ n \in \mathbb{Z}\}$, respectively. The problem is to determine when the Stone Theorem holds for these groups, that is, to determine when they are the Fourier-Stieltjes transform of a spectral measure defined on the dual group. For our purposes, it suffices to consider Stone's Theorem for the pair of (dual) groups $\mathbb{Z}$ and $\mathbb{T}$ (see [8], [11] for example).

Let X be a locally convex Hausdorff space, always assumed to be quasi-complete. The space of continuous linear operators on X with the strong operator topology is denoted by $L(X)$. The spectrum of an operator $T \in L(X)$ is taken in the sense of [11; p.270]. The σ-algebra of Borel sets of $\mathbb{T}$ is denoted by $\mathcal{B}$. Let $\mathcal{P}$ denote the space of trigonometric polynomials in $C(\mathbb{T})$ If $\psi \in C(\mathbb{T})$, then $\hat{\psi}$ denotes its Fourier transform.

<u>STONE'S THEOREM</u>. *Let the space X be barrelled and $T \in L(X)$ have an inverse in $L(X)$. Suppose that one of the following conditions is satisfied.*

(i) *For each $x \in X$, the subset*

$$A_T(x) = \{p(T)x;\ p \in \mathcal{P},\ \|p\|_\infty \le 1\},$$

of X, is relatively weakly compact.

(ii) *The group $\{T^n;\ n \in \mathbb{Z}\}$ is an equicontinuous part of $L(X)$ and*

$$B_T(x) = \left\{k^{-1}\sum_{m=0}^{k-1}\sum_{n=-m}^{m}\hat{\psi}(-n)T^n x;\ k = 1,2,\ldots,\ \psi \in C(\mathbb{T}),\ \|\psi\|_\infty \le 1\right\}$$

is a relatively weakly compact subset of X, *for each* $x \in X$.

Then there exists a regular spectral measure $E : \mathcal{B} \to L(X)$ *such that*

$$T^n = \int_{\mathbb{T}} z^n dE(z), \quad n \in \mathbb{Z}. \tag{1}$$

In particular, T *is a scalar-type spectral operator with* $\sigma(T) \subseteq \mathbb{T}$.

Conversely, if T *is a scalar-type spectral operator with* $\sigma(T) \subseteq \mathbb{T}$, *then* (i) *and* (ii) *are satisfied and* (1) *holds.*

Criterion (i) is a vector version of the well known Bochner-Schoenberg test characterizing those complex sequences on $\mathbb{Z}$ which are the Fourier-Stieltjes transform of a regular Borel measure on $\mathbb{T}$, [2]. Similarly, criterion (ii), stated in terms of Fejér means, is also a vector generalisation of an analogous statement characterising those complex sequences on $\mathbb{Z}$ which are a Fourier-Stieltjes transform, [12].

Given an operator $T \in L(X)$ with $\sigma(T) \subseteq \mathbb{T}$, it may happen, of course, that the group

$$n \mapsto T^n, \quad n \in \mathbb{Z}, \tag{2}$$

is not the Fourier-Stieltjes transform of any $L(X)$-valued spectral measure. That is, T is not a scalar-type spectral operator in the sense of N. Dunford, [1]. Nevertheless, there may exist a space Y containing X such that when interpreted as a part of Y, each of the sets $A_T(x)$, $x \in X$ (or $B_T(x)$, $x \in X$) is relatively weakly compact and T has a natural extension to an operator $T_Y \in L(Y)$. By applying Stone's Theorem to the group $n \mapsto T_Y^n$, $n \in \mathbb{Z}$, it is often possible to deduce that the extended operator T_Y is a scalar-type spectral operator. Accordingly, T_Y admits a rich functional calculus.

More precisely, a locally convex space Y is said to be admissible for a densely defined operator T in X, with domain $D(T)$, [11], if there is a continuous linear injection $\iota : X \to Y$ and an operator $T_Y \in L(Y)$, such that

$\iota(X)$ is dense in Y, the space Y is the completion or quasi-completion of $\iota(X)$ and

$$T_Y(\iota x) = \iota(Tx), \quad x \in D(T).$$

If $T \in L(X)$, then a locally convex space Y is said to be admissible for the group (2) if it is an admissible space for each operator T^n, $n \in \mathbb{Z}$, or equivalently, if it is admissible for T and T^{-1}, and if $\{T_Y^n ; n \in \mathbb{Z}\}$ is an equicontinuous part of $L(Y)$; this need not follow from the equicontinuity of (2).

If Y is an admissible space for an operator $T \in L(X)$, then $\sigma(T_Y)$ can be vastly different from $\sigma(T)$, [11; §2]. Even if $\sigma(T) = \sigma(T_Y)$, particular points of $\sigma(T)$ may be of a different type when considered as points of $\sigma(T_Y)$. For example, if $X = l^1(\mathbb{N})$ and $T \in L(X)$ is given by

$$Tx = (x_2, x_1+x_3, x_2+x_4, x_3+x_5, \ldots), \quad x \in X,$$

then $\sigma(T) = [-2,2]$. The points ± 2 belong to the continuous spectrum of T and the remaining points are in the residual spectrum, [4; pp.29-36]. Let $Y = l^2(\mathbb{N})$. Then Y is an admissible space for T. If T_Y is the natural extension of T to Y, then $\sigma(T_Y) = \sigma(T)$. However, now all the points of $\sigma(T_Y)$ belong to the continuous spectrum of T_Y [5; pp.231-232].

The following two examples illustrate how the suitable choice of an admissible space Y for the group generated by a "non-spectral operator" $T \in L(X)$ with $\sigma(T) \subseteq \mathbb{T}$, that is, the group (2), can often be used to show that T is a spectral operator when considered to be acting in Y rather than X.

<u>EXAMPLE 1</u>. Let T denote the bilateral unit shift in the space $X = l^p(\mathbb{Z})$, $1 < p < 2$, that is, $Tx = y$, $x \in X$, where $y_n = x_{n-1}$ for each

$n \in \mathbb{Z}$. Then the group (2) does not satisfy the criteria of Stone's Theorem, [3, Theorem 5.7]. Let $q > 0$ satisfy $p^{-1} + q^{-1} = 1$. Define a continuous linear operator $F : X \to L^q(\mathbb{T})$ by $F\xi = f$, $\xi \in X$, where $\hat{f} = \xi$. If the range W, of F, is equipped with the norm

$$\|f\| = \|f\|_q + \|\hat{f}\|_X, \quad f \in W,$$

then W is a Banach space and F is an isomorphism. Let $S \in L(W)$ be the operator FTF^{-1}, that is, $Sf = g$, $f \in W$, where $g(z) = zf(z)$, $z \in \mathbb{T}$. Then T is of scalar-type if and only if S is of scalar-type.

By an arc in $\mathbb{T}$ is meant a subset of the form $\{e^{it};\ t \in J\}$, where J is an interval in $\mathbb{R}$. Let $\mathcal{A}$ denote the collection of all arcs in $\mathbb{T}$ and $\mathcal{M}$ the ring generated by $\mathcal{A}$. The map $Q : \mathcal{M} \to L(W)$ given by

$$Q(\tau)f = \chi_\tau f, \quad f \in W,$$

for each $\tau \in \mathcal{M}$, is additive, multiplicative and uniformly bounded on $\mathcal{A}$. However, Q is not uniformly bounded on $\mathcal{M}$, [11; Example 2.8]. Let $E : \mathcal{M} \to L(X)$ be given by

$$E(\tau) = F^{-1}Q(\tau)F, \quad \tau \in \mathcal{M}.$$

Then the group (2), which "ought to be" the Fourier-Stieltjes transform of E, fails to be so because E cannot be extended to a σ-additive, $L(X)$-valued measure on $\mathcal{B}$. In fact, the subsets $A_T(x)$, $x \in X$, of X, are in general unbounded.

However, the space $Y = l^2(\mathbb{Z})$ is admissible for the group (2). Furthermore, if T_Y denotes the natural extension of T to Y, then the Stone Theorem applied in Y implies that $\{T_Y^n;\ n \in \mathbb{Z}\}$ is a Fourier-Stieltjes transform. In fact, if $E_Y(\tau)$ denotes the natural extension of the operator $E(\tau)$ to Y, for each $\tau \in \mathcal{M}$, and E_Y denotes the extension of the so defined measure from $\mathcal{M}$ to $\mathcal{B}$, then

$$T_Y^n = \int_{\mathbb{T}} z^n dE_Y(z), \qquad n \in \mathbb{Z}. \tag{3}$$

In particular, T_Y is a scalar-type spectral operator.

Unfortunately, it is not always possible to choose the space Y to be a Banach space.

EXAMPLE 2. Let $X = l^\infty(\mathbb{Z})$ and $T \in L(X)$ be the operator given by $Tx = y$, $x \in X$, where $y_n = e^{in}x_n$ for each $n \in \mathbb{Z}$. Then the sets $A_T(x)$, $x \in X$, although bounded in X, are not necessarily relatively weakly compact. Accordingly, the group (2) is not a Fourier-Stieltjes transform.

For each $\tau \in \mathcal{B}$, let $E(\tau) \in L(X)$ be the operator given by $E(\tau)x = y$, $x \in X$, where $y_n = \chi_\tau(e^{in})x_n$ for each $n \in \mathbb{Z}$. The group (2), which "ought to be" the Fourier-Stieltjes transform of E, again fails to be so because E is not σ-additive. In this case however, all the projections needed for the "spectral measure" are available, as distinct from Example 1, but the topology of X is too strong for E to be σ-additive.

However, the Fréchet space $Y = \mathbb{C}^{\mathbb{Z}}$ (pointwise convergence topology) is admissible for the group (2). Furthermore, if T_Y denotes the natural extension of T to Y, then $\{T_Y^n;\ n \in \mathbb{Z}\}$ satisfies Stone's Theorem. In fact, if $E_Y(\tau)$ denotes the natural extension of $E(\tau)$ to Y, for each $\tau \in \mathcal{B}$, then (3) holds. In particular, T_Y is a scalar-type spectral operator.

Given an operator $T \in L(X)$, there is no general procedure for finding an admissible space Y for the group (2) in which the extended group $\{T_Y^n;\ n \in \mathbb{Z}\}$ is a Fourier-Stieltjes transform. Some methods, applicable to a large class of examples, are discussed in [11; §4]. The aim is to find an admissible space Y for the group (2), such that the set of operators

$$\{p(T_Y);\ p \in \mathcal{P},\ \|p\|_\infty \le 1\}$$

or the set of operators

$$\{k^{-1}\sum_{m=0}^{k-1}\ \sum_{n=-m}^{m} \hat{\psi}(-n)T_Y^n;\ k = 1,2,\ldots,\ \psi \in C(\mathbb{T}),\ \|\psi\|_\infty \le 1\},$$

is an equicontinuous part of $L(Y)$ and such that each set $A_T(x)$, $x \in X$, (respectively, $B_T(x)$, $x \in X$) is relatively weakly compact in Y. It then follows (cf. proof of [6; Theorem 2]) that each set $A_{T_Y}(y)$, $y \in Y$, (respectively, $B_{T_Y}(y)$, $y \in Y$) is relatively weakly compact.

The approach suggested by the above discussion can, of course, be adopted for operators which do not necessarily have their spectrum in $\mathbb{T}$ or $\mathbb{R}$. Many operators T have naturally associated with them a large family of commuting projections which are in a certain sense dense in the prospective resolution of the identity for T. It is often possible to find an admissible space for T in which the associated family of projections can be extended to a spectral measure (see [10], [11] for example). In this way many important operators of analysis which "ought to be" spectral operators, as pointed out by N. Dunford in the survey [1], are in fact so when considered to be acting in a suitable admissible space for the given operator. We conclude with such an example.

<u>EXAMPLE 3</u>. Let γ be a complex number and p be a $\mathbb{C}$-valued function satisfying $\int_0^\infty e^{\varepsilon t}|p(t)|dt < \infty$, for some $\varepsilon > 0$. Consider the operator T given by

$$-\frac{d^2}{dt^2} + p(t), \qquad t \ge 0,$$

together with the boundary condition

(4) $$f'(0) - \gamma f(0) = 0.$$

The domain of T consists of those functions $f \in X = L^2([0,\infty))$, having derivatives f' absolutely continuous in bounded intervals of $[0,\infty)$ satisfying (4), such that $Tf \in X$.

The spectrum of T consists of the continuous spectrum $[0,\infty)$ and of a finite number of eigenvalues $\lambda_k = \mu_k^2$, $1 \le k \le r$, with $\mathrm{Im}(\mu_k) > 0$, which are zeros of some function Φ holomorphic in the half-plane $\mathrm{Im}(z) > -\frac{1}{2}\varepsilon$, [9]. It can happen that Φ also has real zeros. They too can only be finite in number. If these real zeros are denoted by $\sigma_1,\ldots,\sigma_l$, then the positive numbers $\tilde{\lambda}_j = \sigma_j^2$, $1 \le j \le l$, are called the spectral singularities of T. The "eigenfunctions" corresponding to the spectral singularities are not elements of the space X.

Assume that p and γ are such, that $r = 0$ (eg. $p \equiv 0$, $\gamma = -i$). Denote by M the δ-ring of all Borel sets in $\sigma(T)$ which have positive distance from $\Lambda = \{\tilde{\lambda}_1,\ldots,\tilde{\lambda}_l\}$. Then there exists a multiplicative, σ-additive map $E : M \to L(X)$ which commutes with T. However, if $\{\tau_n\}_{n=1}^{\infty}$ is a sequence of sets from M whose distance from Λ tends to zero as $n \to \infty$, then the sequence of norms $\{\|E(\tau_n)\|;\ n = 1,2,\ldots\}$ is unbounded, [9]. Accordingly, E cannot be extended to an $L(X)$-valued measure on the Borel sets, $\mathcal{B}(\mathbb{C})$, of $\mathbb{C}$.

Let Y denote the projective limit of the system $\{(E(\tau)X, E(\tau));\ \tau \in M\}$. Then there is an $L(Y)$-valued spectral measure E_Y, on $\mathcal{B}(\mathbb{C})$, such that $E_Y(\tau)$ is the unique extension of $E(\tau)$ for each $\tau \in M$. The operator T has an extension to a scalar-type spectral operator T_Y, in Y, with spectral resolution of the identity E_Y, [9; Theorem 5.7].

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