CHARACTERISATIONS OF DENSE SUBMANIFOLDS OF FINITE DEFECT AND THEIR EXISTENCE IN INFINITE-DIMENSIONAL SPACES

SERGEY S. AJIEV

Abstract. We show that a finite-dimensional linear submanifold of linear functionals on an infinite-dimensional linear topological space X does not contain any nontrivial continuous linear functionals if, and only if, the intersection of the kernels of the functionals from this submanifold is dense in X. The algebraic complementability of the finite dimensional subspaces of X by the above submanifolds implied by the existence of a countable bounded linearly independent subset of X is equivalent to the latter property in the setting of X with a countable base. In particular, every bounded subset of X is finite-dimensional if every linear functional is continuous. We also provide specific examples and an application to the characterisation of bounded algebraic operators on Banach spaces.

1. Introduction.

The classical characterisation [2, Section 1.9.6] of the continuity of a linear functional on a linear topological space in terms of its kernel is a remarkable example of the interrelation between purely algebraic and topological properties. We extend this result to the n-dimensional setting and study the algebraic complementability of the finite-dimensional subspaces by dense linear submanifolds in terms of the existence of bounded countable linearly independent subsets, paying special attention to the settings of Banach (normed) and countably-normed spaces, as well as the linear topological spaces with a countable base in general. When the topology of a linear topological space is so rich that every linear functional becomes continuous, it appears that the purely topological condition of the boundedness of a subset implies the purely algebraic finiteness restriction on its dimension.

2. Definitions

We shall **always** assume that a linear topological space X is infinite-dimensional and over $\mathbb{R}$. For a subset $A \subset X$, let $\overline{A}$ and $\mathrm{lin}(A)$ be its topological closure and linear envelope/span respectively. If $T : X \to Y$ is a linear mapping between linear spaces X and Y, $\mathrm{Ker}\, T$ and $\mathrm{Im}\, T$ are its kernel and range/image correspondingly. Assume also that X' is the space of all linear (not necessarily continuous) functionals, while X^* is the

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space of all linear continuous functionals. For a subset $M \subset X'$, let $M^{\perp} = \{x \in X : f(x) = 0 \text{ for } f \in M\}$ be its orthogonal complement. Recall that $\{x_i\}_{i=1}^n \subset X$ and $\{g_i\}_{i=1}^n \subset X'$ are bi-orthogonal if $g_i(x_j) = \delta_{i,j}$ for $1 \leq i, j \leq n$.

3. Dense linear submanifold of finite codimension: characterisation

Theorem 3.1. *Let M be a finite-dimensional linear submanifold of X'. Then the intersection of the kernels $M^{\perp} = \bigcap_{g \in M} \operatorname{Ker} g$ is dense in X if, and only if, $M \setminus \{0\} \subset X' \setminus X^*$.*

Proof of Theorem 3.1. Let us recall (see [2, Section 1.9.6]) that $f \in X^*$ if, and only if, $\operatorname{Ker} f$ is a closed submanifold (subspace) of X. Thus, the inclusion $M^{\perp} \subset \operatorname{Ker} g$ for $g \in M$ gives us the "only if" implication.

To approach the "if" part, we note that $M^{\perp} = \bigcap_{i=1}^n \operatorname{Ker} g_i$, where $\{g_i\}_{i=1}^n$ is a basis in M. Let us assume that $\overline{M^{\perp}}$ is a proper subspace of X. The Hahn-Banach theorem gives us a non-trivial $f \in X^*$ with

$$\operatorname{Ker} f \supset M^{\perp} = \bigcap_{i=1}^{n} \operatorname{Ker} g_i.$$

Therefore, $f \in M$ thanks to Lemma 3.2, contradicting the inclusion $M \setminus \{0\} \subset X' \setminus X^*$ and, thus, finishing the proof. □

Since the next two lemmas are very likely to be known to the reader, we present, perhaps, the simplest proof of the most needed one and postpone the detailed discussion till the end of this section. The next lemma helps to compute the codimension of $M^{\perp}$.

Lemma 3.1 (see [2, 3, 4, 5])**.** *Let $\{f_i\}_{i=1}^n$ be a basis in a linear submanifold $M \subset X'$. Then there exists a bi-orthogonal system $\{x_i\}_{i=1}^n \subset X$.*

Proof of Lemma 3.1. Since $\{f_i\}_{i=1}^n$ are linearly independent, Lemma 3.2 below implies that, for every i, there exists $x_i' \in \bigcap_{k \neq i} \operatorname{Ker} f_k \setminus \operatorname{Ker} f_i$. The normalisation $x_i := x_i'/f_i(x_i')$ gives us the bi-orthogonal system $\{x_i\}_{i=1}^n$ sought for, finishing the proof of the lemma. □

Corollary 3.1. *Let M be a finite-dimensional linear submanifold of X'. Then $M^{\perp} = \bigcap_{g \in M} \operatorname{Ker} g$ is dense in X if, and only if, $\operatorname{Ker} g$ is dense in X for every $g \in M$. Moreover, the codimension of $M^{\perp}$ coincides with the dimension of M.*

Proof of Corollary 3.1. The first part of the corollary follows from Theorem 3.1 and the above-mentioned equivalence (contained, for example, in [2, Section I.9.6])

$$f \in X^* \iff \overline{\operatorname{Ker} f} = \operatorname{Ker} f.$$

The application of Lemma 3.1 yields the codimension of $M^{\perp}$. □

Originally knowing the next lemma as Problem 6 in [3, Section 4.13], the author had learnt from the anonymous referee that it was established by Phillips in 1940.

Lemma 3.2 (see [2, 3, 4, 5]). *For $n \in \mathbb{N}$, let $\{f_i\}_{i=0}^n \subset X'$ and $\bigcap_{i=1}^n \operatorname{Ker} f_i \subset \operatorname{Ker} f_0$. Then there exist $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}$ satisfying $f_0 = \sum_{i=1}^n \alpha_i f_i$.*

Lemmas 3.1 and 3.2 are treated in equivalent terms with the aid of the mathematical induction as Lemma 5 and its corollary in [4, Section II.3], as Propositions 1.14.2 and 1.4.3(2) in [2, Section I.1.4], and as Proposition 1.1 and its corollary in [5, Section IV.1.1]. In the latter source, Lemma 3.2 is deduced from Lemma 3.1, thus, showing that they are actually equivalent.

4. Dense linear submanifold of finite codimension: existence

Theorem 4.1. *For a linear topological space X, let us consider the following properties:*

(i) the space X contains a bounded countable linearly independent system;

(ii) for every $n \in \mathbb{N}$ and a linearly independent subset $\{x_j\}_{j=1}^n$, there exists an n-dimensional linear submanifold $M \subset X'$ satisfying $M \setminus \{0\} \subset X' \setminus X^*$, $\overline{M^\perp} = X$, $M^\perp + \operatorname{lin}\left(\{x_j\}_{j=1}^n\right) = X$ and $M^\perp \cap \operatorname{lin}\left(\{x_j\}_{j=1}^n\right) = \{0\}$.

Then we have the implications:

$a)$ $(i) \Longrightarrow (ii)$;

$b)$ if the topology of X possesses a countable base, then $(i) \Longleftrightarrow (ii)$.

Proof of Theorem 4.1. Let $S' = \{e'_i\}_{i\in\mathbb{N}}$ be the bounded linearly independent set mentioned in (i). To establish Part $a)$, we first form another bounded countable linearly independent set $S = \{e_k\}_{k\in\mathbb{N}} = \{x_j\}_{j=1}^n \cup (S' \setminus W)$, where $W = S' \cap \operatorname{lin}\left(\{x_j\}_{j=1}^n\right)$. According to M. Zorn's lemma, the set of all linearly independent systems containing S and partially ordered by inclusion possesses a least upper bound $H = \{e_i\}_{i\in I}$ with $\mathbb{N} \subset I$ (because $S \subset H$). Hence, H is a Hamel basis of X.

There exists a decomposition $\bigcup_{j=1}^n I_j = \mathbb{N}$ satisfying $x_j \subset I_j$ for $1 \le j \le n$ and $I_k \cap I_l = \varnothing$ for $k \ne l$. For every $1 \le i \le n$, we define the linear functional $f_i = g_i / g_i(x_i)$, where g_i is the linear extension of the function $\phi_i : H \to \mathbb{R}$ defined by

$$\phi_i(e_k) = \begin{cases} k & \text{for } k \in I_i; \\ 0 & \text{for } k \in I \setminus I_i. \end{cases}$$

Note that the linear span M of $\{f_i\}_{i=1}^n$ does not contain nontrivial continuous linear functionals and is also bi-orthogonal to $\{x_i\}_{i=1}^n$. Indeed, a non-trivial $f \in M$ is not bounded on the bounded set $\{e_i\}_{i\in I_k} \subset S$, where α_k is a non-zero coefficient in the expansion $f = \sum_{i=1}^n \alpha_i f_i \ne 0$. The application of Theorem 3.1 finishes the proof of $a)$ and the same implication in $b)$.

To establish the opposite implication in $b)$, we start with a countable base $\tau'_0 = \{G'_i\}_{i\in\mathbb{N}}$ of open vicinities of the origin, form its nested counterpart $\tau_0 = \{G_i\}_{i\in\mathbb{N}}$ with $G_i = \bigcap_{j=1}^i G'_j$ and construct a bounded linearly independent sequence $\{e_k\}_{k\in\mathbb{N}}$ by induction as follows.

According to (ii) with $n = 1$, there exists $x_1 \in G_1$ and $f'_1 \in X' \setminus \{0\}$ with $f_1(X_1) = 1$ and the dense $\operatorname{Ker} f_1 \subset X$ satisfying $\operatorname{lin}(\{x_1\}) + \operatorname{Ker} f_1 = X$ and $\operatorname{lin}(\{x_1\}) \cap \operatorname{Ker} f_1 = \{0\}$.

At the kth step, we start with the system $\{x_j\}_{j=1}^{k-1}$ and $M_{k-1} \setminus \{0\} \subset X' \setminus X^*$ chosen already to satisfy $x_j \in G_j$ for $1 \le j \le k-1$, $\operatorname{lin}\left(\{x_j\}_{j=1}^{k-1}\right) +$

$\overline{M_{k-1}^{\perp}} = X$ and $\operatorname{lin}\left(\{x_j\}_{j=1}^{k-1}\right) \cap M_{k-1}^{\perp} = \{0\}$, with $M_{k-1}^{\perp}$ being dense in X. The latter density allows us to find $x_k \in G_k \cap M_{k-1}^{\perp}$ meaning the linear independence of $\{x_j\}_{j=1}^{k}$ and implying, due to the validity of (ii) with $n = k$, the existence of $M_k^{\perp}$ satisfying $M_k \setminus \{0\} \subset X' \setminus X^*$, $\overline{M_k^{\perp}} = X$, $M_k^{\perp} + \operatorname{lin}\left(\{x_j\}_{j=1}^{k}\right) = X$ and $M_k^{\perp} \cap \operatorname{lin}\left(\{x_j\}_{j=1}^{k}\right) = \{0\}$.

Therefore, the resulting sequence $\{x_i\}_{i\in\mathbb{N}}$ satisfies $\{x_j\}_{j\geq i} \subset G_i$ for every $i \in \mathbb{N}$ and, thus, bounded in X. □

The succeeding immediate corollary to Theorem 4.1 shows that, if the topology of a linear topological space is so rich that the algebraic property of the linearity of a linear functional implies the topological property of its automatic continuity, then the topological property of the boundedness of a subset of such space implies the algebraic property of its finite dimension.

Corollary 4.1. *Let X be a linear topological space with $X' = X^*$, and let $A \subset X$ be bounded. Then the dimension of* $\operatorname{lin}(A)$ *is finite.*

5. Application and examples

In [1, Theorem 4.4], we have extended Kaplansky's characterisation of continuous algebraic operators on a Banach space by showing that every closed and locally algebraic operator T with a non-empty resolvent set $\rho(T)$ is a bounded algebraic operator. Example 5.1 combined with Theorem 4.1, $b)$ implies that, for a real or complex Banach space X with $\dim(X) = \infty$ and every polynomial q with $q(0) = 0$ and, correspondingly, real or complex coefficients, there exists a discontinuous $T : X \to X$ with $\overline{\operatorname{Ker} T} = X$ and $\dim(\operatorname{Im} T) = \deg(q)$ satisfying $q(T) = 0$, where q is a polynomial of the minimal degree for the last identity to hold. Thus, the conditions of the closedness of the operator and $\rho(T) \neq \varnothing$ in [1, Theorem 4.4], implying the closedness of its kernel, are essential. (Note that $\rho(T) \neq \varnothing$ in $\mathbb{C}$ implies the closedness.)

Indeed, for $n = deg(q)$, Example 5.1 and Theorem 4.1, $b)$ produce the system $\{f_j\}_{j=1}^{n} \subset \operatorname{lin}\left(\{f_j\}_{j=1}^{n}\right) \setminus \{0\} \subset X' \setminus X^*$ with a bi-orthogonal system $\{x_i\}_{i=1}^{n} \subset X$ added by Lemma 3.1. Taking an arbitrary matrix $A = (a_{ij})_{i,j=1}^{n}$ with the minimal polynomial q, we define $T : X \to X$ sought for by $T(x) := \sum_{i,j=1}^{n} a_{ij} f_j(x) x_i$. Note that we can choose $A \in \mathbb{R}^{n\times n}$ if q is real.

Example 5.1. *Either F. Riesz' lemma, or A. N. Kolmogorov's characterisation of the normed spaces combined with the existence of a Hamel basis implies the existence of a countable linearly independent subset of the unit sphere of every infinite-dimensional normed space X. If X is an infinite-dimensional Banach space, a combination of the Hahn-Banach theorem and a local compactness argument show that its unit sphere contains even a Schauder basic sequence* (*i.e. a normalised Schauder basis in a subspace of* X).

Example 5.2. *If X is an infinite-dimensional locally convex linear topological space with a countable base τ_0 of convex open vicinities of the origin,* *meaning that the topology of X is generated by the Minkowski functionals $\{p_j\}_{j\in\mathbb{N}}$ of the elements of τ_0, then X contains a bounded countable linearly independent system. Indeed, every Hamel basis of X contains a countable linearly independent system $S' = \{e'_m\}_{m\in\mathbb{N}}$. Since every element of τ_0 is absorbing, we transform S' into the bounded linearly independent system $S = \{e_m\}_{m\in\mathbb{N}}$ by means of the normalisation*

$$e_m = \frac{e'_m}{\max_{i=1}^m p_i(e'_m)} \text{ for } m \in \mathbb{N}.$$

The last example in the style of [2] illustrates the phenomenon described in Corollary 4.1.

Example 5.3. *Assume that $\mathcal{M}$ is the uncountable set of all strictly increasing integer sequences $m = \{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Let the locally convex linear topological space Y be defined as the linear submanifold of c_0 of all finitely non-zero sequences $x = \{x_k\}_{k\in\mathbb{N}} \in c_0$ (i.e. with the finite $\operatorname{supp} x = \{k \in \mathbb{N} : x_k \neq 0\}$) endowed with the topology generated by the system $\{p_m\}_{m\in\mathcal{M}}$ of seminorms $p_m(x) := \sup_{k\in\mathbb{N}} m_k|x_k|$. Then Y possesses the following properties.*

a) Every linear functional on Y is continuous, i.e. $Y' = Y^*$.

b) For every bounded $A \subset X$, the linear span (envelope) $\operatorname{lin}(A)$ is of finite dimension.

Proof of Example 5.3. Part *a*) follows from the observation that $f \in Y'$ is defined by its values $\{f(e_i)\}_{i\in\mathbb{N}}$ on the natural coordinate basis $\{e_i\}_{i\in\mathbb{N}}$ of c_0 (and Y). Namely, we have the continuity estimate $|f(x)| \le p_m(x)$ for $x \in Y$ whenever the (strictly increasing) sequence $m \in \mathcal{M}$ is chosen to satisfy $m_k \ge |f(e_k)|\,(\pi k)^2/6$ for $k \in \mathbb{N}$. Indeed, for $x = \sum_k x_k e_k$, one has

$$|f(x)| \le \sum_{k\in\operatorname{supp} x} |x_k||f(e_k)| \le \sum_{k\in\operatorname{supp} x} 6|x_k|m_k/(\pi k)^2 < p_m(x) \sum_{k\in\mathbb{N}} 6/(\pi k)^2 = p_m(x).$$

While *a*) implies *b*) with the aid of Corollary 4.1, we give a direct proof. Indeed, if $A \subset Y$ is bounded, then the union $\bigcup_{x\in A} \operatorname{supp} x$ is finite. Otherwise, one would have a sequence $\{x^j\}_{j\in\mathbb{N}} \subset A$ with $x^j_{k_j} \neq 0$ and, thus, also $m' \in \mathcal{M}$ satisfying $m'_{k_j} > 1/|x^j_{k_j}|$ for $j \in \mathbb{N}$ that is impossible because the set $\{p_m(x)\}_{x\in A}$ must be bounded, particularly, for $m = \{km'_k\}_{k\in\mathbb{N}} \in \mathcal{M}$. □

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School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW, 2522, Australia.

E-mail address: sajiev@uow.edu.au