VIRTUAL TECHNIQUE FOR ORBIFOLD FREDHOLM SYSTEMS

BOHUI CHEN, AN-MIN LI AND BAI-LING WANG

ABSTRACT. In this paper, we review the the theory of virtual manifold/orbifolds developed by the first named author and Tian and develop the virtual technique for any orbfiold Fredholm system with compact moduli space $\mathcal{M}$. This provides a description of $\mathcal{M}$ in terms of a virtual orbifold system

$$\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I)\}$$

Here $\{\mathcal{V}_I\}$ is a virtual orbifold, and $\{\mathbf{E}_I\}$ is a finite rank virtual orbifold bundle with a virtual section $\{\sigma_I\}$ such that the zero sets $\{\sigma_I^{-1}(0)\}$ form a cover of the underlying moduli space $\mathcal{M}$. A virtual orbifold system can be thought as a special class of Kuranishi structures on a moduli problem developed by Fukaya and Ono. Under some assumptions which guarantee the existence of a partition of unity and a virtual Euler form, we show that the virtual integration is well-defined for the resulting virtual orbifold system.

CONTENTS

1. INTRODUCTION AND STATEMENTS OF MAIN THEOREMS

Pseudo-holomorphic curves in a symplectic manifold (X, ω) with a compatible almost complex structure J were first introduced by Gromov in his seminal paper [10]. This has been followed by deep results in symplectic topology. The Gromov-Witten invariants "*count* " stable pseudo-holomorphic curves of genus g and n-marked points in a symplectic manifold (X, ω, J). The Gromov-Witten invariants for semi-positive symplectic manifolds were defined by Ruan in [30] and Ruan-Tian in [33] and [34]. These invariants can be applied to define a quantum product on the cohomolgy groups of X which leads the notion of quantum cohomolog in [33] for semi-positive symplectic manifolds. Since then the Gromov-Witten invariants have found many applications in symplectic geometry and symplectic topology, see the book by McDuff-Salamon [26] (and references therein).

The main analytical difficulty in defining the Gromov-Witten invariants for general symplectic manifolds is the failure of the transversality of the compactified moduli space of pseudo-holomorphic curves. The foundation to resolve this issue is to construct a virtual fundamental cycle for the compactified moduli space. For smooth projective varieties, the construction of this virtual fundamental cycle was carried out by Li-Tian in [19] where they showed that the Gromov-Witten invariants can be defined purely algebraically. For general symplectic manifolds, the virtual fundamental cycle was constructed by Fukaya-Ono in [11], Li-Tian [20], Liu-Tian [21] and Siebert [29]. Ruan in [32] proposed a virtual neighbourhood technique as a dual approach using the Euler class of a virtual neighbourhood, in which the compactified moduli space is treated as a zero set of a smooth section of a finite dimensional orbfiold vector bundles over an open orbifold.

Further developments in Gromov-Witten theory and its applications require more refined structures on the moduli spaces involved. Some of the analytical details have been provided by Ruan [32], Li-Ruan [15] and Fukaya-Oh-Ohta-Ono [12]. Other methods like the polyfold theory by Hofer-Wysocki-Zehnder [14] are developed to deal with this issue. Recently, there are some renew interests on a variety of technical issues in Gromov-Witten theory by McDuff-Wehrheim [27] and Fukaya-Oh-Ohta-Ono [13]. These technical issues include the differentiable structures on Kuranishi models of the Gromov-Witten moduli spaces and the non-differentiable issue of the action of automorphism groups in the infinite dimensional non-linear Fredholm framework.

Our aim is motivated by how to define the K-theoretical Gromov-Witten invariants for general symplectic manifolds. This amounts to the full machinery of the virtual neighborhood technique to study the Gromov-Witten moduli spaces using the virtual manifold/orbifolds developed in [9]. The language of virtual orbifolds provides an alternative and simpler approach to establish the required differentiable structure on moduli spaces arising from the Gromov-Witten theory. We remark that the theory of virtual manifold/orbifolds and the general framework of the virtual neighborhood technique were established by the first author and Tian in [9].

The virtual neighborhood technique avoids some of subtle and difficult issues when further perturbations are needed. We use the following example to demonstrate the idea that further perturbations are not needed to define Euler invariants. Let E be a real oriented vector bundle over a compact smooth manifold U, and let Θ be a vertically compactly supported Thom form of E then for $\omega \in \Omega^*(U)$

$$\int_U s^*\Theta \wedge \omega = \int_{s^{-1}(0)} \iota^*\omega \tag{1.1}$$

for any section s of E which is transverse to the zero section. Here $\iota : s^{-1}(0) \to U$ is the inclusion map. This formula says that for those integrands from the ambient space U, there is no need to perturb the section to achieve the transversality in order to calculate the right hand side of (1.1), as we know that the final result is given by the left hand side of (1.1)

$$\int_U s^*\Theta \wedge \omega \tag{1.2}$$

for any (not necessarily transversal) section s. Similar results hold for a compact orbifold U with an oriented real orbifold vector bundle. Note that U can be replaced by a non-compact orbifold, then the integrand $s^*\Theta \wedge \omega$ needs to be compactly supported.

In the original proposal of Ruan in [32], it was proposed that the compactified moduli space $\mathcal{M}$ of stable maps in a closed symplectic manifold X can be realised as $s^{-1}(0)$ for a section s of an orbifold vector bundle E over a finite dimensional C^1-orbifold U. The triple (U, E, s) is called a virtual neighbourhood of $\mathcal{M}$ in [32]. In practice, it turns out that such an elegant triple is too idealistic to obtain an orbifold Fredholm system for general cases. Here an **orbifold Fredholm system** consists of a triple

$$(\mathcal{B}, \mathcal{E}, S),$$

where $\mathcal{E}$ is a Banach orbifold bundle over a Banach orbifold $\mathcal{B}$ together with a Fredholm section S. There does not exist a single virtual neighbourhood (U, E, s) even for an orbifold Fredholm system. We remark that for certain orbifold Fredholm systems, the virtual Euler cycles and their properties have been developed systematically in an abstract setting by Lu and Tian in [22].

To remedy this, the theory of virtual manifolds and virtual orbifolds were developed by the first author and Tian, see [9]. We will give a self-contained review of the theory of virtual manifolds and virtual orbifolds in Section 2 and further establish the integration theory on virtual orbifolds using the language of proper étale groupoids. The language of groupiods is very useful when an orbifold atlas can not be clearly described, in particular, in the case of stable maps where the patching data (arrows in the language of groupoids) are induced from biholomorphisms appearing in the universal families of curves over the Teichmüller spaces.

It was proposed in [9] that the single virtual neighbourhood proposed in [32] should be replaced by a system of virtual neighbourhoods

$$\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I)\}$$

indexed by a partial ordered index set $\mathcal{I} = \{I \subset \{1, 2, \cdots, n\} | I \text{is non-empty}\}$. Here $\{\mathcal{V}_I\}$ is a virtual orbifold, and $\{\mathbf{E}_I\}$ is a finite rank virtual orbifold bundle with a virtual section $\{\sigma_I\}$ such that the zero sets $\{\sigma_I^{-1}(0)\}$ form a cover of the underlying moduli space $\mathcal{M}$. If the sections $\{\sigma_I\}$ were all transversal, then the moduli space $\mathcal{M}$ would be a smooth orbifold. In general, this system of virtual neighbourhoods will be called a **virtual system** (Cf. Definitions 3.2 and 4.2). The invariants associated to the moduli space can be obtained by applying the integration theory on $\{\mathcal{V}_I\}$ to certain virtual differential forms. In particular, the collection of Euler forms for $\{\mathbf{E}_I\}$, called a **virtual Euler form**, is a virtual differential form on $\{\mathcal{V}_I\}$. In essence, this virtual Euler form is the dual version of the so-called virtual fundamental cycle.

To demonstrate this idea, global stabilizations(**cf.** §**3.3.2**) were introduced in [9] to get a virtual system from a single Fredholm system $(\mathcal{B}, \mathcal{E}, S)$. These will be reviewed and further developed to orbifold cases in terms of proper étale groupoids in Section 3. We also explain how the integration theory for virtual orbifolds (proper étale virtual groupoids) can be applied to get a well-defined invariant from the virtual system. This process of going from a Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ to a well-defined invariant will be called the **virtua neighbourhood technique**. The main result in Section 3 is to establish the following theorem for any orbifold Fredholm system.

Theorem (Theorem 4.6) *Given an orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ such that $\mathcal{M} = S^{-1}(0)$ is compact, then there exists a finite dimensional virtual orbifold system for $(\mathcal{B}, \mathcal{E}, S)$ which is a collection of* *triples*

$$\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I) | I \subset \{1, 2, \cdots, n\}\}$$

indexed by a partially ordered set ($\mathcal{I} = 2^{\{1,2,\cdots,n\}}, \subset$)*, where*

(1) $\mathcal{V} = \{\mathcal{V}_I\}$ *is a finite dimensional proper étale virtual groupoid,*
(2) $\mathbf{E} = \{\mathbf{E}_I\}$ *is a finite rank virtual orbifold vector bundle over* $\{\mathcal{V}_I\}$
(3) $\sigma = \{\sigma_I\}$ *is a virtual section of* $\{\mathbf{E}_I\}$ *whose zero sets* $\{\sigma_I^{-1}(0)\}$ *form a cover of* $\mathcal{M}$.

Moreover, under Assumptions 4.3 and 4.8, there is a choice of a partition of unity $\eta = \{\eta_I\}$ *on* $\mathcal{V}$ *and a virtual Euler form* $\theta = \{\theta_I\}$ *of* $\mathbf{E}$ *such that each* $\eta_I \theta_I$ *is compactly supported in* $\mathcal{V}_I$. *Therefore, the virtual integration*

$$\int_{\mathcal{V}}^{vir} \alpha = \sum_I \int_{\mathcal{V}_I} \eta_I \theta_I \alpha_I$$

is well-defined for any virtual differential form $\alpha = \{\alpha_I\}$ *on* $\mathcal{V}$ *when both* $\mathcal{V}$ *and* $\mathbf{E}$ *are oriented.*

This paper is organised as follows. In Section 2 we review and summarize the theory of virtual manifolds/orbifolds and the integration theory on proper étale virtual groupoids. Examples and properties of virtual manifolds and virtual vector bundles are listed here for later use. In Section 3, we introduce a notion of virtual system for a Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ satisfying and develop a full machinery of virtual neighborhood techniques. In Section 4, we generalise the virtual neighbourhood technique to any orbifold Fredholm system satisfying Assumptions 4.3 and 4.8. In Section 5, we introduce a notion of proper étale weak Lie groupoids and construct a weak orbifold virtual system for any weak orbifold Fredholm system satisfying Assumptions 5.8.

2. Theory of Virtual manifolds and virtual orbifolds

In this section, we give a self-contained review of the theory of virtual manifolds/orbifolds in [9] and further develop the integration theory on proper étale virtual groupoids.

2.1. Basic Definitions.

Definition 2.1. A *virtual manifold* $\{X_I, \Phi_{I,J}, \phi_{I,J}\}_{I\in\mathcal{I}}$ is a collection of smooth manifolds $\{X_I\}_{I\in\mathcal{I}}$ indexed by

$$\mathcal{I} = \{I \subseteq \{1, 2, \cdots, n\} | I \text{ is non-empty}\}$$

such that whenever I is a proper subset of J, there is a patching datum

$$\begin{array}{ccc} X_{J,I} & \xrightarrow{\subset} & X_J \\ \Phi_{I,J} \big\downarrow \big\uparrow \phi_{I,J} & & \\ X_{I,J} & \xrightarrow{\subset} & X_I. \end{array}$$

where $X_{I,J}$ and $X_{J,I}$ are open sub-manifolds of X_I and X_J respectively, and $X_{J,I}$ is a real vector bundle over $X_{I,J}$ with the projection map $\Phi_{I,J}$ and the zero section $\phi_{I,J}$. Moreover, the patching data $\{(\Phi_{I,J}, \phi_{I,J})\}$ satisfies the following coherence conditions:

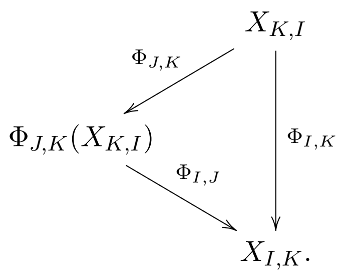
(1) Given any ordered triple $I \subset J \subset K$ with the patching data

$$\begin{array}{ccc} X_{J,I} & \xrightarrow{\subset} & X_J \\ \Phi_{I,J} \big\downarrow \big\uparrow \phi_{I,J} & & \\ X_{I,J} & \xrightarrow{\subset} & X_I, \end{array} \qquad \begin{array}{ccc} X_{K,J} & \xrightarrow{\subset} & X_K \\ \Phi_{J,K} \big\downarrow \big\uparrow \phi_{J,K} & & \\ X_{J,K} & \xrightarrow{\subset} & X_J, \end{array} \qquad \begin{array}{ccc} X_{K,I} & \xrightarrow{\subset} & X_K \\ \Phi_{I,K} \big\downarrow \big\uparrow \phi_{I,K} & & \\ X_{I,K} & \xrightarrow{\subset} & X_I, \end{array}$$

we have

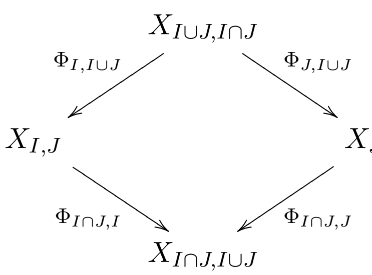
- $X_{K,I} \subset X_{K,J}$ in X_K,
- $X_{I,K} \subset X_{I,J}$ in X_I,
- $\Phi_{J,K}(X_{K,I}) = \Phi^{-1}_{I,J}(X_{I,K})$ in X_J,
- and the cocycle condition $\Phi_{I,K} = \Phi_{I,J} \circ \Phi_{J,K}$ as given by the following commutative diagram

(2.1)



(2) Given a pair I and J with nonempty intersection $I \cap J$, then $X_{I\cup J, I\cap J} = X_{I\cup J,I} \cap X_{I\cup J,J}$ and $X_{I\cap J,I\cup J} = X_{I\cap J,I} \cap X_{I\cap J,J}$ hold, and the diagram

(2.2)



commutes and is a fiber product of vector bundles over $X_{I\cap J,I\cup J}$. Here $X_{I,J} = \Phi_{I,I\cup J}(X_{I\cup J,I\cap J})$ and $X_{J,I} = \Phi_{J,I\cup J}(X_{I\cup J,I\cap J})$ for any pair I and J.

Remark 2.2. (1) Definition 2.1 is equivalent to Definitions 2.1 and 2.2 in [9] except we remove the empty set from the index set, as it does not play any role in the study of virtual technique

(2) We can replace the collection of smooth manifolds $\{X_I\}$ by a collection of topological spaces where the patching data $\{\Phi_{I,J} : X_{J,I} \to X_{I,J}\}$ consist of topological vector bundles, then the coherence conditions (2.1) and (2.2) still make sense in the category of topological vector bundles. The resulting object is called a *topological virtual space*.

Definition 2.3. (*Vector bundles and virtual vector bundles*) Let $\{X_I\}$ be a virtual manifold $\{X_I, \Phi_{I,J}, \phi_{I,J}\}$.

(1) A **real vector bundle** over a virtual manifold $\{X_I\}$ is a virtual manifold $\mathcal{F} = \{F_I\}$ such that F_I is a vector bundle over X_I for each I, and for any ordered pair $I \subset J$, the patching datum for $\mathcal{F}$ is given by

$$F_{I,J} = F_I|_{X_{I,J}}, \qquad F_{J,I} = F_J|_{X_{J,I}}, \qquad F_J|_{X_{J,I}} \cong \Phi^*_{I,J}(F_{I,J}). \tag{2.3}$$

A section of a vector bundle $\mathcal{F}$ over $\{X_I\}$ is a collection of sections $\{S_I : X_I \to F_I\}$ such that for any ordered pair $I \subset J$, under the identification (2.3),

$$S_J|_{X_{J,I}} = \Phi^*_{I,J}\big(S_I|_{X_{I,J}}\big). \tag{2.4}$$

A section $\{S_I\}$ of a vector bundle $\mathcal{F}$ is transverse if each S_I is a transverse section of $F_I \to X_I$.

(2) A **virtual vector bundle** over a virtual manifold $\{X_I\}$ is a virtual manifold $\mathbf{E} = \{E_I \to X_I\}$ such that E_I is a vector bundle over X_I for each I, and for any ordered pair $I \subset J$, the patching datum for $\mathbf{E}$ is given by

$$E_{I,J} = E_I|_{X_{I,J}}, \qquad E_{J,I} = E_J|_{X_{J,I}}, \qquad E_J|_{X_{J,I}} \cong \Phi^*_{I,J}(X_{J,I} \oplus E_I|_{X_{I,J}}). \tag{2.5}$$

A section of a virtual vector bundle $\mathbf{E}$ over $\{X_I\}$ is a collection of sections $\{\sigma_I : X_I \longrightarrow E_I, \}$ called a virtual section of $\mathbf{E}$ if any ordered pair $I \subset J$, under the identification (2.5),

$$\sigma_J|_{X_{J,I}} = \big(s_{X_{J,I}}, \sigma_I|_{X_{I,J}} \circ \Phi_{I,J}\big), \tag{2.6}$$

where $s_{X_{J,I}}$ is the canonical section of the bundle $\Phi^*_{I,J}(X_{J,I})$ over $X_{J,I}$. A section $\sigma = \{\sigma_I\}$ of a virtual vector bundle $\mathbf{E}$ is transverse if each σ_I is a transverse section of $E_I \to X_I$.

Example 2.4. (1) For a smooth compact manifold X with an open cover $\{U_i | i = 1, 2, \cdots, n\}$, there is a canonical virtual manifold structure $\{U_I, \Phi_{I,J}, \phi_{I,J}\}$ given by $U_I = \bigcap_{i\in I} U_i$ for any nonempty $I \subseteq \{1, 2, \cdots, n\}$,

$$U_{I,J} = U_{J,I} = U_I \cap U_J$$

and for $I \subset J$, $\Phi_{I,J} = \phi_{I,J} = Id_{U_I \cap U_J}$. Both coherence conditions (2.1) and (2.2) are trivially satisfied. A virtual vector bundle over the virtual manifold $\{U_I\}$ corresponds to a usual vector bundle over X.

(2) (*Example 1 in* [9]) The following example of virtual manifold structure on a compact manifold X plays a central role in the study of moduli spaces arising from Fredholm systems in Section 3.

Let $\{U_i^{(1)} | i = 1, 2, \cdots, n\}$ be an open cover of X such that for the closure of each $U_i^{(1)}$ is contained in an open set $U_i^{(2)}$. For any nonempty $I \subset \{1, 2, \cdots, n\}$, set

$$X_I = \big(X \cap \bigcap_{i\in I} U_i^{(1)}\big) \setminus \big(\bigcup_{j\notin I} \overline{U_i^{(2)}}\big). \tag{2.7}$$

Define $X_{I,J} = X_{J,I} = X_I \cap X_J$ and $\Phi_{I,J} = \phi_{I,J} = Id_{X_I \cap X_J}$ for any $I \subset J$. Then $\{X_I, \Phi_{I,J}, \phi_{I,J}\}$ is a virtual manifold.

(3) Given a virtual manifold $\{X_I, \Phi_{I,J}, \phi_{I,J} | I \subset J \in 2^{\{1,2,\cdots,n\}}\}$, its tangent bundle

$$\{TX_I \longrightarrow X_I\}$$

is a virtual vector bundle as for any ordered pair $I \subset J$

$$TX_J|_{X_{J,I}} \cong \Phi^*_{I,J}(X_{J,I} \oplus TX_I|_{X_{I,J}}).$$

We remark that the cotangent bundle $\{T^*X_I\}$ is not virtual vector bundle unless $X_{J,I} \to X_{I,J}$ is equipped with an inner product so that the dual bundle $X^*_{J,I} \to X_{I,J}$ is identified $X_{J,I}$. Though the k-th exterior bundle $\{\bigwedge^k(T^*X_I)\}$ are neither vector bundles nor virtual vector bundles, a section $\omega = \{\omega_I\}$ of $\{\bigwedge^k(T^*X_I)\}$ will be called a degree k **differential form** on $\{X_I\}$ if the following condition holds

$$\omega_J|_{X_{J,I}} = \Phi^*_{I,J}\big(\omega_I|_{X_{I,J}}\big).$$

The following proposition plays a key role in the construction of virtual system in this paper.

Proposition 2.5. *Let $\{X_I, \Phi_{I,J}, \phi_{I,J}\}$ be a virtual manifold.*

(1) *Given a vector bundle $\mathcal{F} = \{F_I \to X_I\}$ with a transverse section $\{S_I\}$, then the collections of zero sets*

$$\{Z_I := S_I^{-1}(0)\}$$

admits a canonical virtual manifold structure. If $\{S_I\}$ is not transverse to the zero section, then $\{Z_I\}$ is a topological virtual space.

(2) *Given a virtual vector bundle $\{E_I \to X_I\}$ with a transverse section $\{\sigma_I\}$, then the collection of zero sets*

$$\{Y_I := \sigma_I^{-1}(0)\},$$

under the induced patching data, forms a smooth manifold. In the absence of transversality, $\{Y_I\}$ forms a topological space under the homeomorphism $\Phi_{I,J} : Y_{J,I} \cong Y_{I,J}$.

Proof. (1) Being a transverse section $\{S_I\}$, each zero set $Z_I = S_I^{-1}(0)$ is a smooth manifold. For $I \subset J$, we have

- $Z_{I,J} = Z_I \cap X_{I,J} = (S_I|_{X_{I,J}})^{-1}(0)$,
- $Z_{J,I} = Z_J \cap X_{J,I} = (S_J|_{X_{J,I}})^{-1}(0)$,
- The conditions (2.3) and (2.4) imply that $\Phi_{I,J}|_{Z_{J,I}} : Z_{I,J} \to Z_{I,J}$ is a vector bundle.

It is easy to check that the induced patching data satisfy the coherence conditions (2.1) and (2.2). If $\{S_I\}$ is not transversal to the zero section, then each $Z_I = S_I^{-1}(0)$ is only a topological space. The induced patching data defines a virtual topological structure. That is, $\{Z_I = S_I^{-1}(0)\}$ is only a topological virtual space.

(2) In this case, the collection of zero sets $\{Y_I = \sigma_I^{-1}(0)\}$ consists of smooth manifolds of the same dimention. Note that for $I \subset J$, we have

- $Y_{I,J} = Y_I \cap X_{I,J} = (\sigma_I|_{X_{I,J}})^{-1}(0)$,
- $Y_{J,I} = Y_J \cap X_{J,I} = (\sigma_J|_{X_{J,I}})^{-1}(0)$, under $\Phi_{I,J}$, is diffeomorphic to $Y_{I,J}$.

Therefore, $\{Y_I = \sigma_I^{-1}(0)\}$ forms a smooth manifold.

Without the transversality condition, obviously, $\{Y_I = \sigma_I^{-1}(0)\}$ forms a topological space, as $Y_{J,I} = Y_J \cap X_{J,I} = (\sigma_J|_{X_{J,I}})^{-1}(0)$, under $\Phi_{I,J}$, is only homeomorphic to $Y_{I,J}$.

□

Remark 2.6. As explained in [9], we can define a virtual manifold with boundary. Here we require that each manifold X_I is a manifold with a boundary ∂X_I, and the patching data satisfy the following condition

$$\partial X_{J,I} = \Phi_{I,J}^{-1}(\partial X_{I,J}).$$

Then $\{\partial X_I\}_I$ is a virtual manifold. Moreover, Proposition 2.5 takes the following form. Let

$$\{X_I, \Phi_{I,J}, \phi_{I,J}\}$$

be a virtual manifold with a boundary $\{\partial X_I\}_I$.

(1) Given a vector bundle $\mathcal{F} = \{F_I \to X_I\}$ with a transverse section $\{S_I\}$, then the collections of zero sets

$$\{Z_I := S_I^{-1}(0), \partial Z_I\}$$

is a virtual manifold with boundary. where If $\{S_I\}$ is not transverse to the zero section, then $\{Z_I\}$ is only a topological virtual space with boundary.

(2) Given a virtual vector bundle $\{E_I \to X_I\}$ with a transverse section $\{\sigma_I\}$, then the collection of zero sets

$$\{Y_I := \sigma_I^{-1}(0)\},$$

under the induced patching data, forms a smooth manifold with boundary. In the absence of transversality, $\{Y_I\}$ forms a topological space with boundary.

Remark 2.7. The definitions of virtual manifold and virtual vector bundle can be easily generalised to equivariant cases under Lie group actions and to orbifold cases. The virtual manifold with boundary can be defined in these set-up too.

(1) (*Virtual G-manifold*) Given a Lie group G, a virtual G-manifold is a collection of G-manifolds $\{X_I\}_{I\in\mathcal{I}}$, together with patching data

$$\{(\Phi_{I,J}, \phi_{I,J})|I, J \in \mathcal{I}, I \subset J\},$$

where $\Phi_{I,J} : X_{J,I} \to X_{I,J}$ is a G-equivariant vector bundle with the zero section $\phi_{I,J} : X_{I,J} \to X_{J,I}$ for open G-invariant sub-manifolds $X_{I,J}$ and $X_{J,I}$ of X_I and X_J respectively, whenever I is a proper subset of J. Moreover, the patching data $\{(\Phi_{I,J}, \phi_{I,J})|I \subset J\}$ satisfy the coherence conditions (2.1) and (2.2) in the G-equivariant sense.

(2) (*Virtual orbifold*) A virtual orbifold is a collection of orbifolds $\{\mathcal{X}_I\}_{I\in\mathcal{I}}$, together with patching data

$$\{(\Phi_{I,J}, \phi_{I,J})|I, J \in \mathcal{I}, I \subset J\},$$

where $\Phi_{I,J} : X_{J,I} \to X_{I,J}$ is an orbifold vector bundle with the zero section $\phi_{I,J} : \mathcal{X}_{I,J} \to \mathcal{X}_{J,I}$ for open sub-orbifolds $\mathcal{X}_{I,J}$ and $\mathcal{X}_{J,I}$ of $\mathcal{X}_I$ and X_J respectively, whenever I is a proper subset of J. Moreover, the patching data $\{(\Phi_{I,J}, \phi_{I,J})|I \subset J\}$ satisfy the coherence conditions (2.1) and (2.2) in the category of orbifold vector bundles.

It becomes more evident that the language of proper étale groupoids provides a convenient and economical way to describe orbifolds. We briefly recall the definition of a proper étale groupoid.

Definition 2.8. (*Lie groupoids and proper étale groupoids*) A Lie groupoid $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1)$ consists of two smooth manifolds $\mathcal{G}^0$ and $\mathcal{G}^1$, together with five smooth maps (s, t, m, u, i) satisfying the following properties.

(1) The source map and the target map $s, t : \mathcal{G}^1 \to \mathcal{G}^0$ are submersions.
(2) The composition map

$$m : \mathcal{G}^{[2]} := \{(g_1, g_2) \in \mathcal{G}^1 \times \mathcal{G}^1 : t(g_1) = s(g_2)\} \longrightarrow \mathcal{G}^1$$

written as $m(g_1, g_2) = g_1 \circ g_2$ for composable elements g_1 and g_2, satisfies the obvious associative property.
(3) The unit map $u : \mathcal{G}^0 \to \mathcal{G}^1$ is a two-sided unit for the composition.
(4) The inverse map $i : \mathcal{G}^1 \to \mathcal{G}^1$, $i(g) = g^{-1}$, is a two-sided inverse for the composition.

In this paper, a groupoid $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1)$ will be denoted by $\mathcal{G} = (\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$ where $\mathcal{G}^0$ will be called the space of objects or units, and $\mathcal{G}^1$ will be called the space of arrows. A Lie groupoid $\mathcal{G} = (\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$ is **proper** if $(s, t) : \mathcal{G}^1 \to \mathcal{G}^0 \times \mathcal{G}^0$ is proper, and called **étale** if s and t are local diffeomorphisms. Given a proper étale groupoid $(\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$, for any $x \in \mathcal{G}^0$, $(s, t)^{-1}(x, x) = s^{-1}(x) \cap t^{-1}(x)$ is a finite group, called the **isotropy group** at x.

Remark 2.9. Given a proper étale Lie groupoid $\mathcal{G}$, there is a canonical orbifold structure on its orbit space $|\mathcal{G}|$ ([1, Prop. 1.44]). Here the orbit space $|\mathcal{G}|$ is the quotient space of $\mathcal{G}^0$ by the equivalence relation defined by $\mathcal{G}^1$. Two Morita equivalent proper étale Lie groupoids define two diffeomorphic orbifolds ([1, Theorem 1.45]). Conversely, given an orbifold $\mathcal{X}$ with a given orbifold atlas, there is a canonical proper étale Lie groupoid $\mathcal{G}_{\mathcal{X}}$, locally given by the action groupoid of the orbifold charts (See [28] and [23]). Two equivalent orbifold atlases define two Morita equivalent proper étale Lie groupoids. Due to this correspondence, a proper étale Lie groupoid will also be called an orbifold groupoid. Often this same notion will be used particularly for the proper étale Lie groupoid $\mathcal{G}_{\mathcal{X}}$ constructed from an orbifold atlas on an orbifold $\mathcal{X}$.

An orbifold vector bundle $E \to \mathcal{X}$ corresponds to a vector bundle over the groupoid $\mathcal{G}_{\mathcal{X}}$, which is a vector bundle $\pi : E^0 \to \mathcal{G}^0$ with a fiberwise linear action

$$\mu : \mathcal{G}^1 \times_{(s,\pi)} E^0 \longrightarrow E^0$$

covering the canonical action of $\mathcal{G}^1$ on $\mathcal{G}^0$ and satisfying some obvious compatibility conditions. Here

$$\mathcal{G}^1 \times_{(s,\pi)} E^0 = \{(g, v) \in \mathcal{G}^1 \times E^0 | s(g) = \pi(v)\}.$$

In general, a vector bundle over a Lie groupoid $(\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$ is a vector bundle E^0 over $\mathcal{G}^0$ with a fiberwise linear action

$$\mu : \mathcal{G}^1 \times_{(s,\pi)} E^0 \longrightarrow E^0, \tag{2.8}$$

and a section of a vector bundle over $\mathcal{G}$ is a section of E^0 which is invariant under the action of (2.8).

Note that the action (2.8) is defined by a section ξ of the bundle $Iso(s^*E^0, t^*E^0) \to \mathcal{G}^1$ where $Iso(s^*E^0, t^*E^0)$ is the bundle of bundle isomorphisms from s^*E to t^*E. That means, given an arrow $\alpha \in \mathcal{G}^1$, there is a linear isomorphism of vector spaces

$$\xi(\alpha) : E^0_{s(\alpha)} \longrightarrow E^0_{t(\alpha)}$$

such that $\xi(\alpha \circ \beta) = \xi(\alpha) \circ \xi(\beta)$. In fact, the action (2.8) defines a Lie groupoid

$$E_1 := \mathcal{G}^1 \times_{(s,\pi)} E^0 \rightrightarrows E^0$$

with the source map $\tilde{s}$ given by the projection, and the target map $\tilde{t}$ given by the action. This motivates the following characterisation of vector bundles in the language of Lie groupoids.

Proposition 2.10. *Given a Lie groupoid $\mathcal{G} = (\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$, a Lie groupoid $(E^1 \rightrightarrows E^0)$ is a vector bundle over $\mathcal{G}$ if and only if there is a strict Lie groupoid morphism $\pi : (E^1 \rightrightarrows E^0) \to (\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$ given by the commutative diagram*

$$\begin{array}{ccc} E^1 & \xrightarrow{\pi_1} & \mathcal{G}^1 \\ \downdownarrows & & \downdownarrows \\ E^0 & \xrightarrow[\pi_0]{} & \mathcal{G}^0 \end{array} \tag{2.9}$$

in the category of Lie groupoids with strict morphisms (this means, the maps π_1 and π_0 commute with the source maps and the target maps, and are compatible to the composition, unit and inverse maps), such that

(1) *the diagram (2.9) is a pull-back groupoid diagram,, that is, for any $v_1, v_2 \in E^0$, the set of arrows from v_1 to v_2 in E^1 agrees with the set of arrows from $\pi_0(v_1)$ to $\pi_0(v_2)$ in $\mathcal{G}^1$,*

(2) *both* $\pi_1 : E^1 \to \mathcal{G}^1$ *and* $\pi_0 : E^0 \to \mathcal{G}^0$ *are vector bundles.*

(3) *for* $\alpha \in \mathcal{G}^1$, $x = s(\alpha)$ *and* $y = t(\alpha)$, *the fiber of* E^1 *at* α

$$E^1_\alpha = \{(v_x, \alpha, v_y) | (v_x, v_y) \in E^0_{s(\alpha)} \times E^0_{t(\alpha)}\}$$

is defined by a linear isomorphism $\xi(\alpha) : E^0_{s(\alpha)} \to E^0_{t(\alpha)}$ *sending* v_x *to* x_y.

A section s *of a vector bundle* $(E^1 \rightrightarrows E^0)$ *over* $(\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$ *is given by a pair of sections* (s_1, s_0) *such that the following diagram commutes in the category of Lie groupoids with strict morphism*

$$\begin{array}{ccc} E^1 & \underset{\pi_1}{\overset{s_1}{\rightleftarrows}} & \mathcal{G}^1 \\ \downdownarrows & & \downdownarrows \\ E^0 & \underset{\pi_0}{\overset{s_0}{\rightleftarrows}} & \mathcal{G}^0. \end{array}$$

Proof. The proof is straightforward, so it is omitted. □

With this proposition understood, the definition of virtual G-manifolds and virtual orbifolds given in Remark 2.1 can be written in terms of Lie groupoids. The resulting geometric object is called a virtual Lie groupoid. We leave the explicit definition of general virtual Lie groupoids to the readers. In this paper, we will only be interested in the study of virtual G-manifolds and virtual orbifolds (or equivalently, proper étale virtual groupoids). Here a **proper étale virtual groupoid** is a collection of proper étale groupoids $\{\mathcal{G}_I\}_{I\in\mathcal{I}}$ indexed by non-empty subsets of $\{1, 2, \cdots, n\}$, together with patching data

$$\{(\Phi_{I,J}, \phi_{I,J}) | I \subset J\},$$

where $\Phi_{I,J} : \mathcal{G}_{J,I} \to \mathcal{G}_{I,J}$ is a vector bundle with the zero section $\phi_{I,J} : \mathcal{G}_{I,J} \to \mathcal{G}_{J,I}$ for open subgroupoids $\mathcal{G}_{I,J}$ and $\mathcal{G}_{J,I}$ of $\mathcal{G}_I$ and $\mathcal{G}_J$ respectively, whenever $I \subset J$. Moreover, the patching data $\{(\Phi_{I,J}, \phi_{I,J}) | I \subset J\}$ satisfy the coherence conditions (2.1) and (2.2) in the category of proper étale groupoids with the strict morphisms.

We remark that a finite rank (virtual) vector bundle over a proper étale virtual groupoid can be defined in a similar manner. Examples in Example 2.4 all have their proper étale virtual groupoid counterparts. In particular, we have the foliowing proposition whose proof is is analogous to the proof of Proposition 2.5. There is also a version of this proposition for a proper étale virtual groupoid with boundary.

Proposition 2.11. *Let* $\{\mathcal{G}_I, \Phi_{I,J}, \phi_{I,J} | I \subset J \in 2^{\{1,2,\cdots,n\}}\}$ *be a proper étale virtual groupoid.*

(1) *Given a vector bundle* $\mathcal{F} = \{F_I \to \mathcal{G}_I\}$ *with a transverse section* $\{S_I\}$, *then the collections of zero sets*

$$\{Z_I := S_I^{-1}(0)\}$$

defines a canonical proper étale virtual groupoid. where If $\{S_I\}$ *is not transverse to the zero section, then* $\{Z_I\}$ *is only a topological proper étale virtual groupoid.*

(2) *Given a virtual vector bundle* $\{E_I \to \mathcal{G}_I\}$ *with a transverse section* $\{\sigma_I\}$, *then the collection of zero sets*

$$\{Y_I := \sigma_I^{-1}(0)\},$$

under the induced patching data, forms a smooth proper étale groupoid. In the absence of transversality, $\{Y_I\}$ *forms a topological proper étale groupoid, under the homeomorphism* $\Phi_{I,J} : Y_{J,I} \cong Y_{I,J}$.

As virtual manifolds are special cases of virtual orbifolds, we only develop the integration theory for virtual orbifolds using the language of proper étale groupoids, following closely the corresponding integration theory for virtual manifolds in [9]. This will done in Subsection 2.2 after we discuss the orientation on proper étale virtual groupoids.

2.2. **Differential forms and orientations on proper étale virtual groupoids.**

A differential form on a proper étale groupoid $\mathcal{G} = (\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$ is a differential form on $\mathcal{G}^0$ which is invariant under the action of $\mathcal{G}^1$. By Proposition 2.10, a differential k-form ω on $\mathcal{G}$ is a pair of sections (ω_0, ω_1) of the k-th exterior power of the cotangent bundles such that the diagram commutes

$$\begin{array}{ccc} \left(\bigwedge^k T^*\mathcal{G}\right)^1 & \underset{\pi_1}{\overset{\omega_1}{\leftrightarrows}} & \mathcal{G}^1 \\ \downdownarrows & & \downdownarrows \\ \left(\bigwedge^k T^*\mathcal{G}\right)^0 & \underset{\pi_0}{\overset{\omega_0}{\leftrightarrows}} & \mathcal{G}^0. \end{array} \tag{2.10}$$

We remark that $\left(\bigwedge^k T^*\mathcal{G}\right)^0 = \bigwedge^k T^*\mathcal{G}^0$ and $\left(\bigwedge^k T^*\mathcal{G}\right)^1$ defined as in Proposition 2.10 agrees with $\left(\bigwedge^k T^*\mathcal{G}^1\right)$, the existence of the section ω_1 is guaranteed by ω_0 and the invariance property under the action $\mathcal{G}^1$.

A differential form ω on $\mathcal{G}$ is **compactly supported** if the support of ω is $\mathcal{G}$-compact in the sense that the $\mathcal{G}^1$-quotient of the support is compact in $|\mathcal{G}|$. The space of differential k-forms (with $\mathcal{G}$-compact support, respectively) will be denoted by $\Omega^k(\mathcal{G})$ ($\Omega^k_c(\mathcal{G})$ respectively).

In order to define the integral of certain differential forms (to be specified later) over proper étale virtual groupoids, we need to define a notion of orientation on it. First we recall the orientable condition on a proper étale groupoid. Given a proper étale groupoid $\mathcal{G}$, the tangent bundle $T\mathcal{G}$, by applying Proposition 2.10, is a vector bundle over $\mathcal{G}$ given by a strict morphism $\pi : T\mathcal{G} \to \mathcal{G}$, that is, a commutative diagram in the category of étale virtual groupoids

$$\begin{array}{ccc} (T\mathcal{G})^1 & \xrightarrow{\pi_1} & \mathcal{G}^1 \\ \downdownarrows & & \downdownarrows \\ (T\mathcal{G})^0 & \xrightarrow[\pi_0]{} & \mathcal{G}^0 \end{array}$$

where $(T\mathcal{G})^0 = T\mathcal{G}^0$ is the usual tangent bundle of $\mathcal{G}^0$. A proper étale groupoid $\mathcal{G}$ is orientable if and only if the determinant (the highest exterior power) bundle

$$\pi : \left((\det T\mathcal{G})^1 \rightrightarrows (\det T\mathcal{G})^0\right) \longrightarrow \left(\mathcal{G}^1 \rightrightarrows \mathcal{G}^0\right)$$

of the tangent bundle $T\mathcal{G}$, with $(\det T\mathcal{G})^0 = \det(T\mathcal{G}^0)$ and $(\det T\mathcal{G})^1 = \det(T\mathcal{G}^1)$, is trivial. When $\mathcal{G}$ is orientable, a trivialisation of $T\mathcal{G}$ defines an orientation of $\mathcal{G}$. That is, there exist a pair of nowhere vanishing sections $\mathbf{o}_0$ and $\mathbf{o}_1$ of $\pi_i : (\det T\mathcal{G})^i \to \mathcal{G}^i, i = 1, 2$, such that the diagram

$$\begin{array}{ccc} (\det T\mathcal{G})^1 & \underset{\pi_1}{\overset{\mathbf{o}_1}{\leftrightarrows}} & \mathcal{G}^1 \\ \downdownarrows & & \downdownarrows \\ (\det T\mathcal{G})^0 & \underset{\pi_0}{\overset{\mathbf{o}_0}{\leftrightarrows}} & \mathcal{G}^0 \end{array} \tag{2.11}$$

commutes in the category of groupoids with strict morphisms. An orientation on $\mathcal{G}$ will be denoted by a nowhere vanishing section $\mathbf{o}_{\mathcal{G}} : \mathcal{G} \longrightarrow \det T\mathcal{G}$.

Similarly, given a real vector bundle E over a groupoid $\mathcal{G}$, E is orientable if and only if $\det E$ is trivializable in the sense that there is a nowhere vanishing section $\mathbf{o}_E : \mathcal{G} \to \det E$ (defining an orientation on E).

Now we can give a definition of an orientation on a proper étale virtual groupoid.

Definition 2.12. Given a proper étale virtual groupoid $\{(\mathcal{G}_I, \Phi_{I,J}, \phi_{I,J}) | I \subset J\}$, we say that it is oriented if there exists a system of nowhere vanishing sections (called an orientation on $\{\mathcal{G}_I\}$)

$$\mathbf{o}_I : \mathcal{G}_I \longrightarrow \det T\mathcal{G}_I, \qquad \mathbf{o}_{I,J} : \mathcal{G}_{I,J} \longrightarrow \det \mathcal{G}_{J,I}$$

for every I and every ordered pair $I \subset J$, satisfying

$$\mathbf{o}_J|_{\mathcal{G}_{J,I}} = \Phi_{I,J}^*(\mathbf{o}_I|_{\mathcal{G}_{I,J}} \otimes \mathbf{o}_{I,J}),$$

under the isomorphisms

$$(\det T\mathcal{G}_J)|_{\mathcal{G}_{J,I}} \cong \Phi_{I,J}^*\big((\det T\mathcal{G}_I)|_{\mathcal{G}_{I,J}} \otimes (\det \mathcal{G}_{J,I})\big).$$

A virtual vector bundle $\{E_I\}$ over an oriented proper étale virtual groupoid $\{\mathcal{G}_I\}$ is called oriented if there is a collection of orientations $\{\mathbf{o}_{E_I}\}$ such that the bundle isomorphisms (2.5) preserve the orientations.

Given an oriented proper étale groupoid $\mathcal{G}$, introduce the notation

$$\Omega_c^*(\mathcal{G}) = \bigoplus_k \Omega_c^k(\mathcal{G}).$$

Then there is a well-defined linear functional

$$\int_{\mathcal{G}} : \Omega_c^*(\mathcal{G}) \longrightarrow \mathbb{R}$$

given by

$$\int_{\mathcal{G}} \omega = \int_{\mathcal{G}} \omega^{top}, \tag{2.12}$$

where ω^{top} is the degree m component of ω with $m = \dim \mathcal{G}_0$. We now recall the definition of integration over a proper étale groupoid for example as in [1]. Denote by $|\mathcal{G}|$ the quotient space of $\mathcal{G}^0$ by the equivalence relation from $\mathcal{G}^1$. Let $\{(G_i \ltimes \tilde{U}_i \rightrightarrows \tilde{U}_i)\}$ be a collection of sub-groupoids with each G_i being a finite group such that the collection of quotient spaces $\{U_i = \tilde{U}_i/G_i\}$ forms an open cover of $|\mathcal{G}|$. There is a collection of smooth functions $\{\rho_i\}$ with the following property

(1) ρ_i is a G_i-invariant function on $\tilde{U}_i$, hence defines a continuous function $\bar{\rho}_i$ on U_i.
(2) $supp(\rho_i) \subset \tilde{U}_i$ is compact.
(3) $0 \leq \rho_i \leq 1$.
(4) For every $x \in |\mathcal{G}|$, $\sum_i \bar{\rho}_i(x) = 1$.

A partition of unity subordinate to $\{(\tilde{U}_i, G_i)\}$ can be obtained from a routine construction as the manifold case. Then the integration (2.12)

$$\int_{\mathcal{G}} \omega = \sum_i \frac{1}{|G_i|} \int_{\tilde{U}_i} \rho_i \cdot \omega \tag{2.13}$$

is independent of the choice of $\{\rho_i\}$ and the covering sub-groupoids $\{(G_i \ltimes \tilde{U}_i \rightrightarrows \tilde{U}_i)\}$. For simplicity, we write

$$\frac{1}{|G_i|}\int_{\tilde{U}_i} \rho_i \cdot \omega = \int_{U_i} \bar{\rho}_i \cdot \omega. \tag{2.14}$$

In order to develop a meaningful integration theory on proper étale virtual groupoids, we will introduce a special class of differential forms, called twisted virtual forms below. Recall that a **Thom form** for a rank m oriented vector bundle $\pi : \mathcal{E} = (E^1 \rightrightarrows E^0) \to \mathcal{G} = (\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$ is a closed differential m-form $\Theta_{\mathcal{E}}$ on $\mathcal{E}$ compactly supported in the vertical direction such that under the integration along the fiber

$$\pi_* : \Omega^k_{cv}(\mathcal{E}) \longrightarrow \Omega^{k-m}(\mathcal{G}),$$

we have

$$\pi_*(\Theta_{\mathcal{E}}) = 1 \in \Omega^0(\mathcal{G}).$$

Here $\Omega^k_{cv}(\mathcal{E})$ is the space of differential k-forms on $\mathcal{E}$ with compact support in the vertical direction. The corresponding cohomology will be denoted by $H^*_{cv}(\mathcal{E})$. The construction of Thom forms with support in an arbitrary small neighbourhood of the zero section is quite standard, one can adapt the construction in Chapter 1.6 in [2] to the cases of oriented proper étale groupoids.

The Thom form plays an important role in differential geometry through the Thom isomorphism, the projective and product formulae.

(1) (**Thom isomorphism**) The map sending ω in $\Omega^*(\mathcal{G})$ to $\pi^*\omega \wedge \Theta_{\mathcal{E}} \in \Omega^{*+k}_{cv}(\mathcal{E})$ defines a functorial isomorphism
$$H^*(\mathcal{G}) \cong H^{*+m}_{cv}(\mathcal{E})$$
whose inverse map is π_*.

(2) (**Projection formula**) Suppose that $\mathcal{G}$ is oriented, then for every closed differential form $\omega \in \Omega^{*+k}_c(\mathcal{G})$,
$$\int_{\mathcal{E}} \Theta_{\mathcal{E}} \wedge \pi^*\omega = \int_{\mathcal{G}} \omega \tag{2.15}$$

(3) (**Product formula**) Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two oriented vector bundles over $\mathcal{G}$, and π_1 and π_2 be two projections from $\mathcal{E}_1 \oplus \mathcal{E}_2$ to $\mathcal{E}_1$ and $\mathcal{E}_2$. If $\Theta_{\mathcal{E}_1}$ and $\Theta_{\mathcal{E}_2}$ are Thom forms of $\mathcal{E}_1$ and $\mathcal{E}_2$ with compact vertical support, then
$$\pi_1^*\Theta_{\mathcal{E}_1} \wedge \pi_2^*\Theta_{\mathcal{E}_2}$$
is a Thom form of $\mathcal{E}_1 \oplus \mathcal{E}_2$ with compact vertical support.

We remark that Thom isomorphism and the Projective/Product formulae follow from the same proofs for smooth manifold cases as in [3], as the existence of a partition of unity and the Mayer-Vietoris principle hold for proper étale groupoids.

Definition 2.13. Let $\{\mathcal{G}_I\} = \{(\mathcal{G}_I, \Phi_{I,J}, \phi_{I,J}) | I \subset J\}$ be a proper étale oriented virtual groupoid with an orientation, that is, a coherent orientation system $\{\mathbf{o}_I, \mathbf{o}_{I,J} : I \subset J\}$ as in Definition 2.12.

(1) A collection of Thom forms $\Theta = \{\Theta_{I,J} | I \subset J\}$, where $\Theta_{I,J}$ is a Thom form of the bundle $\Phi_{I,J} : \mathcal{G}_{J,I} \to \mathcal{G}_{I,J}$, is called a **transition Thom form** for $\{\mathcal{G}_I\}$, if the following two conditions are satisfied.
   (a) For any ordered triple $I \subset J \subset K$, $\Theta_{I,K} = \Theta_{J,K} \wedge \Phi^*_{J,K}\big(\Theta_{I,J}\big)$ as differential forms on $X_{K,I}$.
   (b) For any I and J, $\Theta_{I\cap J, I\cup J} = \Phi^*_{I,I\cup J}\big(\Theta_{I\cap J,I}\big) \wedge \Phi^*_{J,I\cup J}\big(\Theta_{I\cap J,J}\big)$ on $\mathcal{G}_{I\cup J, I\cap J}$.

(2) Given a transition Thom form $\Theta = \{\Theta_{I,J} | I \subset J\}$, a collection of differential forms

$$\omega = \{\omega_I \in \Omega^k_c(\mathcal{G}_I)\}$$

is called a **Θ-twisted differential form** on $\{\mathcal{G}_I\}$ if $\{\omega_I\}$ satisfy the following condition

$$\omega_J|_{\mathcal{G}_{J,I}} = \Phi^*_{I,J}(\omega_I|_{\mathcal{G}_{I,J}}) \wedge \Theta_{I,J}. \tag{2.16}$$

From the projective formula for Thom forms, we know that for any Θ-twisted compactly supported differential form $\{\omega_I\}$ on $\{\mathcal{G}_I\}$,

$$\int_{\mathcal{G}_{J,I}} \omega_J = \int_{\mathcal{G}_{I,J}} \omega_J.$$

The space of all Θ-twisted differential forms with compact support on $\{\mathcal{G}_I\}$ is denoted by $\Omega^*_c(\{\mathcal{G}_I\}, \Theta)$. The de Rham differential is well-defined on $\Omega^*_c(\{\mathcal{G}_I\}, \Theta)$, so $(\Omega^*_c(\{\mathcal{G}_I\}, \Theta), d)$ is a complex, whose cohomology

$$H^k_c(\{\mathcal{G}_I\}, \Theta) = \frac{\{\text{closed } \Theta\text{-twisted differential forms on } \{\mathcal{G}_I\}\}}{\{\text{exact } \Theta\text{-twisted differential forms on } \{\mathcal{G}_I\}\}},$$

is called the Θ-twisted cohomology of $\{\mathcal{G}_I\}$.

2.3. **Integration on proper étale oriented virtual groupoids.** In this subsection, we define a notion of integration map on the space of Θ-twisted virtual differential forms

$$\int_{\{\mathcal{G}_I\}} : \Omega^*_c(\{\mathcal{G}_I\}, \Theta) \longrightarrow \mathbb{R}$$

for an oriented proper étale virtual groupoid $\{\mathcal{G}_I\}$ with a transition Thom form $\Theta = \{\Theta_{I,J} | I \subset J\}$. We need to assume that a partition of unity exists for a topological space obtained from $\{\mathcal{G}_I\}$.

For any proper étale virtual groupoid $\{\mathcal{G}_I\}$, there is a topological space

$$|\{\mathcal{G}_I\}| = \bigsqcup_I |\mathcal{G}_I| / \sim \tag{2.17}$$

where $\sim$ is an equivalence relation: for $x \in |\mathcal{G}_I|$ and $y \in |\mathcal{G}_J|$, $x \sim y$ if and only if there exists a $K \subset I \cap J$ such that

$$|\Phi_{K,I}|(x) = |\Phi_{K,J}|(y) \tag{2.18}$$

where $|\Phi_{K,I}| : |\mathcal{G}_{I,K}| \to |\mathcal{G}_{K,I}|$ and $|\Phi_{K,J}| : |\mathcal{G}_{J,K}| \to |\mathcal{G}_{K,J}|$ are the quotient maps of $\Phi_{K,I}$ and $\Phi_{K,J}$ respectively. Denote by $\pi_I : |\mathcal{G}_I| \to |\{\mathcal{G}_I\}|$ the obvious projection map. We call the space $|\{\mathcal{G}_I\}|$ the *support* of $\{\mathcal{G}_I\}$, a section of a (virtual) vector bundle over $\{\mathcal{G}_I\}$ is called *compactly supported* if it has compact support in $|\{\mathcal{G}_I\}|$. We remark that the above constructions can be applied to topological virtual spaces.

Definition 2.14. Let $\{\mathcal{G}_I\}$ be a proper étale virtual groupoid. A collection of smooth functions $\{\rho_I \in C^\infty(\mathcal{G}_I)\}$ is called a partition of unity for $\{\mathcal{G}_I\}$ if the following conditions are satisfied.

(1) each ρ_I is invariant under the above equivalence relation, hence defines a continuous function $\bar\rho_I : \pi_I(|\mathcal{G}_I|) \to \mathbb{R}$.
(2) $supp(\bar\rho_I)$ is compact, this condition can be omitted when the Θ-twisted form ω_I can be chosen such that the product $\rho_I\omega_I$ is compactly supported on $\mathcal{G}_I$.
(3) For every $x \in |\{\mathcal{G}_I\}|$, $\sum_I \bar\rho_I(x) = 1$.

Under the assumption that a transition Thom form Θ and a partition of unity $\{\rho_I \in C^\infty(\mathcal{G}_I)\}$ exist for an oriented proper étale virtual groupoid $\{\mathcal{G}_I\}$, we define

$$\int_{\{\mathcal{G}_I\}} : \Omega^*_c(\{\mathcal{G}_I\}, \Theta) \longrightarrow \mathbb{R} \tag{2.19}$$

to be

$$\int_{\{\mathcal{G}_I\}} \omega = \sum_I \int_{\mathcal{G}_I} \rho_I \cdot \omega_I$$

for any Θ-twisted differential form $\omega = \{\omega_I \in \Omega^*_c(\mathcal{G}_I)\} \in \Omega^*_c(\{\mathcal{G}_I\}, \Theta)$. In practice, one needs to establish the existence of a transition Thom form Θ and a partition of unity $\{\rho_I \in C^\infty(\mathcal{G}_I)\}$ exist for a proper étale oriented virtual groupoid $\{\mathcal{G}_I\}$.

Theorem 2.15. *Given an oriented proper étale virtual groupoid $\{\mathcal{G}_I\}$, assume that a transition Thom form Θ and a partition of unity $\{\rho_I \in C^\infty(\mathcal{G}_I)\}$ exist. The integration map (2.19) is well-defined, that is, it is independent of the choice of a partition of unity $\{\rho_I \in C^\infty(\mathcal{G}_I)\}$ and the choice of the transition Thom form. Assume further that $\{\mathcal{G}_I\}$ is an oriented proper étale virtual groupoid with boundary. Then the Stokes' formula hold*

$$\int_{\{\mathcal{G}_I\}} d\omega = \int_{\{\partial\mathcal{G}_I\}} \iota^*\omega$$

for $\omega \in \Omega^*_c(\{\mathcal{G}_I\}, \Theta)$. Here $\iota : \{\partial\mathcal{G}_I\} \to \{\mathcal{G}_I\}$ is the inclusion map.

Proof. It suffices to show that for every sufficiently small open subset U of $|\{\mathcal{G}_I\}|$,

$$\sum_I \int_{\pi_I^{-1}(U)} \rho_I \cdot \omega_I$$

is independent of the choice of a partition of unity $\{\rho_I \in C^\infty(\mathcal{G}_I)\}$ for any $\omega = \{\omega_I \in \Omega^*_c(\mathcal{G}_I)\} \in \Omega^*_c(\{\mathcal{G}_I\}, \Theta)$.

When U small enough, there exists an I_0 such that $\pi_{I_0}(\pi_{I_0}^{-1}(U)) = U$. Then we claim that

$$\sum_I \int_{\pi_I^{-1}(U)} \rho_I \omega_I = \int_{\pi_{I_0}^{-1}(U)} \omega_{I_0}. \tag{2.20}$$

As ω_{I_0} is a compactly supported differential form on $\mathcal{G}_{I_0}$, the integration map is defined on $\Omega^*_c(\{\mathcal{G}_I\}, \Theta)$. For simplicity, we introduce the notation

$$U_I = \pi_I^{-1}(U), \quad U_{I_0,I} = \pi_{I_0}^{-1}(\pi_I(U_I)).$$

Then we claim that

$$\int_{U_I} \rho_I \omega_I = \int_{U_{I_0,I}} \rho_I \omega_{I_0}. \tag{2.21}$$

In fact, by the definition of Θ and Θ-twisted virtual differential form, we have

$$\int_{U_I} \rho_I \omega_I = \int_{\Phi_{I_0\cap I, I}(U_I)} \rho_I \omega_{I_0\cap I} = \int_{\Phi_{I_0\cap I, I_0}(U_{I_0,I})} \rho_I \omega_{I_0\cap I} = \int_{U_{I_0}} \rho_I \omega_{I_0}.$$

Hence, the left hand side of (2.20) becomes

$$\sum_I \int_{U_{I_0,I}} \rho_I \omega_{I_0} = \sum_I \int_{U_{I_0}} \rho_I \omega_{I_0} = \int_{U_{I_0}} \omega_{I_0}.$$

The same argument implies that the integration map is independent of the choice of the partition of unity for $\{\mathcal{G}_I\}$ and the choice of the transition Thom form. The proof of the Stokes' formula is straightforward, as the Stokes' formula holds for orbifold integrations. □

Remark 2.16. The assumption of the existence of a transition Thom form and and a partition of unity in Theorem 2.15 will automatically be satisfied for those proper étale virtual groupoids when we apply the virtual neighborhood technique to study the Fredholm system (cf. §3).

3. Virtual technique for smooth Fredholm systems

In this section, we will introduce a notion of virtual systems arising from Fredholm systems following closely [9, Sections 5 and 6]. Recall that a **Fredhom system** in [9] is a triple $(\mathcal{B}, \mathcal{E}, S)$, consisting of

(1) a smooth Banach manifold $\mathcal{B}$,
(2) a smooth Banach bundle $\pi : \mathcal{E} \to \mathcal{B}$,
(3) a smooth Fredholm section $S : \mathcal{B} \to \mathcal{E}$.

The zeros of the section S, denoted by $M = S^{-1}(0)$, is called the moduli space of the Fredholm system $(\mathcal{B}, \mathcal{E}, S)$. We remark that a section S is called Fredholm if the differential in the fiber direction as a linear operator

$$D_x S : T_x\mathcal{B} \xrightarrow{dS_x} T_{S(x)}\mathcal{E} \longrightarrow \mathcal{E}_x,$$

for any $x \in M$ is a Fredholm operator. Here dS_x is the differential of S at x and $T_{S(x)}\mathcal{E} \longrightarrow \mathcal{E}_x$ is the natural projection defined the zero section at x. When $D_x S$ is surjective for any $x \in M$, $(\mathcal{B}, \mathcal{E}, S)$ is called a **regular** Fredholm system.

Remark 3.1. The operator $D_y S$ depends on a choice of a connection on $\mathcal{E}$, in particular, when $y \notin M$. But once a connection is chosen, for any $x \in M$ we may assume that $D_y S$ is Fredholm when y is close enough to x. In practical, in geometric applications, the operator $D_x S$ is Fredholm for all $x \in \mathcal{B}$ with a natural connection on $\mathcal{E}$.

If the section S is transverse to the zero section, that is, the operator

$$D_x S : T_x\mathcal{B} \longrightarrow \mathcal{E}_x$$

is surjective at $x \in M$, then M is a smooth manifold of dimension given by the Fredholm index of $D_x S$. When the transversality of S fails, one needs to apply the virtual neighborhood technique to get a virtual manifold as originally proposed in [9]. We assume that $M = S^{-1}(0)$ is compact.

3.1. From smooth Fredholm systems to finite dimensional virtual systems.

Definition 3.2. (**Finite dimensional virtual system**) A *finite dimensional virtual system* is a collection of triples

$$\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I) | I \subset \{1, 2, \cdots, n\}\}$$

indexed by a partially ordered set $(\mathcal{I} = 2^{\{1,2,\cdots,n\}}, \subset)$, where

(1) $\mathcal{V} = \{\mathcal{V}_I\}$ is a fiinite dimensional virtual manifold;
(2) $\mathbf{E} = \{\mathbf{E}_I\}$ is a finite rank virtual vector bundle over $\{cV_I\}$;
(3) $\sigma = \{\sigma_I\}$ is a virtual section of the virtual vector bundle $\{\mathbf{E}_I\}$.

Then by the arguments in Example 2.4 (5), the zero sets

$$\{\sigma_I^{-1}(0) | I \subset \{1, 2, \cdots, n\}\}$$

forms a topological space, denoted by $|\{\sigma_I^{-1}(0)\}|$.

Let $(\mathcal{B}, \mathcal{E}, S)$ be a Fredholm system such that $M = S^{-1}(0)$ is compact. We will construct a finite dimensional virtual system $\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I) | I \subset \{1, 2, \cdots, n\}\}$ using local and global stabilizations such that there exist a collection of homeomorphisms

$$\{\psi_I : \sigma_I^{-1}(0) \longrightarrow U_I\}$$

from $\sigma_I^{-1}(0)$ to an open subset $U_I \subset M$, where $\{U_I\}$ is an open cover of M.

3.1.1. *Local stabilizations.*

Definition 3.3. Given a topological space M and a point $x \in M$, a *local virtual neighbourhood* at x is a 4-tuple

$$(\mathcal{V}_x, \mathbf{E}_x, \sigma_x, \psi_x)$$

consisting of

(1) a smooth finite dimensional manifold $\mathcal{V}_x$,
(2) a smooth vector bundle $\mathbf{E}_x$ with a section σ_x and
(3) a homeomorphism $\psi_x : \sigma^{-1}(0) \to U_x$ for an open neighbourhood U_x of x in M.

Given a Fredholm system $(\mathcal{B}, \mathcal{E}, S)$, for $x \in M = S^{-1}(0)$, setting $F = \mathcal{E}_x$ (the fiber of $\mathcal{E}$ at x), there is a neighborhood U of x such that $\mathcal{E}$ is trivialized over U, that is, a bundle isomorphism

$$\phi_x : \mathcal{E}|_U \to U \times F. \tag{3.1}$$

Denote by $\phi_{x;y} : \mathcal{E}_y \to F$ the isomorphism induced by (3.1) for $y \in U$.

Consider the Fredholm operator $DS_x : T_x\mathcal{B} \to F$. Choose $K_x \subset F$ to be a finite dimensional subspace of F such that

$$K_x + D_xS(T_x\mathcal{B}) = F. \tag{3.2}$$

Define a thickened Fredholm system $(U \times K_x, \mathcal{E}|_U \times K_x, \tilde{S}_x)$, where

$$\tilde{S}_x(y, k) = S(y) + \phi_{x;y}^{-1}(k) \in \mathcal{E}_y = (\mathcal{E}|_U \times K_x)_{(y,k)}.$$

Then by the upper semi-continuity for the dimension of the cokernels of any continuous family of Fredholm operators, there exist a small neighbourhood U_x of x such that $\overline{U}_x \subset U$ and a small constant $\epsilon > 0$ such that $\tilde{S}_x$ is transverse to the zero section over $U_x \times K_x(\epsilon)$. Here $K_x(\epsilon)$ denotes the ϵ-ball in K_x centred at the origin. We will choose U_x and ϵ such that

$$(U_x \times K_x(\epsilon), \mathcal{E}|_{U_x} \times K_x(\epsilon), \tilde{S}_x)$$

is a regular Fredholm system. Let

$$\mathcal{V}_x = \tilde{S}_x^{-1}(0)$$

then $\mathcal{V}_x$ is a smooth finite dimensional manifold of dimension given by $\mathbf{Index}(D_xS) + \dim K_x$. There is a trivial vector bundle $\mathbf{E}_x = \mathcal{V}_x \times K_x \to \mathcal{V}_x$ with a canonical section σ_x given by the component of K_x in $\mathcal{V}_x$. Then there is a homeomorphism

$$\psi_x : \sigma_x^{-1}(0) \longrightarrow M \cap U_x.$$

So we obtain a local virtual neighbourhood

$$(\mathcal{V}_x, \mathbf{E}_x, \sigma_x, \psi_x) \tag{3.3}$$

of $x \in M$ in the sense of Definition 3.3. We point out that the virtual neighbourhood (3.3) provides a Kuranishi chart of M at x as in the sense of [11].

To assemble these local virtual neighbourhood structures to get a virtual system for M, we impose the following assumption on $\mathcal{B}$.

Assumption 3.4. *For any $x \in M$, $\mathcal{B}$ admits a smooth cutoff function supported in any open neighbourhood U_x of x, that is, there exist open neighbourhoods $U_x^{(1)}$ and $U_x^{(2)}$ of x such that*

$$U_x^{(1)} \subset \overline{U_x^{(1)}} \subset U_x^{(2)} \subset \overline{U_x^{(2)}} \subset U_x,$$

and a smooth function $\beta_x : U_x \to [0,1]$ satisfying $\beta_x \equiv 1$ on $U_x^{(1)}$ and $\beta_x \equiv 0$ on $U_x \backslash U_x^{(2)}$.

Choose a smooth cutoff function β_x as in Assumption 3.4 supported in U_x and a smooth function $\gamma_x : U_x \to [0, \epsilon^{-1}]$ such that $\gamma_x = \epsilon^{-1}$ on $U_x^{(2)}$ and its support is contained in an open neighbourhood $U_x^{(3)}$ of x such that

$$\overline{U_x^{(2)}} \subset U_x^{(3)} \subset \overline{U_x^{(3)}} \subset U_x.$$

Definition 3.5. Given the cutoff functions β_x and γ_x, let C_x be an open subset of $U_x \times K_x$ consists of points (y, v) satisfying $\gamma_x(y)\|v\| < 1$. A *trimmed local stabilization* of $(\mathcal{B}, \mathcal{E}, S)$ at $x \in M$ is defined to be the restriction to C_x of the thickened Fredholm system

$$(U_x \times K_x(\epsilon), \mathcal{E}|_{U_x} \times K_x(\epsilon), S_x)$$

with $S_x(y, v) = S(y) + \beta_x(y)\phi_{x;y}^{-1}(v)$. We denote the resulting Fredholm system by (C_x, F_x, T_x).

The trimmed local stabilization process turns the Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ at each point $x \in M$ to a Fredholm system (C_x, F_x, T_x) which contains a regular Fredholm system

$$(U_x^{(1)} \times K_x(\epsilon), \mathcal{E}|_{U_x^{(1)}} \times K_x, S_x).$$

Hence, we have established the existence of local stablizability for any Fredholm system which satisfies Assumption 3.4.

3.1.2. *Global stabilizations.* To patch together local virtual neighbourhoods at every point $x \in M$ to get a virtual system on M, we need to introduce a notion of global stablization for any Fredholm system which satisfies Assumption 3.4.

As M is compact, there exist finite points, say $x_i, 1 \leq i \leq n$, with a collection of trimmed local stabilizations

$$\{(C_{x_i}, F_{x_i}, T_{x_i})\}$$

associates to a collection of nested open neighbourhoods

$$\{U_{x_i}^{(1)} \subset \overline{U_{x_i}^{(1)}} \subset U_{x_i}^{(2)} \subset \overline{U_{x_i}^{(2)}} \subset U_{x_i}^{(3)} \subset \overline{U_{x_i}^{(3)}} \subset U_{x_i}\}$$

and a collection of cut-off functions $\{\beta_{x_i}\}$ as in Assumption 3.4 and smooth functions $\{\gamma_{x_i}\}$ as in Definition 3.5. Moreover, we require $M \subset \bigcup_{i=1}^n U_{x_i}^{(1)}$.

For any non-empty $I \subset \{1, 2, \ldots, n\}$, define

$$X_I' = \left(\bigcap_{i\in I} U_{x_i}\right) \setminus \left(\bigcup_{j\notin I} \overline{U_{x_i}^{(3)}}\right).$$

Then $\beta_{x_j} = \gamma_{x_j} = 0$ on X_I' for any $j \notin I$, and $M \subset \bigcup_I X_I'$.

Define $X^{(1)} = \bigcup_{i=1}^n U^{(1)}_{x_i}$ and $X_I = X'_I \cap X^{(1)}$. Then $\{X_I\}_I$ is an open covering of $X^{(1)}$. Set $X_{I,J} = X_{J,I} = X_I \cap X_J$, then $\{X_I\}$ gives a virtual manifold structure for $X^{(1)}$ (Cf. Example 2.4 (2)).

For any non-empty $I \subset \{1, 2, \ldots, n\}$, setting $K_I = \prod_{i\in I} K_{x_i}$, we construct a thickened Fredholm system as follows. Let C_I be an open subset of $X_I \times K_I$ defined by

$$C_I = \{(y, (k_i)_{i\in I}) | \gamma_{x_i}(y) k_i < 1, \text{for any } i \in I\}.$$

with the obvious projection $p_I : C_I \to X_I$. Define F_I to be the pull-back bundle $p_I^*(\mathcal{E}|_{X_I})$. Then F_I is a Banach bundle over C_I with an induced section $S_I : C_I \to F_I$ given by

$$S_I(y, (k_i)_{i\in I}) = S(y) + \sum_{i\in I} \beta_{x_i}(y) \phi^{-1}_{x_i;y}(k_i) \in \mathcal{E}_y.$$

Then (C_I, F_I, S_I) is a Fredholm system for a sufficiently small $\epsilon > 0$.

Proposition 3.6. (1) *The collection $\mathcal{C} = \{C_I\}$ forms a virtual Banach manifold with $C_{I,J} = p_I^{-1}(X_{I,J})$ and $C_{J,I} = p_I^{-1}(X_{J,I})$ for any pair $I \subset J$.*

(a) *$C_{I,J} = X_{I,J} \times K_I$ for any pair I, J, and*

(b) *$\Phi_{I,J} : C_{J,I} = C_{I,J} \times \prod_{i\in J\setminus I} K_{x_i} \to C_{I,J}$ is a vector bundle for any pair $I \subset J$.*

(2) *$\mathcal{F} = \{F_I\}$ is a Banach bundle over $\mathcal{C}$ in the sense of Definition 2.3.*

(3) *$\mathcal{S} = \{S_I\}$ is a* transverse Fredholm *section of $\mathcal{F}$.*

This collection $\{(\mathcal{C}, \mathcal{F}, \mathcal{S})\}$ is a Fredhom system of infinite dimensional virtual manifolds and is called a **global stabilization** *of the original Fredholm system $(\mathcal{B}, \mathcal{E}, S)$.*

Proof. (1) Note that $\{X_I\}$ is a virtual Banach manifold with with $X_{I,J} = X_{J,I} = X_I \cap X_J$ and the patching data given by the identity map. From the definition, we know that for each non-empty I, C_I is a Banach manifold, and

$$C_{J,I} = C_{I,J} \times \prod_{i\in J\setminus I} K_{x_i}.$$

The latter identification follows from the fact that $\gamma_{x_i} = 0$ on $X_{I,J}$ for $i \in J \setminus I$. So $\mathcal{C} = \{C_I\}$ forms a virtual Banach manifold with $C_{I,J} = p_I^{-1}(X_{I,J})$ and $C_{J,I} = p_J^{-1}(X_{J,I})$ for any pair $I \subset J$, and the patching data $\Phi_{I,J} : C_{J,I} \to C_{I,J}$ given by the obvious bundle projection. The coherence conditions (2.1) and (2.2) hold on the nose.

(2) For $I \subset J$, it is straightforward to check that $F_J|_{C_{J,I}} = \Phi^*_{I,J}(F_I|_{C_{I,J}})$, so $\mathcal{F} = \{F_I\}$ is a Banach bundle over the virtual Banach manifold $\mathcal{C}$.

(3) We can check by a direct calculation that

$$S_J|_{C_{J,I}} = \Phi^*_{I,J}(S_I|_{C_{I,J}}),$$

that is, for $(x, (k_j)_{j\in J}) \in C_{J,I} = C_{I,J} \times \prod_{i\in J\setminus I} K_{x_i}$, we have

$$\begin{aligned} S_J(x, (k_j)_{j\in J}) &= S(x) + \textstyle\sum_{j\in J} \beta_{x_j}(x) \phi^{-1}_{x_j;y}(k_j) \\ &= S(x) + \textstyle\sum_{i\in I} \beta_{x_i}(x) \phi^{-1}_{x_i;y}(k_i) \\ &= S_I \circ \Phi_{J,I}(x, (k_j)_{j\in J}). \end{aligned} \tag{3.4}$$

Here we use the fact that $\beta_{x_j}(x) = 0$ when $j \notin I$ and $x \in X_{I,J}$. The Fredholm property of $\mathcal{S}$ follows from the Fredholm property of S in the Fredholm system $(\mathcal{B}, \mathcal{E}, S)$. The transversality of each S_I is due to the choices of K_I's and the fact that there exists $i \in I$ such that $\beta_{x_i}(x) \neq 0$ for $x \in X_I$. This completes the proof.

□

Given a global stabilization $\{(\mathcal{C},\mathcal{F},\mathcal{S})\}$ of a Fredholm system $(\mathcal{B},\mathcal{E},S)$, there is a canonical virtual vector bundle $\mathcal{O}=\{O_I\}$ over $\mathcal{C}$, defined by

$$O_I = C_I \times K_I,$$

with a canonical section $\sigma=\{\sigma_I\}:\mathcal{C}\to\mathcal{O}$ given by

$$\sigma_I(x,(k_i)_{i\in I}) = (x,(k_i)_{i\in I},(k_i)_{i\in I})$$

for $(x,(k_i)_{i\in I})\in C_I = X_I\times K_I$. Define

$$\mathcal{V}_I = S_I^{-1}(0)\subset C_I,\quad \mathcal{V}_{I,J} = S_I^{-1}(0)\cap C_{I,J}. \tag{3.5}$$

Being a transverse Fredholm section of the vector bundle $\mathcal{F}$ over a virtual manifold $\mathcal{C}$, its zero set

$$\{\mathcal{V}_I = S_I^{-1}(0)\} \tag{3.6}$$

is a collection of finite dimensional smooth manifolds. Note that from the identity (3.4) we have

$$\mathcal{V}_{J,I} = \mathcal{V}_{I,J}\times\prod_{j\in J\setminus I} K_{x_i}$$

We still denote the bundle projection map $\mathcal{V}_{J,I}\to\mathcal{V}_{I,J}$ by $\Phi_{I,J}$. Then $\mathcal{V}=\{\mathcal{V}_I\}$ is a virtual manifold.

By restricting the virtual bundle $\mathcal{O}$ and the section σ to $\mathcal{V}$, we get a virtual bundle

$$\mathbf{E} = \{\mathbf{E}_I = O|_{\mathcal{V}_I}\} \tag{3.7}$$

over $\mathcal{V}$ and a section $\sigma=\{\sigma_I\}$. For a given I, there is a canonical inclusion

$$\psi_I:\sigma_I^{-1}(0)\cong\{x\in X_I|S_I(x)=0\}\longrightarrow X_I\cap M,$$

and $M=\bigcup_I\psi_I\big(\sigma_I^{-1}(0)\big)$.

In summary, we obtain the following finite dimensional virtual system from a Fredholm system $(\mathcal{B},\mathcal{E},S)$.

Theorem 3.7. *Given a Fredholm system $(\mathcal{B},\mathcal{E},S)$ with a global stabilization $\{(\mathcal{C},\mathcal{F},\mathcal{S})\}$ of $M=S^{-1}(0)$. Then the collection of triples*

$$\{(\mathcal{V}_I,\mathbf{E}_I,\sigma_I)|I\subset\{1,2,\dots,n\}\}$$

defines a virtual system in the sense of Definition 3.2. Moreover, there exists a collection of inclusions

$$\{\psi_I:\sigma_I^{-1}(0)\longrightarrow M|I\subset\{1,2,\dots,n\}\}$$

such that $M=\bigcup_I\psi_I\big(\sigma_I^{-1}(0)\big)$.

3.2. **Integration and invariants for virtual systems.** In Section 2, the integration on an oriented virtual manifold (a special case of an oriented proper étale virtual groupoid) is defined under the assumption of the existence of a partition of unity and a transition Thom form. In this subsection, we show that a partition of unity and a transition Thom form naturally exist for the finite dimensional virtual system

$$\{(\mathcal{V}_I,\mathbf{E}_I,\sigma_I)|I\subset\{1,2,\dots,n\}\}$$

from applying the global stabilization of a Fredholm system $(\mathcal{B},\mathcal{E},S)$.

3.2.1. *Partition of unity.* In this subsection, we will construct a partition of unity for a virtual system $\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I)\}$ in Theorem 3.7.

For any $z \in M \subset \bigcup_I X_I \subset \mathcal{B}$, there exists an I_z (which will be fixed) such that $z \in U_z^{(2)}$, an open neighbourhood of z in X_{I_z}. By Assumption 3.4, there exists a **smooth** cut-off function η'_z supported in $U_z^{(2)} \subset X_{I_z}$ and $\eta'_z \equiv 1$ in an open neighborhood $U_z^{(1)}$ of z. Hence, $\eta'_z \in C^\infty(X_{I_z})$.

Since M is compact, there exist finitely many those points $z_k, 1 \le k \le m$, with $\eta'_{z_k} \in C^\infty(X_{I_{z_k}})$, such that

$$M \subset \mathcal{U} = \bigcup_{k=1}^m U_{z_k}^{(1)}.$$

On $\mathcal{U}$, since $\sum_{k=1}^m \eta'_{z_k} \neq 0$, we define

$$\eta_{z_k} = \frac{\eta'_{z_k}}{\sum_{l=1}^m \eta'_{z_l}}. \tag{3.8}$$

Then

$$\sum_{k=1}^m \eta_{z_k}(y) = 1, \quad y \in \mathcal{U}. \tag{3.9}$$

Note that $\eta'_{z_k} \in C^\infty(X_{I_k})$ for some $I_k = I_{z_k}$, then η_{z_k} is a function on $\mathcal{U} \cap X_{I_k}$. By composing with the projection onto $\mathcal{U} \cap X_{I_k}$, we get a function on

$$C'_{I_k} := (\mathcal{U} \cap X_{I_k}) \times K_{I_k} \subset C_{I_k} = X_{I_k} \times K_{I_k},$$

which will still be denoted by η_{z_k}. It is clear that η_{z_k} is an invariant function on C'_{I_k} under the equivalence relation (2.18), that is, for any $J \subset I_k$, the cut-off function η_{z_k} is constant along the fiber of the bundle

$$C'_{I_k,J} \longrightarrow C'_{J,I_k}.$$

Lemma 3.8. *For any $I \subset \{1, 2, \cdots, n\}$, define $\eta_I = \sum_{k \in \{k | I_k = I\}} \eta_{z_k} \in C^\infty(C'_I)$, then $\{\eta_I\}$ forms a partition of unity on the virtual manifold $\mathcal{C}' := \{C'_I\}$.*

Proof. Each η_I is an invariant function on C'_{I_k} under the equivalence relation (2.18) as each $\eta_{z_k} \in C^\infty(C'_I)$ satisfies this property. Now for any point $x \in |\{C'_I\}|$, we need to check that

$$\sum_I \bar{\eta}_I(x) = 1, \tag{3.10}$$

where $\bar{\eta}_I$ is the induced function on $\pi_{I_0}(C'_I) \subset |\{C'_I\}|$.

Choose $(y, k) \in C'_{I_0} = (\mathcal{U} \cap X_{I_0}) \times K_{I_0}$ such that $\pi_{I_0}(y, k) = x$. We write

$$\{J | y \in X_J\} = \{J_\ell | \ell = 1, 2 \cdots, p\} \subset \{I_1, I_2, \cdots, I_m\}.$$

Then among $\{J_1, J_2 \cdots, J_p\}$ there exists a smallest element, say J_1, in the sense that $J_1 \subset J_\ell$, for any ℓ. This follows from the fact that for any pair X_I and X_J containing y, then $y \in X_{I \cap J}$. Note that in [9], X_{J_1} is called the support of y.

Setting $(y, k') = \Phi_{J_1, I_0}(y, k)$ under the bundle map $\Phi_{J_1, I_0} : C'_{I_0, J_1} \to C'_{J_1, I_0}$, then the condition (3.10) is equivalent to

$$\sum_{\ell=1}^p \bar{\eta}_{J_\ell}\big([\Phi^{-1}_{J_1, J_\ell}(y, k')]\big) = 1. \tag{3.11}$$

In this expression, $[\Phi^{-1}_{J_1,J_\ell}(y,k')]$ denotes the equivalence class in $\pi_{J_\ell}(C'_{J_\ell}) \subset |\{C'_I\}|$. This is exactly $x \in \pi_{J_\ell}(C'_{J_\ell}) \subset |\{C'_I\}|$, so $\bar\eta_{J_\ell}(x) = \eta_{z_k}(y)$ where z_k is determined by the relation

$$\{J_\ell | \ell = 1, 2 \cdots, p\} \subset \{I_1, I_2, \cdots, I_k, \cdots, I_m\}.$$

Hence (3.11) follows directly from (3.9). □

Here, we do not require that the support of η_I is compact as in Definition 2.14, since we can find a well-controlled twisted form $\theta = \{\theta_I\}$ such that $\eta_I\theta_I$ is compactly supported.

Let $\mathcal{V}'_I = \mathcal{V}_I \cap C'_I$. Then $\{\mathcal{V}'\}$ is a virtual submanifold of $\{\mathcal{V}_I\}$.

Corollary 3.9. *$\{\eta_I\}$ forms a partition of unity of $\mathcal{V}' := \{\mathcal{V}'_I\}$.*

3.2.2. *Transition Thom forms and Θ-twisted forms.* In the local/global stabilizations, we have chosen a collection of finite points $\{x_i \in M | i = 1, 2, \cdots, n\}$ and a finite dimensional $K_{x_i} \subset \mathcal{E}_{x_i}$ for each x_i. Let $\Theta_i, 1 \le i \le n$ be a volume form on K_{x_i} that is supported in a small ϵ-ball B_ϵ of the origin in K_{x_i}. Then the volume form $\bigwedge_{i\in I}\Theta_i$ on $K_I = \oplus_{i\in I}K_{x_i}$ defines a Thom form Θ_I on the bundle $O_I = C_I \times K_I \to C_I$.

Recall that $C_{J,I} = C_{I,J} \times \prod_{i\in J\setminus I} K_{x_i}$ for $I \subset J$, so the volume form $\bigwedge_{i\in J\setminus I}\Theta_i$ defines a Thom form $\Theta_{I,J}$ of the bundle $\Phi_{I,J} : C_{J,I} \to C_{I,J}$. Then the collection of Thom forms $\{\Theta_{I,J} | I \subset J\}$ is a transition Thom form in the sense of Definition 2.13.

Under the inclusions,

$$\mathcal{V}_I = S_I^{-1}(0) \subset C_I \subset O_I$$

where the last inclusion is given by the zero section of O_I, we can restrict the Thom form Θ_I to $\mathcal{V}_I$ to get an Euler form for the bundle $\mathbf{E}_I = O_I|_{\mathcal{V}_I}$. Denote this Euler form of $\mathbf{E}_I$ by θ_I.

Lemma 3.10. (1) *$\Theta = \{\Theta_{I,J}|_{\mathcal{V}_{J,I}} | I \subset J\}$ is a the transition Thom form for $\{\mathcal{V}_I\}$*

(2) *$\theta = \{\theta_I\}$ is a Θ-twisted virtual differential form on $\{\mathcal{V}_I\}$ with respect to the transition Thom form $\Theta = \{\Theta_{I,J}|_{\mathcal{V}_{J,I}} | I \subset J\}$. This twisted virtual differential form will be called a* **virtual Euler form** *of the virtual vector bundle $\{\mathbf{E}_I\}$.*

Proof. The proof is just a straightforward calculation. □

The following assumption is crucial in finishing off the virtual neighborhood technique for a Fredholm system.

Assumption 3.11. *[virtual convergence] Given a sequence $\{(y_N, k_N) | N = 1, 2, \cdots, \}$ in $C_I = X_I \times K_I$ such that $S_I(y_N, k_N) = 0$ and $k_N \to 0$, then the sequence $\{y_N\}$ converges in X_I.*

Theorem 3.12. *Under the Assumption 3.11, ϵ in the ϵ-balls in K_{x_i}'s can be chosen sufficiently small so that the support of θ_I is contained in $\mathcal{V}'_I = \mathcal{V}_I \cap C'_I$. In particular, the support of $\Theta_i, 1 \le i \le n$, can be chosen sufficiently small so that $\eta_I\theta_I$ is compactly supported in $\mathcal{V}'_I \subset \mathcal{V}_I = S_I^{-1}(0) \subset C_I$.*

Proof. For simplicity, suppose $I = \{1, \ldots, \ell\}, \ell \le n$. If the claim of the theorem is not true, then for any $\epsilon = 1/N$, there exists a point (y_N, k_N) that solves the equation $S_I(y_N, k_N) = 0$, $\|k_N\| < 1/N$ and $y_N \notin (X_I \cap \mathcal{U}) \subset \mathcal{U}$.

As $N \to \infty$, $k_N \to 0$, by the Assumption 3.11, y_N converges to $y_\infty \in \mathcal{B}$ and $S(y_\infty) = S_I(y_\infty, 0) = 0$. Therefore, $y_\infty \in M \cap \mathcal{U} \subset X_I \cap \mathcal{U}$. This contradicts to the fact that $y_N \notin X_I \cap \mathcal{U}$. □

Let $\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I) | I \subset \{1, 2, \ldots, n\}\}$ be the virtual system obtained in Theorem 3.7. We assume that $\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I)\}$ is **oriented** in the sense that both $\{\mathcal{V}_I\}$ and $\{\mathbf{E}_I\}$ are oriented. Let $\eta = \{\eta_I\}$ be a partition of unity and $\theta = \{\theta_I\}$ be a Θ-twisted differential form constructed in Lemma 3.10.

Given $\alpha = \{\alpha_I\}$ a degree d virtual differential form on $\mathcal{V} = \{\mathcal{V}_I\}$, that is, a smooth section of $\bigwedge^d (T^*\mathcal{V})$ such that under the bundle map $\Phi_{I,J} : \mathcal{V}_{J,I} \to \mathcal{V}_{I,J}$ for $I \subset J$, we have

$$\alpha_J|_{\mathcal{V}_{J,I}} = \Phi^*_{I,J}\big(\alpha_I|_{\mathcal{V}_{I,J}}\big).$$

Definition 3.13. Define a virtual integration of α to be

$$\int_{\mathcal{V}}^{vir} \alpha = \sum_I \int_{\mathcal{V}_I} \eta_I \cdot \theta_I \cdot \alpha_I. \tag{3.12}$$

Now given a cohomology class $H^d(\mathcal{B})$, suppose that under the following smooth maps

$$\mathcal{V}_I \subset C_I = X_I \times K_I \longrightarrow X_I \subset \mathcal{B},$$

for each $I \subset \{1, 2, \cdots, n\}$, the pull-back cohomology class can be represented by a closed degree d virtual differential form α_I. Then we can define an invariant associated to the Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ to be

$$\Phi(\alpha) = \int_{\mathcal{V}}^{vir} \alpha. \tag{3.13}$$

4. Fredholm orbifold systems and their virtual orbifold systems

In this subsection, we generalise the virtual neighborhood technique for any Fredholm system satisfying Assumptions 3.11 and 3.11 to the orbifold case using the language of proper étale groupoids.

By an orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$, we mean that

(1) $\mathcal{B}$ is a proper étale groupoid $\mathcal{B} = (\mathcal{B}^1 \rightrightarrows \mathcal{B}^0)$, that is, $|\mathcal{B}|$ is a Banach orbifold,
(2) $\mathcal{E} = (\mathcal{E}^1 \rightrightarrows \mathcal{E}^0) \to \mathcal{B}$ is a Banach vector bundle in the sense of Proposition 2.10,
(3) S is a Fredholm section of $\mathcal{E}$ given by a pair sections (S_1, S_0) such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}^1 & \underset{\pi_1}{\overset{S_1}{\rightleftarrows}} & \mathcal{B}^1 \\ \downdownarrows & & \downdownarrows \\ \mathcal{E}^0 & \underset{\pi_0}{\overset{S_0}{\rightleftarrows}} & \mathcal{B}^0. \end{array} \tag{4.1}$$

Here S is Fredholm if S_0 is Fredholm. We say that an orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ is transverse if S_0 is transverse to the zero section of $\mathcal{E}_0 \to \mathcal{B}_0$.

Lemma 4.1. *Given a transverse orbifold Fredholm system* $(\mathcal{B}, \mathcal{E}, S)$*, the zero set*

$$\mathcal{M} = S^{-1}(0) = \big(S_1^{-1}(0) \rightrightarrows S_0^{-1}(0)\big)$$

with the induced groupoid structure from $\mathcal{B}$ *is a smooth orbifold.*

Proof. As S_0 is a transverse section, $S_0^{-1}(0)$ is a finite dimensional smooth manifold. Using the étale properties of $\mathcal{B}$ and $\mathcal{E}$, S_1 is also a transverse section. Hence $S_1^{-1}(0)$ is a finite dimensional smooth manifold. The diagram (4.1) implies that $\mathcal{M}$ is a proper étale groupoid. $\square$

Consider a general orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ when S is not transverse to the zero section. The virtual neighborhood technique can still be developed for this orbifold Fredholm system to get a finite dimensional virtual orbifold system.

Definition 4.2. (**Finite dimensional virtual orbifold system**) A finite dimensional virtual orbifold system is a collection of triples

$$\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I) | I \subset \{1, 2, \cdots, n\}\}$$

indexed by a a partially ordered set $(\mathcal{I} = 2^{\{1,2,\cdots,n\}}, \subset)$, where

(1) $\mathcal{V} = \{\mathcal{V}_I\}$ is a finite dimensional proper étale virtual groupoid,
(2) $\mathbf{E} = \{\mathbf{E}_I\}$ is a finite rank virtual vector bundle over $\{\mathcal{V}_I\}$
(3) $\sigma = \{\sigma_I\}$ is a virtual section of the virtual vector bundle $\{\mathbf{E}_I\}$.

4.0.3. *Local stabilizations.* Let $\pi : \mathcal{M} \to |\mathcal{M}|$ be the quotient map. Given $x \in |\mathcal{M}|$, let $\tilde{x} \in \pi^{-1}(x)$ with its isotropy group $G_{\tilde{x}} = (s,t)^{-1}(\tilde{x}, \tilde{x})$. Then there exists a small neighbourhood U_x of x in $|\mathcal{B}|$, and a $G_{\tilde{x}}$-invariant neighbourhood $U_{\tilde{x}}$ of $\tilde{x}$ in $\mathcal{B}^0$ such that the triple

$$(U_{\tilde{x}}, G_{\tilde{x}}, U_x)$$

is an orbifold chart at $x \in |\mathcal{B}|$ and $\mathcal{E}|_{U_{\tilde{x}}}$ has a $G_{\tilde{x}}$-equivariant trivialisation

$$\phi_{\tilde{x}} : \mathcal{E}^0|_{U_{\tilde{x}}} \longrightarrow U_{\tilde{x}} \times F_{\tilde{x}}$$

where $F_{\tilde{x}}$ is the fiber of $\mathcal{E}$ at $\tilde{x}$. Let $\phi_{\tilde{x},y} : \mathcal{E}^0_y \to F_{\tilde{x}}$ be the induced isomorphism for $y \in U_{\tilde{x}}$. Then the local stabilization in §3.1.1 can be carried over to a $G_{\tilde{x}}$-invariant local stabilization of $(\mathcal{B}, \mathcal{E}, S)$ at $\tilde{x}$ under the following assumption.

Assumption 4.3. *For any $\tilde{x} \in \pi^{-1}(x)$, there are open $G_{\tilde{x}}$-invariant neighborhoods $U^{(i)}_{\tilde{x}}$ of $\tilde{x}$ for $i = 1, 2, 3$ such that*

$$U^{(1)}_{\tilde{x}} \subset \overline{U^{(1)}_{\tilde{x}}} \subset U^{(2)}_{\tilde{x}} \subset \overline{U^{(2)}_{\tilde{x}}} \subset U^{(3)}_{\tilde{x}} \subset \overline{U^{(3)}_{\tilde{x}}} \subset U_{\tilde{x}} \tag{4.2}$$

and a **smooth** *$G_{\tilde{x}}$-invariant cut-off function $\beta_{\tilde{x}} : U^{(3)}_{\tilde{x}} \to [0,1]$ such that $\beta_{\tilde{x}} \equiv 1$ on $U^{(1)}_{\tilde{x}}$ and is supported in $U^{(2)}_{\tilde{x}}$. Here $\overline{U^{(i)}_{\tilde{x}}}$ is the closure of $U^{(i)}_{\tilde{x}}$.*

We choose these nested open $G_{\tilde{x}}$-invariant neighborhoods

$$U^{(1)}_{\tilde{x}} \subset U^{(2)}_{\tilde{x}} \subset U^{(3)}_{\tilde{x}} \subset U_{\tilde{x}}$$

such that under the quotient map π, we get

$$U^{(1)}_x \subset U^{(2)}_x \subset U^{(3)}_x \subset U_x \tag{4.3}$$

which are **independent** of the choices of $\tilde{x} \in \pi^{-1}(x)$. Under Assumption 4.3, the cut-off functions $\{\beta_{\tilde{x}} | \tilde{x} \in \pi^{-1}(x)\}$ can be chosen to be invariant under the action of $\mathcal{B}^1$.

The $G_{\tilde{x}}$-invariant local stabilization of $(\mathcal{B}, \mathcal{E}, S)$ at $\tilde{x}$ is given by a thickened $G_{\tilde{x}}$-invariant Fredholm system

$$(U^{(3)}_{\tilde{x}} \times K_{\tilde{x}}, \mathcal{E}|_{U^{(3)}_{\tilde{x}}} \times K_{\tilde{x}}, S_{\tilde{x},0})$$

with $S_{\tilde{x},0}(y,k) = S_0(y) + \beta_{\tilde{x}}(y)\psi^{-1}_{\tilde{x},y}(k)$. Here $K_{\tilde{x}}$ is an even dimensional $G_{\tilde{x}}$-invariant linear subspace of $F_{\tilde{x}}$ such that $S_{\tilde{x},0}$ is transverse to the zero section on $U^{(1)}_{\tilde{x}} \times K_{\tilde{x}}$. Such a $K_{\tilde{x}}$ can be found by setting $K_{\tilde{x}} = \sum_{g \in G_{\tilde{x}}} g \cdot K$ for some K defined by the Fredholm property of $S_0 : \mathcal{B}^0 \to \mathcal{E}^0$ as in (3.2). We further require that $\{K_{\tilde{x}}\}_{\tilde{x} \in \pi^{-1}(x)}$ is invariant under the action of $\mathcal{B}^1$.

Denote the $G_{\tilde{x}}$-invariant zero set by

$$\mathcal{V}_{\tilde{x}} = S^{-1}_{\tilde{x},0}(0) \cap \left(U^{(1)}_{\tilde{x}} \times K_{\tilde{x}}\right),$$

then the $G_{\tilde{x}}$-invariant local stabilization of $(\mathcal{B}, \mathcal{E}, S)$ at $\tilde{x}$ provides a $G_{\tilde{x}}$-invariant local virtual neighbourhood

$$(\mathcal{V}_{\tilde{x}}, \mathbf{E}_{\tilde{x}} = \mathcal{V}_{\tilde{x}} \times K_{\tilde{x}}, \sigma_{\tilde{x}}, \psi_{\tilde{x}})$$

where $\mathbf{E}_{\tilde{x}} \to \mathcal{V}_{\tilde{x}}$ is a $G_{\tilde{x}}$-equivariant vector bundle over $\mathcal{V}_{\tilde{x}}$.

Due to the $G_{\tilde{x}}$-invariance, this local stabilization of $(\mathcal{B}, \mathcal{E}, S)$ at $\tilde{x}$ can be extended to a $\mathcal{B}^1$-invariant local stabilization of $(\mathcal{B}, \mathcal{E}, S)$ at $x \in |\mathcal{M}|$ in the following sense. We can choose U_x small enough such that

$$\{U_{\tilde{x}} | \tilde{x} \in \pi^{-1}(x)\}$$

are all disjoint. Denote by

$$\iota_x : \widetilde{U}^0_x = \bigsqcup_{\tilde{x} \in \pi^{-1}(x)} U_{\tilde{x}} \subset \mathcal{B}^0,$$

the obvious inclusion map, then $\widetilde{U}^0_x$ is $\mathcal{B}^1$-invariant. Let

$$\widetilde{U}^1_x = \mathcal{B} \times_{s, \iota_x} \widetilde{U}^0_x = \{(\gamma, y) | y \in \widetilde{U}^0_x, \gamma \in \mathcal{B}, s(\gamma) = y\}$$

Then $\widetilde{\mathbf{U}}_x = (\widetilde{U}^1_x \rightrightarrows \widetilde{U}^0_x)$, with the source and target maps given by $(\gamma, y) \mapsto y$ and $(\gamma, y) \mapsto \gamma \cdot y$, is a sub-groupoid of $\mathcal{B}$. In fact, this groupoid is Morita equivalent to the action groupoid $G_{\tilde{x}} \ltimes U_{\tilde{x}} \rightrightarrows U_{\tilde{x}}$. So both groupoids define the same orbifold structure on U_x. We will call $\widetilde{\mathbf{U}}_x = (\widetilde{U}^1_x \rightrightarrows \widetilde{U}^0_x)$ the sub-groupoid of $\mathcal{B}$ generated by the $\mathcal{B}^1$-action on $\widetilde{U}^0_x$. Similarly, given a $\mathcal{B}$-manifold X, then the action groupoid $(\mathcal{B} \ltimes X \rightrightarrows X)$ is called the groupoid generated by the $\mathcal{B}^1$-action on X. As $\mathcal{B}$ is proper and étale, the groupoid $(\mathcal{B} \ltimes X \rightrightarrows X)$ is also proper and étale.

Applying the local stabilization to

$$(\widetilde{\mathbf{U}}_x, \mathcal{E}|_{\widetilde{\mathbf{U}}_x}, S),$$

we get the following local stabilization theorem for an orbifold Fredholm system.

Theorem 4.4. *Let $x \in |\mathcal{M}|$, under Assumption 4.3 for any $\tilde{x} \in \pi^{-1}(x)$, we get a local orbifold Fredholm system*

$$(\mathcal{K}_x, \widetilde{\mathcal{E}}_x, S_x)$$

consisting of

(1) *$\mathcal{K}_x$ is the groupoid generated by the $\mathcal{B}^1$-action on $\bigsqcup_{\tilde{x} \in \pi^{-1}(x)} \left(U^{(3)}_{\tilde{x}} \times K_{\tilde{x}}\right)$,*
(2) *$\widetilde{\mathcal{E}}_x$ is the Banach bundle over $\mathcal{K}_x$ generated by the $\mathcal{B}^1$-action on $\bigsqcup_{\tilde{x} \in \pi^{-1}(x)} \left(\mathcal{E}|_{U^{(3)}_{\tilde{x}}} \times K_{\tilde{x}}\right)$,*
(3) *S_x is a section of $\widetilde{\mathcal{E}}$ defined by $\bigsqcup_{\tilde{x} \in \pi^{-1}(x)} S_{\tilde{x},0}$.*

Define $\mathcal{V}_x$ to be the groupoid generated by the $\mathcal{B}^1$-action on

$$\bigsqcup_{\tilde{x} \in \pi^{-1}(x)} \mathcal{V}^0_{\tilde{x}} = \bigsqcup_{\tilde{x} \in \pi^{-1}(x)} \left(S^{-1}_{\tilde{x},0}(0) \cap \left(U^{(1)}_{\tilde{x}} \times K_{\tilde{x}}\right)\right).$$

Let $\mathbf{E}_x$ be the restriction of the vector bundle generated by the $\mathcal{B}^1$-action on

$$\bigsqcup_{\tilde{x} \in \pi^{-1}(x)} \left(\left(U^{(1)}_{\tilde{x}} \times K_{\tilde{x}}\right) \times K_{\tilde{x}}\right).$$

Note that $\mathbf{E}_x$ has a canonical section σ_x. Then we get a finite dimensional local orbifold system (see Definition 4.2)

$$(\mathcal{V}_x, \mathbf{E}_x, \sigma_x)$$

of $(\mathcal{B}, \mathcal{E}, S)$ at $x \in |\mathcal{B}|$. Moreover, there is an inclusion

$$\psi_x : |\sigma_x^{-1}(0)| \longrightarrow |\mathcal{M}| \cap U_x^{(1)},$$

where $U_x^{(1)}$ is the quotient space $U_{\tilde{x}}^{(1)}/G_{\tilde{x}}$ for any $\tilde{x} \in \pi^{-1}(x)$.

Remark 4.5. Note that the orbifold Fredholm system $(\mathcal{K}_x, \widetilde{\mathcal{E}}_x, S_x)$ is Morita equivalent to the orbifold Fredholm system defined by the $G_{\tilde{x}}$-invariant Fredholm system

$$(U_{\tilde{x}}^{(3)} \times K_{\tilde{x}}, \mathcal{E}|_{U_{\tilde{x}}^{(3)}} \times K_{\tilde{x}}, S_{\tilde{x},0})$$

for any $\tilde{x} \in \pi^{-1}(x)$. In fact, the latter should be thought as a **slice** of the $\mathcal{B}$-action on $(\mathcal{K}_x, \widetilde{\mathcal{E}}_x, S_x)$. On the one hand, the advantage of the more involved orbifold Fredholm system is the convenience of considering coordinate changes for different points in $|\mathcal{M}|$. On the other hand, the integration and a partition of unity are defined in terms of these slices, see (2.13).

4.0.4. *Global stabilization.* Given an orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$, assume that $|\mathcal{M}| = |S^{-1}(0)$ is compact, then the local stabilization process provides a finite collection of local transverse orbifold Fredholm systems

$$\{(\mathcal{K}_{x_i}, \widetilde{\mathcal{E}}_{x_i}, S_{x_i})|i = 1, 2, \cdots, n\},$$

and its corresponding orbifold systems

$$\{(\mathcal{V}_{x_i}, \mathbf{E}_{x_i}, \sigma_{x_i})|i = 1, 2, \cdots, n\}$$

such that the images of $\{\psi_{x_i} : |\sigma_{x_i}^{-1}(0)| \to |\mathcal{M}|\}_i$ form a open cover of $|\mathcal{M}|$.

Theorem 4.6. *There exists a finite dimensional virtual orbifold system for $(\mathcal{B}, \mathcal{E}, S)$ which is a collection of triples*

$$\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I)|I \subset \{1, 2, \cdots, n\}\}$$

indexed by a partially ordered set $(\mathcal{I} = 2^{\{1,2,\cdots,n\}}, \subset)$, where

1. $\{\mathcal{V}_I|I \subset \{1, 2, \cdots, n\}\}$ *is a finite dimensional proper étale virtual groupoid,*
2. $\{\mathbf{E}_I\}$ *is a finite rank virtual vector bundle over* $\{\mathcal{V}_I\}$
3. $\{\sigma_I\}$ *is a section of the virtual vector bundle* $\{\mathbf{E}_I\}$ *whose zeros* $\{\sigma_I^{-1}(0)\}$ *form a cover of* $\mathcal{M}$.

Proof. The proof is parallel to that of Proposition 3.6 and Theorem 3.7, with some extra care to deal with orbifold structures using the language of groupoids. The proof that follows is presented to spell out the difference to the proof of Theorem 3.7.

Under Assumption 4.3, for each x_i, we have a local stabilization

$$(\mathcal{K}_{x_i}, \widetilde{\mathcal{E}}_{x_i}, S_{x_i})$$

provided by Theorem 4.4, where $\mathcal{K}_{x_i}$ is a proper étale groupoid with its unit space (the space of objects) given by

$$\bigsqcup_{\tilde{x}_i \in \pi^{-1}(x_i)} (U_{\tilde{x}_i}^{(3)} \times K_{\tilde{x}_i}),$$

and $\widetilde{\mathcal{E}}_{x_i}$ is a proper étale groupoid with its unit space given by

$$\bigsqcup_{\tilde{x}_i \in \pi^{-1}(x_i)} (\mathcal{E}|_{U_{\tilde{x}_i}^{(3)}} \times K_{\tilde{x}_i}).$$

Note that $\widetilde{\mathcal{E}}_{x_i}$ is the Banach bundle over $\mathcal{K}_{x_i}$ with a section S_{x_i} which is transverse on

$$\bigsqcup_{\tilde{x}_i\in\pi^{-1}(x_i)} \big(U^{(1)}_{\tilde{x}_i}\times K_{\tilde{x}_i}\big),$$

The choices of $\{x_1, x_2, \cdots, x_n\}$ are such that

$$\bigcup_{i=1}^{n}\left(\bigsqcup_{\tilde{x}_i\in\pi^{-1}(x_i)} U^{(1)}_{\tilde{x}_i}\right)\supset\mathcal{M}. \tag{4.4}$$

which implies $|\mathcal{M}|\subset\bigcup_{i=1}^{n}U^{(1)}_{x_i}\subset|\mathcal{B}|$.

Notice that, from the choices of nested neighbourhoods in (4.2) and (4.3), we have the following inclusions of proper étale groupoids

$$\widetilde{\mathbf{U}}^{(1)}_{x_i}\subset\widetilde{\mathbf{U}}^{(2)}_{x_i}\subset\widetilde{\mathbf{U}}^{(3)}_{x_i}$$

which are defined by the inclusions of unit spaces

$$\bigsqcup_{\tilde{x}_i\in\pi^{-1}(x_i)} U^{(1)}_{\tilde{x}_i}\subset\bigsqcup_{\tilde{x}_i\in\pi^{-1}(x_i)} U^{(2)}_{\tilde{x}_i}\subset\bigsqcup_{\tilde{x}_i\in\pi^{-1}(x_i)} U^{(3)}_{\tilde{x}_i}.$$

Similarly we have

$$\mathcal{K}^{(1)}_{x_i}\subset\mathcal{K}^{(2)}_{x_i}\subset\mathcal{K}_{x_i}$$

which are defined by the inclusions of unit spaces

$$\bigsqcup_{\tilde{x}_i\in\pi^{-1}(x_i)} \big(U^{(1)}_{\tilde{x}_i}\times K_{\tilde{x}_i}\big)\subset\bigsqcup_{\tilde{x}_i\in\pi^{-1}(x_i)} \big(U^{(2)}_{\tilde{x}_i}\times K_{\tilde{x}_i}\big)\subset\bigsqcup_{\tilde{x}_i\in\pi^{-1}(x_i)} \big(U^{(3)}_{\tilde{x}_i}\times K_{\tilde{x}_i}\big).$$

Therefore, we have the inclusions of local Fredholm orbifold systems

$$(\mathcal{K}^{(1)}_{x_i},\widetilde{\mathcal{E}}^{(1)}_{x_i},S_{x_i})\subset(\mathcal{K}^{(2)}_{x_i},\widetilde{\mathcal{E}}^{(2)}_{x_i},S_{x_i})\subset(\mathcal{K}_{x_i},\widetilde{\mathcal{E}}_{x_i},S_{x_i}).$$

Define

$$\mathcal{A}'_I=\left(\bigcap_{i\in I}\widetilde{\mathbf{U}}_{x_i}\right)\setminus\left(\bigcup_{j\notin I}\overline{\widetilde{\mathbf{U}}^{(3)}_{x_i}}\right), \tag{4.5}$$

where $\overline{\widetilde{\mathbf{U}}^{(3)}_{x_i}}$ is the closure of the groupoid $\widetilde{\mathbf{U}}^{(3)}_{x_i}$. Then $\mathcal{A}'_I$ is the groupoid generated by the $\mathcal{B}^1$-action on

$$\pi^{-1}(\left(\bigcap_{i\in I}U_{x_i}\right)\setminus\left(\bigcup_{j\notin I}\overline{U^{(3)}_{x_i}}\right)).$$

Set

$$\mathcal{A}_I=\mathcal{A}'_I\cap\left(\bigcup_{i}\widetilde{\mathbf{U}}^{(1)}_{x_i}\right). \tag{4.6}$$

Let $\mathcal{A}_{I,J}=\mathcal{A}_{J,I}=\mathcal{A}_I\cap\mathcal{A}_J$ with $\Phi_{I,J}=Id$ for any $I\subset J$. Then $\{\mathcal{A}_I\}$ covers $\mathcal{M}$, so $\{\mathcal{A}_I\}$ provides a virtual orbifold structure for $\bigcup_i U^{(1)}_{x_i}$, a small neighbourhood of $|\mathcal{M}|$ in $|\mathcal{B}|$. We remark that the collection of $\{\mathcal{A}_I\}$ plays exactly the same role as $\{X_I\}$ in Proposition 3.6.

Following the constructions in Proposition 3.6, we have a collection of (thickened) **transverse** Fredholm orbifold systems

$$\{(\mathcal{C}=\{\mathcal{C}_I\},\mathcal{F}=\{\mathcal{F}_I\to\mathcal{C}_I\},\mathcal{S}=\{S_I\})\}$$

which is a global stabilization of the original orbifold Fredholm system $(\mathcal{B},\mathcal{E},S)$.

With this understanding of the collection of transverse Fredholm orbifold systems

$$\{(\mathcal{C} = \{\mathcal{C}_I\}, \mathcal{F} = \{\mathcal{F}_I \to \mathcal{C}_I\}, \mathcal{S} = \{S_I\})\},$$

define $\mathcal{V}_I = S_I^{-1}(0) \subset \mathcal{C}_I$. Then $\mathcal{V}_I$ is a finite dimensional proper étale groupoid. There is a canonical finite rank vector bundle $\mathbf{E}_I$ over $\mathcal{V}_I$ whose fiber at

$$(y, (k_{\tilde{x}_i})_{\{\tilde{x}_i\}_{i\in I}}) \in S_I^{-1}(0) \cap \left(\Big(\bigcap_{\{\tilde{x}_i\}_{i\in I}} U_{\tilde{x}_i} \Big) \times K_{\{\tilde{x}_i\}_{i\in I}} \right)$$

is $\prod_{\{\tilde{x}_i\}_{i\in I}} K_{\tilde{x}_i}$. Then $\mathbf{E}_I$ has a canonical section $\sigma_I : (y, (k_{\tilde{x}_i})_{\{\tilde{x}_i\}_{i\in I}}) \mapsto (y, (k_{\tilde{x}_i})_{\{\tilde{x}_i\}_{i\in I}}, (k_{\tilde{x}_i})_{\{\tilde{x}_i\}_{i\in I}})$.

The rest of the proof is just an obvious adaptation of the proof of Theorem 3.7 to the proper étale groupoid cases. □

4.1. **Integration and invariants for virtual orbifold systems.** The existence of a partition of unity for a virtual system arising from a Fredholm system in §3.2 can be also adapted with some minor changes to get a partition of unity for the virtual orbifold system

$$\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I) | I \subset \{1, 2, \cdots, n\}\}$$

obtained in Theorem 4.6.

For any $z \in |\mathcal{M}| \subset \bigcup_I |\mathcal{A}_I| \subset |\mathcal{B}|$, there exists an I_z (which will be fixed) such that $z \in U_z^{(2)}$, an open neighbourhood of z in $|\mathcal{A}_{I_z}|$. By Assumption 4.3, there exists a $\mathcal{B}^1$-invariant, smooth cut-off function $\eta'_z \in C^\infty(\mathcal{A}_{I_z})$ such that the induced function $\bar{\eta}'_z$ on the orbit space $|\mathcal{A}_I|$ is supported in $U_z^{(2)} \subset |\mathcal{A}_{I_z}|$ and $\eta'_z \equiv 1$ in an open neighborhood $U_z^{(1)}$ of z. We denote the corresponding proper étale groupoids by

$$\mathbf{U}_z^{(1)} \subset \mathbf{U}_z^{(2)}$$

as sub-groupoids of $\mathcal{A}_I$.

Since we assume that $|\mathcal{M}|$ is compact, there exist finitely many points $z_k, 1 \le k \le m$, such that

$$\mathcal{M} \subset \mathcal{U} = \bigcup_{k=1}^m \mathbf{U}_{z_k}^{(1)}.$$

On the proper étale groupoid $\mathcal{U}$, since $\sum_{k=1}^m \eta'_{z_k} \neq 0$, define

$$\eta_{z_k} = \frac{\eta'_{z_k}}{\sum_{l=1}^m \eta'_{z_l}}. \tag{4.7}$$

Then

$$\sum_{k=1}^m \eta_{z_k}(y) = 1, \quad y \in \mathcal{U}. \tag{4.8}$$

Suppose that $\eta'_{z_k} \in C^\infty(\mathcal{A}_{I_k})$ for $I_k \subset \{1, 2, \cdots, n\}$, then η_{z_k} is a function on $\mathcal{U} \cap \mathcal{A}_{I_k}$. By composing with the projection onto $\mathcal{U} \cap \mathcal{A}_{I_k}$, we get a smooth function on

$$\mathcal{C}'_{I_k} := p_{I_k}^{-1}(\mathcal{U} \cap \mathcal{A}_{I_k}) \subset \mathcal{C}_{I_k},$$

where $p_{I_k} : \mathcal{C}_{I_k} \to \mathcal{A}_{I_k}$ is the projection map associated to the bundle $\mathcal{C}_{I_k} \to \mathcal{A}_{I_k}$ constructed in the proof of Theorem 4.6.

We will still denote the resulting function on $\mathcal{C}'_{I_k}$ as η_{z_k}. Then we have a collection of smooth functions

$$\{\eta_{z_k} \in C^\infty(\mathcal{C}'_{I_k}) | k = 1, 2, \cdots m\}$$

for a sub-collection $\{I_k|k=1,2,\cdots,m\}$ of the index set $2^{\{1,2,\cdots,n\}}$. It is clear that η_{z_k} is an invariant function on $\mathcal{C}'_{I_k}$ under the equivalence relation (2.18) , that is, for any $J\subset I_k$, the cut-off function η_{z_k} is constant along the fiber of bundle

$$\mathcal{C}'_{I_k,J}\longrightarrow \mathcal{C}'_{J,I_k}.$$

Lemma 4.7. *For any $I\subset\{1,2,\cdots,n\}$, define $\eta_I=\sum_{k\in\{k|I_k=I\}}\eta_{z_k}\in C^\infty(\mathcal{C}'_I)$., then $\{\eta_I\}$ is a partition of unity on the virtual manifold $\mathcal{C}':=\{\mathcal{C}'_I\}$ in the sense of Definition 2.14. Let $\mathcal{V}'_I=\mathcal{C}'_I\cap\mathcal{V}_I$, then the restriction of $\{\eta_I\}$ is a partition of unity on the virtual manifold $\{\mathcal{V}'_I\}$.*

Proof. The proof of this lemma is analogous to the proof of Lemma 3.8. We leave it to interested readers as an exercise. □

Integrations and invariants for a virtual orbifold system can be obtained just as in §3.2, under a technical assumption which is the orbifold version of Assumption 3.11.

Assumption 4.8. *Given a sequence $\{y_N|N=1,2,\cdots,\}$ in $\mathcal{C}^0_I$ (the unit space of $\mathcal{C}_I$) such that*

- $S^0_I(y_N)=0$.
- $\lim_{N\to\infty}\|k_N\|=0$ *where $\|k_N\|$ is the Banach norm of the fiber component of y_N over $p_I(y_N)$. Here p_I is the projection for the bundle $\mathcal{C}_I\to\mathcal{A}_I$.*

Then the sequence $\{p_I(y_N)\}$ converges in $|\mathcal{C}_I|$.

As pointed out in Remark 4.5, to define an integration over a proper étale groupoid $\mathcal{G}=(\mathcal{G}^1\rightrightarrows\mathcal{G}^0)$, we only need to cover the orbit space by a collection of groupoids from a sub-collection of orbifold charts, rather than taking a collection of $\mathcal{G}$-invariant subgroupoids. For this purpose, we fix an $|I|$-tuple

$$\{\tilde{x}_i\}_{i\in I}\in\prod_{i\in I}\pi^{-1}(x_i)$$

such that $\bigcap_{\{\tilde{x}_i\}_{i\in I}}U^{(3)}_{\tilde{x}_i}\neq\emptyset$. Let

$$\mathcal{Y}_I=(\mathcal{Y}^1_I\rightrightarrows\mathcal{Y}^0_I)$$

be the full sub-groupoid of $\mathcal{A}_I$ with the unit space $\mathcal{Y}^0_I=\mathcal{A}^0_I\cap\left(\bigcap_{\{\tilde{x}_i\}_{i\in I}}U^{(3)}_{\tilde{x}_i}\right)$. Denote by

$$K_I:=\prod_{\{\tilde{x}_i\}_{i\in I}}K_{\tilde{x}_i},$$

then $\mathcal{C}^0_I|_{\mathcal{Y}^0_I}=\mathcal{Y}^0_I\times K_I\longrightarrow\mathcal{Y}^0_I$ generates through $\mathcal{A}_I$-action a vector bunlde $\mathcal{C}_I|_{\mathcal{Y}_I}$ over $\mathcal{Y}_I$. For any ordered pair $I\subset J$, we may choose the $|I|$-tuple $\{\tilde{x}_i\}_{i\in I}$ and the $|J|$-tuple $\{\tilde{x}_i\}_{i\in J}$ such that

$$\{\tilde{x}_i\}_{i\in I}\subset\{\tilde{x}_i\}_{i\in J}.$$

We set $\mathcal{Y}^0_{I,J}=\mathcal{Y}^0_{J,I}=\mathcal{Y}^0_I\cap\mathcal{Y}^0_J$, then

$$\mathcal{C}^0_J|_{\mathcal{Y}^0_{J,I}}=\mathcal{C}^0_I|_{\mathcal{Y}^0_{I,J}}\times\prod_{i\in J\setminus I}K_{\tilde{x}_i}, \tag{4.9}$$

generates the bundle $\mathcal{C}_{J,I}|_{\mathcal{Y}_{J,I}}\to\mathcal{C}_{I,J}|_{\mathcal{Y}_{I,J}}$ so that $\{\mathcal{C}_I|_{\mathcal{Y}_I}\}$ is a virtual proper étale groupoid.

Define $\mathcal{W}_I=\mathcal{V}_I\cap\left(\mathcal{C}_I|_{\mathcal{Y}_I}\right)\subset\mathcal{V}_I$, then the virtual orbifold system

$$\{(\mathcal{V}_I,\mathbf{E}_I,\sigma_I)|I\subset\{1,2,\cdots,n\}\}$$

arising from Theorem 4.6 is Morita equivalent to the virtual orbifold system

$$\{(\mathcal{W}_I,\mathbf{E}_I|_{\mathcal{W}_I},\sigma_I)|I\subset\{1,2,\cdots,n\}\}.$$

Given I with a fixed $|I|$-tuple $\{\tilde{x}_i\}_{i\in I} \in \prod_{i\in I} \pi^{-1}(x_i)$ such that $\bigcap_{\{\tilde{x}_i\}_{i\in I}} U^{(3)}_{\tilde{x}_i} \neq \emptyset$. For $i \in I$, let Θ_i be a volume form on the finite dimensional $G_{\tilde{x}_i}$-invariant linear space $K_{\tilde{x}_i}$ that is supported in a small ϵ-ball B_ϵ of 0 in $K_{\tilde{x}_i}$. Then the volume form $\bigwedge_{i\in I}\Theta_i$ on $K_I = \oplus_{i\in I} K_{\tilde{x}_i}$ defines a Thom form Θ_I on bundle $\mathcal{C}_I|_{\mathcal{Y}_I} \to \mathcal{Y}_I$.

By (4.9), the volume form $\bigwedge_{i\in J\setminus I}\Theta_i$, defines a Thom form $\Theta_{I,J}$ of the bundle

$$\mathcal{C}_{J,I}|_{\mathcal{Y}_{J,I}} \to \mathcal{C}_{I,J}|_{\mathcal{Y}_{I,J}}.$$

Then one can check that the collection of Thom forms $\{\Theta_{I,J}|I \subset J\}$ is a transition Thom form for $\{\mathcal{C}_I|_{\mathcal{Y}_I}\}$.

Under the inclusions,

$$\mathcal{W}_I = S_I^{-1}(0) \cap \left(\mathcal{C}_I|_{\mathcal{Y}_I}\right) \subset \mathbf{E}_I|_{\mathcal{W}_I}$$

where the last inclusion is given by the zero section of $\mathbf{E}_I$, we can restrict the Thom form Θ_I to $\mathcal{W}_I$ to get an Euler form for the bundle $\mathbf{E}_I|_{\mathcal{W}_I}$. Denote this Euler form of $\mathbf{E}_I|_{\mathcal{W}_I}$ by θ_I. Then

$$\theta = \{\theta_I | I \subset \{1, 2, \cdots, n\}\}$$

is a twisted virtual form on $\{\mathcal{W}_I\}$ with respect to the transition Thom form $\Theta = \{\Theta_{I,J}|_{\mathcal{W}_{J,I}} : I \subset J\}$. This twisted virtual form will be called a **virtual Euler form** of $\{\mathbf{E}_I\}$.

Theorem 4.9. *Under the Assumption 4.8, for any I, the vertical support of Θ_I can be chosen sufficiently small such that $\eta_I\theta_I$ is compactly supported in $\mathcal{W}_I$.*

Proof. The proof is the same as the proof of Theorem 3.12. We omit it here. ☐

We assume that the virtual orbifold system $\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I)\}$ is **oriented**. Let $\theta = \{\theta_I\}$ and $\eta = \{\eta_I\}$ be the partition of unity and the virtual Euler form constructed as above.

Given $\alpha = \{\alpha_I\}$ a degree d differential form on $\mathcal{V} = \{\mathcal{V}_I\}$, we define a virtual integration of α to be

$$\int^{vir}_{\mathcal{V}} \alpha = \sum_I \int_{\mathcal{W}_I} \eta_I \cdot \theta_I \cdot (\alpha_I)|_{\mathcal{W}_I}. \tag{4.10}$$

Now, given cohomology class $H^d(\mathcal{B})$, suppose that under the following smooth maps

$$\mathcal{V}_I \subset C_I = X_I \times K_I \longrightarrow X_I \subset \mathcal{B},$$

for each $I \subset \{1, 2, \cdots, n\}$, the pull-back cohomology class can be represented by a closed degree d virtual differential form α_I. Then we can define an invariant associated to the Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ to be

$$\Phi(\alpha) = \int^{vir}_{\mathcal{V}} \alpha. \tag{4.11}$$

Remark 4.10. (1) Given two sets of local stabilisations with the resulting orbifold virtual systems denoted by

$$\{(\mathcal{V}'_I, \mathbf{E}'_I, \sigma'_I) | I \subset \{1, 2, \cdots, m\}\} \qquad \text{and} \qquad \{(\mathcal{V}''_I, \mathbf{E}''_I, \sigma''_I) | I \subset \{1, 2, \cdots, n\}\}$$

respectively. We can constructed an orbifold virtual system with boundary from the Fredholm system $(\mathcal{B}\times[0,1], \mathcal{E}\times[0,1], S\times[0,1])$, equipped with a local stabilisation extending two local stabilisations at the two ends. Then we apply the Stokes' formula for the virtual integration in Theorem 2.15 to show that the invariant defied in (4.11) does not depend on the choice of local stabilisations.

(2) When the expected dimension of $\mathcal{M}$, the difference between the dimension of $\mathcal{V}$ and the virtual rank of $\mathbf{E}$, is zero, then $\Phi(1)$ should be thought of a virtual Euler number of $\mathbf{E}$. If it happens that $\mathcal{M}$ consists of a collection of smooth orbifold points $\{(x_1, G_{x_1}), (x_2, G_{x_2}), \cdots, (x_n, G_{x_n})\}$, then

$$\Phi(1) = \sum_{i=1}^{n} \frac{1}{|G_{x_i}|},$$

agrees with the orbifold Euler characteristic of $\mathcal{M}$ (a rational number in general).

5. Proper étale weak Lie groupoids and weak orbifold Fredholm systems

In the Gromov-Witten theory, there are some further technical issues in applying the virtual neighborhood technique to the moduli spaces of stable maps. There are two main issues:

(1) the underlying space of stable maps is often stratified,
(2) some of the stratum (like the domain curves being spheres with less than three marked point) are not smooth due to the fact that the reparametrization group action is not differentiable.

So in general, the virtual neighborhood technique developed in this section should be thought as a guiding principle rather than a complete recipe to define the Gromov-Witten invariants. To fully develop the virtual neighbourhood techniques for the Gromov-Witten invariants, we need to deal with the non-differentiability issue of reparametrisations using the notion of weak proper étale groupoids.

Definition 5.1. (1) A topological groupoid $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1)$ is called a **weak Lie groupoid** if $\mathcal{G}_0$ is a smooth manifold, $\mathcal{G}_1$ is topological manifold and all the structure maps (s, t, m, u, i) are continous maps.

(2) Given a two weak Lie groupoids $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1)$ and $\mathcal{H} = (\mathcal{G}^0, \mathcal{G}^1)$, a strict continuous morphism $f = (f_0, f_1) : \mathcal{G} \to \mathcal{H}$ is called smooth if f_0 is smooth. The collection of objects being weak Lie groupoids of morphisms being smooth morphisms will be called the category of weak Lie groupoids.

(3) In the category of weak Lie groupoids, a topological vector bundle $\pi = (\pi_0, \pi_1) : \mathcal{E} \to \mathcal{G}$ over a weak Lie groupiod $\mathcal{G}$ is called a smooth vector bunlde if $\mathcal{E}$ is a week Lie groupoid and the bundle project π_0 is smooth. A section $s = (s_0, s_1)$ of a smooth vector bunlde $\mathcal{E}$ over a weak Lie groupoid $\mathcal{G}$ is called smooth if s_0 is smooth. A smooth section $s = (s_0, s_1)$ is transversal if s_0 is transversal.

Any topological Lie groupoid $\mathcal{G} = (\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$ is Morita equivalent to a weak Lie groupoid (the covering groupoid associated to a good cover of $\mathcal{G}^0$). Taking an open covering $\mathcal{U} = \{U_i\}$ of $\mathcal{G}^0$ such that each open set U_i is homeomorphic to an open set in an Euclidean space (this impossible as $\mathcal{G}^0$ is a topological manifold). Then the associated covering Lie groupoid

$$\mathcal{G}[\mathcal{U}] = \left(\bigsqcup_{i,j} \mathcal{G}_{U_i}^{U_j} \rightrightarrows \bigsqcup_i U_i \right),$$

where $\mathcal{G}_{U_i}^{U_j} = s^{-1}(U_i) \cap t^{-1}(U_j) \subset \mathcal{G}^1$, is a weak Lie groupoid with the obvious smooth structure on $\bigsqcup_i U_i$.

Lemma 5.2. *Let $\mathcal{G} = (\mathcal{G}^1 \rightrightarrows \mathcal{G}^0)$ be an* **étale** *weak Lie groupoid with the source and target maps s and t respectively.*

(1) *Then $\mathcal{G}$ is a Lie groupoid if and only if two smooth structures on $\mathcal{G}^1$ from the pull-backs of s and t are compatible and the rest of structure maps are smooth.*
(2) *Let X be a smooth manifold, a map $f = (f^1, f^0) : \mathcal{G} \to X$ is smooth if and only if f^1 is smooth with respect to both smooth structures on $\mathcal{G}^1$ (the pull-back smooth structures by s and t).*

Proof. As $\mathcal{G}$ is étale, s and t are local homeomorphisms. Choose a smooth atlas on $\mathcal{G}^0$ which is generated by coordinate charts $\{(U_i, \varphi_i)\}$ such that s and t are homeomorphisms on components of $s^{-1}(U_i)$ and components of $t^{-1}(U_i)$ respectively for each i. Then $\mathcal{G}^1$ is a smooth manifold such that s and t are smooth if and only if

$$\{(U_i^{(j)}, \varphi_i \circ s), (U_i^{(k)}, \varphi_i \circ t) | s^{-1}(U_i) = \sqcup_j U_i^{(j)}, t^{-1}(U_i) = \sqcup_k U_i^{(k)})\}$$

are compatible. Hence Claim (1) follows directly.

For Claim (2), notice that a map f from a étale weak Lie groupoid $\mathcal{G}$ to a space X consists of a pair of maps $f_0 : \mathcal{G}^0 \to X$ and $f_1 : \mathcal{G}^1 \to X$ such that $s^* f_0 = t^* f_0 = f_1$. By definition, f is smooth if f_0 is smooth. Then $f_1 = s^* f_0 = t^* f_0$ is smooth with respect to both the pull-back smooth structures by s and t. □

Definition 5.3. In the category of weak Lie groupoids, a topological vector bundle $\pi = (\pi_0, \pi_1) : \mathcal{E} \to \mathcal{G}$ over a weak Lie groupiod $\mathcal{G}$ is called a smooth vector bunlde if $\mathcal{E}$ is a week Lie groupoid and the bundle project π_0 is smooth. A section $s = (s_0, s_1)$ of a smooth vector bunlde $\mathcal{E}$ over a weak Lie groupoid $\mathcal{G}$ is called smooth if s_0 is smooth. A smooth section $s = (s_1, s_0)$ is transversal if s_0 is transversal.

for a étale weak Lie groupoid $\mathcal{G}$, a smooth section $s = (s_1, s_0)$ of a smooth vector bundle $\mathcal{E}$ over $\mathcal{G}$ implies that s_1 is smooth with respect to both the pull-back smooth structures by s and t on both $\mathcal{G}$ and $\mathcal{E}$ simultaneously. We like to clarify that s_1 is not a smooth section in general as $\mathcal{G}$ and $\mathcal{E}$ are only weak Lie groupoids, are not Lie groupoids per se. Nevertheless, the following proposition says that in the category of weak Lie groupoids, transversal sections are very useful in the constructions of new weak Lie groupoids.

Proposition 5.4 (Transversality). *Let $\mathcal{E} \to \mathcal{G}$ be a smooth vector bundle over an étale* weak *Lie groupoid $\mathcal{G}$. Suppose that $\sigma = (\sigma_0, \sigma_1)$ be a transversal smooth section of $\mathcal{E}$ Then*

$$\sigma^{-1}(0) = (\sigma_1^{-1}(0) \rightrightarrows \sigma_0^{-1}(0))$$

is an étale weak Lie groupoid.

Proof. A priori, the sets of zeros $\sigma^{-1}(0) = (\sigma_1^{-1}(0) \rightrightarrows \sigma_0^{-1}(0))$ is only an étale topological groupoid. Since $\mathcal{E}^0 \to \mathcal{G}^0$ is a smooth bundle over a smooth manifold $\mathcal{G}^0$ and σ_0 is a transversal section, the classical transversality theorem implies that $\sigma_0^{-1}(0)$ is a smooth manifold. Being étale, the pull-back smooth structures by the source maps define smooth structures on $\mathcal{E}$ and $\mathcal{G}$ respectively. With respect to these smooth structures on $\mathcal{E}^1 \to \mathcal{G}^1$, $\sigma_1 = s^*\sigma_0 : \mathcal{G}^1 \to \mathcal{E}^1$ is a transversal section, hence $\sigma_1^{-1}(0)$ is a smooth manifold. In particular, $\sigma_1^{-1}(0)$ is a topological manifold. This ensures that $\sigma^{-1}(0)$ is an étale weak Lie groupoid. □

The same argument in the proof can also be applied to the target maps of $\mathcal{G}$ and $\mathcal{E}$, we get a smooth structure on $(\sigma_1)^{-1}(0)$ which might be different to the smooth structure induced by the pull-back of the sourse map. Hence $\sigma^{-1}(0)$ is only a weak Lie groupoid in general. In dealing with moduli spaces arising from elliptic PDEs, like those in Gromov-Witten theory, we have a pleasant but unsurprised result, the two smooth structures induced by the source and target maps happen to be compatible. So we end up with an étale Lie groupoid $\sigma^{-1}(0) = (\sigma_0^{-1}(0) \rightrightarrows \sigma_1^{-1}(0))$ from a transversal section σ defined by a system of elliptic PDES. This is the key observation in the proposal of weak Lie groupoids for the study of these moduli spaces.

Definition 5.5. A **weak orbifold Fredholm system** is a triple $(\mathcal{B}, \mathcal{E}, S)$ consisting of

(1) a proper étale weak Lie groupoid $\mathcal{B} = (\mathcal{B}^1 \rightrightarrows \mathcal{B}^0)$;
(2) a Banach vector bundle $\mathcal{E} = (\mathcal{E}^1 \rightrightarrows \mathcal{E}^0)$ over $\mathcal{B}$ in the category of proper étale weak Lie groupoids,
(3) $S = (S_1, S_0)$ is a smooth section of $\mathcal{E}$ such that S_0 is Fredholm.

Following from Proposition 5.4, we have the following corollary where a weak orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ is called regular if the section S is transversal.

Corollary 5.6. *Given a regular weak orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$, then the moduli space*

$$\mathcal{M} = (\mathcal{M}^1 \rightrightarrows \mathcal{M}^0) = (S_1^{-1}(0) \rightrightarrows S_0^{-1}(0))$$

is a proper étale weak Lie groupoid of dimension given by the Fredholm index of S_0.

In the category of weak Lie groupoids, the notion of virtual weak Lie groupoids can be defined as follows. A virtual weak Lie groupoid is a collection of weak Lie groupoids $\{\mathcal{X}_I\}_{I\in\mathcal{I}}$ indexed by a partially ordered set $(\mathcal{I} = 2^{\{1,2,\cdots,N\}}, \subset)$, together with patching data

$$\{(\Phi_{I,J}, \phi_{I,J}) | I, J \in \mathcal{I}, I \subset J\},$$

such that the cocycle condition (2.1) and the fiber product condition (2.2) hold in the category of weak Lie groupoids. In particular, a proper étale virtual weak Lie groupoid can be defined in the category of proper étale weak Lie groupoids. Moreover, with the help of Corollary 5.6, we have the following proposition.

Proposition 5.7. *Let $\{\mathcal{X}_I, \Phi_{I,J}, \phi_{I,J} | I \subset J \in 2^{\{1,2,\cdots,N\}}\}$ be a proper étale virtual weak Lie groupoid.*

(1) *Given a vector bundle $\mathcal{F} = \{\mathcal{F}_I \to \mathcal{X}_I\}$ over $\{\mathcal{X}_I\}$ with a transversal section $\{S_I\}$, then the collections of zero sets*
$$\{\mathcal{Z}_I = S_I^{-1}(0)\}$$
is a proper étale virtual weak groupoid. If $\{S_I\}$ is not transversal to the zero section, then $\{\mathcal{Z}_I = S_I^{-1}(0)\}$ is only a topological proper étale virtual groupoid.
(2) *Given a virtual vector bundle $\{\mathbf{E}_I \to \mathcal{X}_I\}$ with a transversal section $\{\sigma_I\}$, then the collection of zero sets*
$$\{\mathcal{Y}_I = \sigma_I^{-1}(0)\},$$
under the induced patching data, forms a proper étale weak Lie groupoid. In the absence of transversality, $\{\mathcal{Y}_I = \sigma_I^{-1}(0)\}$ forms a topological proper étale groupoid under the homeomorphism $\phi_{I,J} : \mathcal{Y}_{J,I} \cong \mathcal{Y}_{I,J}$. The collection of triples $\{\mathcal{X}_I, \mathbf{E}_I, \sigma_I)\}$ is called a **weak orbifold virtual system** *for this topological proper étale groupoid.*

Assumption 5.8. *Given a weak orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$, for any $x \in \mathcal{M}^0$, there exist $\mathcal{B}^1$-invariant open neighborhoods $U_x^{(i)}$ and U_x of x in $\mathcal{B}^0$ for $i = 1, 2, 3$ such that*

$$U_x^{(1)} \subset \overline{U_x^{(1)}} \subset U_x^{(2)} \subset \overline{U_x^{(2)}} \subset U_x^{(3)} \subset \overline{U_x^{(3)}} \subset U_x \tag{5.1}$$

and there is a $\mathcal{B}^1$-invariant **smooth** *cut-off function $\beta_x : U_x^{(3)} \to [0,1]$ such that $\beta_x \equiv 1$ on $U_x^{(1)}$ and is supported in $U_x^{(2)}$. Here $\overline{U_x^{(i)}}$ is the closure of $U_x^{(i)}$.*

The constructions the previous sections can be carried over to establish the following theorem for any a weak orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ with compact moduli space satisfying Assumption 5.8.

Theorem 5.9. *Given a weak orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ satisfying Assumption 5.8 such that its moduli space $\mathcal{M} = S^{-1}(0)$ has a compact coarse space $|\mathcal{M}|$, then there exists a finite dimensional weak orbifold virtual system for $(\mathcal{B}, \mathcal{E}, S)$ which is a collection of triples*

$$\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I) | I \subset \{1, 2, \cdots, N\}\}$$

indexed by a partially ordered set $(\mathcal{I} = 2^{\{1,2,\cdots,N\}}, \subset)$, where

(1) $\mathcal{V} = \{\mathcal{V}_I\}$ *is a finite dimensional weak proper étale virtual Lie groupoid,*
(2) $\mathbf{E} = \{\mathbf{E}_I\}$ *is a finite rank virtual orbifold vector bundle over $\{\mathcal{V}_I\}$*
(3) $\sigma = \{\sigma_I\}$ *is a virtual section of $\{\mathbf{E}_I\}$ whose zero sets $\{\sigma_I^{-1}(0)\}$ form a cover of $\mathcal{M}$.*

REFERENCES

[1] A. Adem, J. Leida and Y. Ruan, *Orbifolds and stringy topology,* Cambridge Tracts in Mathematics 171, Cambridge University Press (2007).
[2] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Spinger.
[3] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982.
[4] B. Chen *Smoothness on bubble tree compactified instanton moduli spaces*. Acta Math. Sin. (Engl. Ser.) 26 (2010), no. 2, 209-240.
[5] B. Chen, S. Hu, *A de Rham model for Chen-Ruan cohomology ring of abelian orbifolds.* Math. Ann., 336, no. 1, 5171, 2006. Math.SG/0408265.
[6] B. Chen and A. Li, *Symplectic virtual localization of Gromov-Witten classes,* arXiv:DG/0610370.
[7] B. Chen, A. Li and B. Wang, Gluing principle for orbifold stratified spaces. Preprint.
[8] B. Chen, A. Li and B. Wang, *Smooth structures on moduli spaces of stable maps,* in preparation.
[9] B. Chen and G. Tian, *Virtual manifolds and localization*, Acta Math. Sinica, 26 (1), 1-24.
[10] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds.* Invent. Math. Vol. 82 (1985), 307-347.
[11] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant.* Topology 38 (1999), no. 5, 933-1048.
[12] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Intersection Theory, Anomaly and Obstruction, Parts I and II.* AMS/IP Studies in Advanced Mathematics, Amer. Math. Soc. and Internat. Press.
[13] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Technical details on Kuranishi structure and virtual fundamental chain*, arXiv:1209.4410.
[14] H. Hofer, K. Wysocki, and E. Zehnder, *Applications of Polyfold theory I: The Polyfolds of Gromov-Witten theory,* arXiv:1107.2097.
[15] A. Li, Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds.* Invent. Math. 145(2001), 151-218.
[16] D. Joyce, *Kuranishi homology and Kuranishi cohomology*, arXiv:0710.5634.
[17] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry.* Commun. Math. Phys. Vol. 164 (1994), 525-562.
[18] A. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds I.* Invent. Math. 145 (2001), no. 1, 151-218.
[19] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties.* J. Amer. Math. Soc. 11 (1998), no. 1, 119-174.
[20] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds.* **Topics in symplectic 4-manifolds** (Irvine, CA, 1996), 47-83, First Int. Press Lect. Ser., I, Internat. Press, Cambridge, MA, 1998.
[21] G. Liu and G. Tian, *Floer homology and Arnold conjecture.* J. Differential Geom. 49 (1998), no. 1, 1-74.

[22] G. Lu and G. Tian, *Constructing virtual Euler cycles and classes*, Int. Math. Res. Surveys (2008)

[23] E. Lupercio, B. Uribe, *Gerbes over orbifolds and twisted K-theory,* Comm. Math. Phys. **245** (2004), no. 3, 449-489.

[24] D. McDuff, *Symplectic manifolds with contact-type boundaries.* Invent. Math Vol. 103, (1991), 651-671.

[25] D. McDuff, *The virtual moduli cycle.* Amer. Math. Soc. Transl. (2) 196 (1999), 73-102.

[26] D. McDuff, D. Salamon, *J-holomorphic curves and Symplectic Topology*, AMS Colloquium Publications, Vol. 52.

[27] D. McDuff, K. Wehrheim, *Smooth Kuranishi atlases with trivial isotropy*, arXiv:1208.1340v3.

[28] I. Moerdijk, D. A. Pronk, *Simplcial cohomolgy of orbifolds*, Indag. Math. (N.S.) Vol. 10, (1999), no. 2, 269-293.

[29] B. Siebert, *Gromov-Witten invariants for general symplectic manifolds*, arXiv:dg-ga/9608005

[30] Y. Ruan, *Symplectic Topology on Algebraic 3-folds.* Jour. of Diff. Geom. 39 (1994), 215-227.

[31] Y. Ruan, *Topological sigma model and Donaldson-type invariants in Gromov theory.* Duke Math. J. , Vol. 83, No. 2 (1996), 461-500. Preprint 1993.

[32] Y. Ruan, *Virtual neighborhoods and pseudo-holomorphic curves.* Proceedings of 6th Gokova Geometry-Topology Conference. Turkish J. Math. 23 (1999), no. 1, 161-231.

[33] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology.* J. Differential Geom. 42 (1995), no. 2, 259-367.

[34] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma model coupled with gravity*. Turkish J. Math. Vol. 20 no. 1 (1996), 75-83.

[35] C. Vafa , *Topological mirrors and quantum rings.* in **Essays on Mirror Manifolds**, International Press, Hong Kong, 1992, 96-119.

[36] E. Witten, *Topological sigma models.* Comm. Math. Phys. 118 (1988), 411-449.

SCHOOL OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, CHINA
E-mail address: bohui@cs.wisc.edu, math_li@yahoo.com.cn

DEPARTMENT OF MATHEMATICS, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA ACT 0200, AUSTRALIA
E-mail address: bai-ling.wang@anu.edu.au