Axisymmetric two-fluid plasma equilibria with momentum sources and sinks

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Introduction (1)

- Basic tool used to describe axisymmetric plasma equilibria (e.g. tokamaks, accretion discs) is *Grad-Shafranov equation*[1], obtained from MHD force balance in absence of equilibrium flows & viscosity ($\nabla p = \mathbf{j} \times \mathbf{B}$):

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \Psi}{\partial R}\right)+\frac{\partial^2 \Psi}{\partial Z^2}=-\mu_0 R^2\frac{dp}{d\Psi}-f\frac{df}{d\Psi}$$

(R,φ,Z) – right-handed cylindrical coordinates; $\Psi(R,Z)$ - poloidal magnetic flux, defined such that

$$\mathbf{B}=\nabla\Psi\times\nabla\varphi+RB_\varphi\nabla\varphi$$

$f = f(\Psi) = RB_\varphi$ - stream function for poloidal current; pressure $p = p(\Psi)$

- Can be generalised to include toroidal rotation – necessary when flow approaches or exceeds local sound speed c_s, e.g. in Joint European Torus (JET)[2] or Mega Ampère Spherical Tokamak (MAST)[3] at Culham

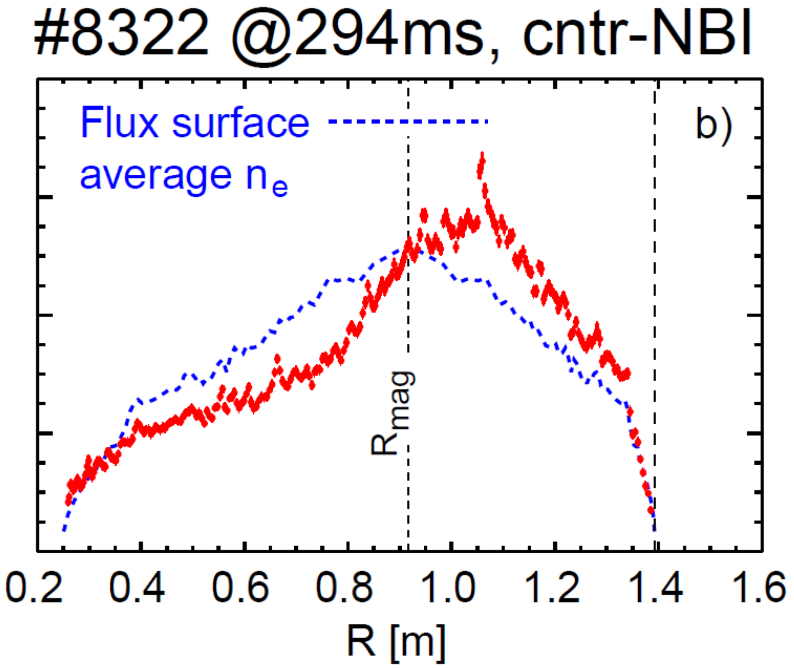
[1] Shafranov Sov. Phys.-JETP **6**, 545 (1958)

[2] de Vries *et al.* Nucl. Fusion **48**, 065006 (2008)

[3] Akers *et al.* Proc. 20th IAEA Fusion Energy Conf., paper EX/4-4 (2005)

Introduction (2)

- ☐ Flows ~350 km s^{-1} occurred in MAST during counter-current beam injection[1]
  - ➢ driven by **j**×**B** torque associated with radial bulk ion current balancing current due to beam ion losses[2]
- ☐ Midplane density profile $n_e(R)$ shifted outboard with respect to temperature (assumed to be flux function due to rapid parallel heat transport)



- ☐ If flow is purely toroidal, T_e & T_i are flux functions, flux surfaces rotate as rigid bodies at rate Ω_ζ, & momentum sources/sinks are neglected, then[3]

$$n_e = n_0(\Psi)\exp[m_i\Omega_\zeta^2 R^2 / 2(T_e + T_i)]$$

[1] Akers *et al.* Proc. 20th IAEA Fusion Energy Conf., paper EX/4-4 (2005)

[2] McClements & Thyagaraja Phys. Plasmas **13**, 042503 (2006)

[3] Maschke & Perrin Plasma Phys. **22**, 579 (1980)

Introduction (3)

- ☐ Rigid body rotation implied by ideal MHD Ohm's law + axisymmetry:

$$\frac{\partial \Phi}{\partial R} = \Omega_\zeta \frac{\partial \Psi}{\partial R}, \qquad \frac{\partial \Phi}{\partial Z} = \Omega_\zeta \frac{\partial \Psi}{\partial Z} \qquad \Rightarrow \qquad \Omega_\zeta = \Omega_\zeta(\Psi)$$

  Φ - electrostatic potential

- ☐ What happens when all possible relevant terms in force balance equation(s) are taken into account?
- ☐ Due to dissipation (in particular neoclassical & turbulent viscosity) flows in tokamaks must be continuously driven
- ☐ Poloidal flows, when measurable, usually found to be very small (~ few km s^{-1}), in accordance with neoclassical predictions, but occasionally observed to be significant fraction of c_s, e.g. close to internal transport barriers in JET[1] – such flows could affect equilibrium[2]
- ☐ In this talk I will consider purely toroidal flows

[1] Crombé *et al*. PRL **95**, 155003 (2005)

[2] McClements & Hole Phys. Plasmas **17**, 082509 (2010)

Plasma coordinates (1)

- ☐ Often convenient to use right-handed plasma-based coordinates (Ψ,θ,ζ) where toroidal angle $\zeta = -\varphi$ & poloidal angle θ is defined such that Jacobian of transformation from laboratory coordinates does not generally vanish in domain of interest:

$$J \equiv \left(\nabla\Psi \times \nabla\theta\right)\cdot\nabla\zeta = \frac{1}{R}\frac{\partial\left(\Psi,\theta\right)}{\partial\left(R,Z\right)} = \frac{\left|\nabla\Psi\right|}{R}\frac{\partial\theta}{\partial\ell}$$

ℓ - arc length along flux surface in (R,Z) plane

- ☐ We set $J = J(\Psi)$ – generalisation of Hamada coordinates (J = constant)[1]; facilitates evaluation of flux-surface averages, since volume element is

$$Rd\ell d\zeta d\Psi / \left|\nabla\Psi\right| = d\theta d\zeta d\Psi / J\left(\Psi\right)$$

Such coordinate systems are *quasi-orthogonal* – $\nabla\Psi\cdot\nabla\theta \neq 0$ in general

- ☐ $\mathbf{B} = \nabla\zeta \times \nabla\Psi + RB_\zeta\nabla\zeta$ where $B_\zeta = -B_\varphi$; we denote RB_ζ by $F = -f$

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[1] Hamada Nucl. Fusion **2**, 23 (1962)

Plasma coordinates (2)

- ❑ In steady-state & in absence of poloidal flows, momentum sources & dissipation, $F = F(\Psi)$ in both ideal MHD[1] & 2-fluid theory (in limit $m_e \to 0$)[2]
- ❑ Here we assume only that F is axisymmetric, i.e. $F = F(R,Z)$ or $F(\Psi,\theta)$
- ❑ Ampère's law ⇒ $$\mathbf{j} = Rj_\zeta \nabla\zeta - \frac{1}{\mu_0}\frac{\partial F}{\partial \Psi}\nabla\zeta \times \nabla\Psi - \frac{1}{\mu_0}\frac{\partial F}{\partial \theta}\nabla\zeta \times \nabla\theta$$

where

$$j_\zeta = \frac{1}{\mu_0 R}\left\{R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \Psi}{\partial R}\right) + \frac{\partial^2 \Psi}{\partial Z^2}\right\} \equiv \frac{1}{\mu_0 R}\Delta^* \Psi$$

Hence $$\boxed{\mathbf{j}\times\mathbf{B} = -\frac{1}{\mu_0 R^2}\left\{\Delta^*\Psi + F\frac{\partial F}{\partial \Psi}\right\}\nabla\Psi - \frac{F}{\mu_0 R^2}\frac{\partial F}{\partial \theta}\nabla\theta + \frac{J}{\mu_0}\frac{\partial F}{\partial \theta}\nabla\zeta}$$

Axisymmetric ideal MHD with purely toroidal flow & no sources/sinks requires that toroidal component of **j**×**B** vanishes ⇒ $F = F(\Psi)$

[1] McClements & Thyagaraja Mon. Not. R. Astron. Soc. **323**, 733 (2001)

[2] Thyagaraja & McClements Phys. Plasmas **13**, 062502 (2006)

Ion momentum balance

- Quasi-neutral plasma with singly-charged ions & electrons, each with scalar pressure; ions have toroidal flow $\mathbf{v}_i = R^2\Omega_\zeta\nabla\zeta$
  - ion momentum balance equation can be written as

$$-\frac{1}{2}m_i n\Omega_\zeta^2\nabla R^2 = -\nabla p_i - ne\nabla\Phi - ne\Omega_\zeta\nabla\Psi + \mathbf{F}^{\text{ext}} + \mathbf{F}^{\text{drag}} + n\mathbf{R}_{\text{ie}}$$

applied torque

momentum loss rate (e.g. due to viscosity)

momentum exchange with electrons

For inductive tokamak operation we can write

$$\mathbf{R}_{\text{ie}} = \frac{eV_L}{2\pi}\nabla\zeta - e\eta\mathbf{j}$$

V_L – loop voltage (we assume uniform toroidal voltage across plasma);

η - resistivity (assumed to be isotropic)

Electron momentum balance

- In limit $m_e \to 0$ electron momentum balance equation (≡ generalised Ohm's law) can be written as

$$\mathbf{0} = -\nabla p_e + ne\nabla\Phi + ne\Omega_\zeta \nabla\Psi + \mathbf{j}\times\mathbf{B} + n\mathbf{R}_{\mathrm{ei}} \longleftarrow$$

momentum exchange with ions

- Momentum conservation $\Rightarrow \mathbf{R}_{\mathrm{ei}} = -\mathbf{R}_{\mathrm{ie}} = -\dfrac{eV_L}{2\pi}\nabla\zeta + e\eta\mathbf{j}$

- Momentum sources & sinks neglected in electron momentum balance – any momentum acquired by electron via interaction with e.g. beam ions very rapidly transferred to bulk ions, hence $\mathbf{F}^{\mathrm{ext}} \approx -\mathbf{F}^{\mathrm{drag}}$ for electrons (if this were not the case, beam injection would produce large numbers of highly superthermal electrons, which are not observed)
- We neglect external current sources (driven e.g. by beams or radio-frequency waves) & bootstrap current (diamagnetic current associated with drift orbits of trapped particles)

Single-fluid momentum balance (1)

- ☐ Adding ion & electron equations yields

$$-\frac{1}{2}m_i n\Omega_\zeta^2\nabla R^2 = -\nabla p + \mathbf{j}\times\mathbf{B} + \mathbf{F}^{\text{ext}} + \mathbf{F}^{\text{drag}} \qquad (1)$$

where $p = p_i + p_e$; to ensure compatibility with neglect of poloidal flows we consider only toroidal components of $\mathbf{F}^{\text{ext}}$ & $\mathbf{F}^{\text{drag}}$, & assume that drag term can be characterised by phenomenological relaxation time τ_ζ:

$$\mathbf{F}^{\text{drag}} = -\frac{m_i n\Omega_\zeta R^2}{\tau_\zeta}\nabla\zeta$$

(for momentum losses arising from neoclassical or turbulent viscosity, more exact expression would involve spatial derivatives of $\mathbf{v}_i$)

- ☐ Substituting our expression for **j**×**B** into (1) yields

$$\frac{1}{\mu_0 R^2}\left\{\Delta^*\Psi + F\frac{\partial F}{\partial\Psi}\right\}\nabla\Psi = -\frac{F}{\mu_0 R^2}\frac{\partial F}{\partial\theta}\nabla\theta + \frac{J}{\mu_0}\frac{\partial F}{\partial\theta}\nabla\zeta + \frac{1}{2}m_i n\Omega_\zeta^2\nabla R^2$$

$$-\nabla p + \left[F_\zeta^{\text{ext}}R - \frac{m_i n\Omega_\zeta R^2}{\tau_\zeta}\right]\nabla\zeta$$

Single-fluid momentum balance (2)

- Regarding R^2 & p as functions of Ψ & θ, & equating components, we obtain

$$\frac{1}{\mu_0 R^2}\left\{\Delta^*\Psi + F\frac{\partial F}{\partial \Psi}\right\} = -\frac{\partial p}{\partial \Psi} + \frac{1}{2} m_i n \Omega_\zeta^2 \frac{\partial R^2}{\partial \Psi} \qquad (1)$$

$$\frac{\partial p}{\partial \theta} = \frac{1}{2} m_i n \Omega_\zeta^2 \frac{\partial R^2}{\partial \theta} - \frac{F}{\mu_0 R^2}\frac{\partial F}{\partial \theta} \qquad (2)$$

$$F_\zeta^{\text{ext}} R + \frac{J}{\mu_0}\frac{\partial F}{\partial \theta} = \frac{m_i n \Omega_\zeta R^2}{\tau_\zeta} \qquad (3)$$

- In limit $\Omega_\zeta \to 0$, $F_\zeta^{\text{ext}} \to 0$ (3) implies $F = F(\Psi)$ & (2) implies $p = p(\Psi)$; (1) then reduces to standard form of Grad-Shafranov equation
- When $\Omega_\zeta \neq 0$ & $F = F(\Psi)$ (1) is equivalent to Grad-Shafranov equation for purely toroidal flow obtained by previous authors[1]
- $F = F(\Psi)$ to leading order in $\left(v_\zeta^2 / c_A^2\right)/\left(\Omega_\zeta \tau_\zeta\right)$

[1] Maschke & Perrin Plasma Phys. **22**, 579 (1980)

Variation of density on flux surfaces (1)

- Relation between momentum drive & dissipation depends on beam deposition & momentum transport; we consider simple cases to illustrate influence of torque & relaxation time on density distribution
- 1st case: $F = F(\Psi)$ & $F_\zeta^{\mathrm{ext}}\tau_\zeta = KR$ where $K = K(\Psi)$; from (2) & (3) on previous slide we obtain $m_i n\Omega_\zeta = K$ &

  $$n^2 = \langle n^2 \rangle + \frac{K^2}{2m_iT}\left(R^2 - \langle R^2 \rangle\right) = \frac{\langle n^2 \rangle}{1 + m_i\Omega_\zeta^2(\langle R^2 \rangle - R^2)/2T}$$

  $\langle \ldots \rangle$ - flux surface average

  - differs from result for rigidly-rotating flux surfaces when momentum sources & sinks are neglected:

  $$n = n_0(\Psi)\exp[m_i\Omega_\zeta^2 R^2/4T]$$

- In both cases n increases with R on flux surface – arises from inertial term in ion momentum balance equation, & is qualitatively consistent with measurements in spherical tokamak plasmas with transonic toroidal flows

Variation of density on flux surfaces (2)

- ☐ 2nd case: F & $F_\zeta^{\text{ext}}\tau_\zeta$ are flux functions → $M \equiv m_i n\Omega_\zeta R$ (toroidal linear momentum per unit volume) is a flux function &

$$n^2 = \frac{M^2}{2m_i T}\ln\left(\frac{R^2}{R_{\min}^2(\Psi)}\right) + n_{\min}^2(\Psi) = \frac{n_{\min}^2}{1-(m_i\Omega_\zeta^2 R^2/2T)\ln(R^2/R_{\min}^2)}$$

- ☐ 3rd case: F & $F_\zeta^{\text{ext}}\tau_\zeta R$ are flux functions → $L \equiv m_i n\Omega_\zeta R^2$ (toroidal angular momentum per unit volume) is a flux function &

$$n^2 = N^2(\Psi) - L^2/2m_i TR^2 = N^2/(1+m_i\Omega_\zeta^2 R^2/2T)$$

- ☐ In all cases we find that at magnetic axis

$$R\frac{d\ln n}{dR} = \frac{m_i\Omega_\zeta^2 R^2}{2T}$$

- agrees well with measurements in National Spherical Torus Experiment (NSTX) at Princeton[1]; but measurements at magnetic axis alone cannot be used to determine density variation on flux surface

[1] Menard *et al*. Nucl. Fusion **43**, 330 (2003)

Temperature – density relation: theory

- Eliminating n we obtain $\Omega_\zeta = \Omega_\zeta(T)$; e.g. in 1st case

$$\Omega_\zeta = \frac{KT^{1/2}}{m_i[IT + K^2R^2/2m_i]^{1/2}}$$

where I, K are flux functions

- Expressions for Ω_ζ in terms of T_i & T_e can be obtained from ion & electron momentum balance equations in limit $F_\zeta^{ext} \to 0$, $\tau_\zeta \to \infty$, $V_L \to 0$;[1] result for rigidly-rotating flux surfaces is

$$\Omega_\zeta = \Omega_{\zeta 0}\left(\frac{T_i + T_e}{2T_0}\right)^{1/2} \exp\left[-\int_{\Psi_0}^{\Psi} \frac{T_e^{'} d\psi}{2(T_i + T_e)}\right]$$

$\Omega_{\zeta 0}$, T_0, Ψ_0 – constants; when $T_i = T_e = T$ this reduces to

$$\Omega_\zeta = \Omega_{\zeta 0}(T/T_0)^{1/4}$$

→ rotation profile predicted to be much flatter than temperature profile

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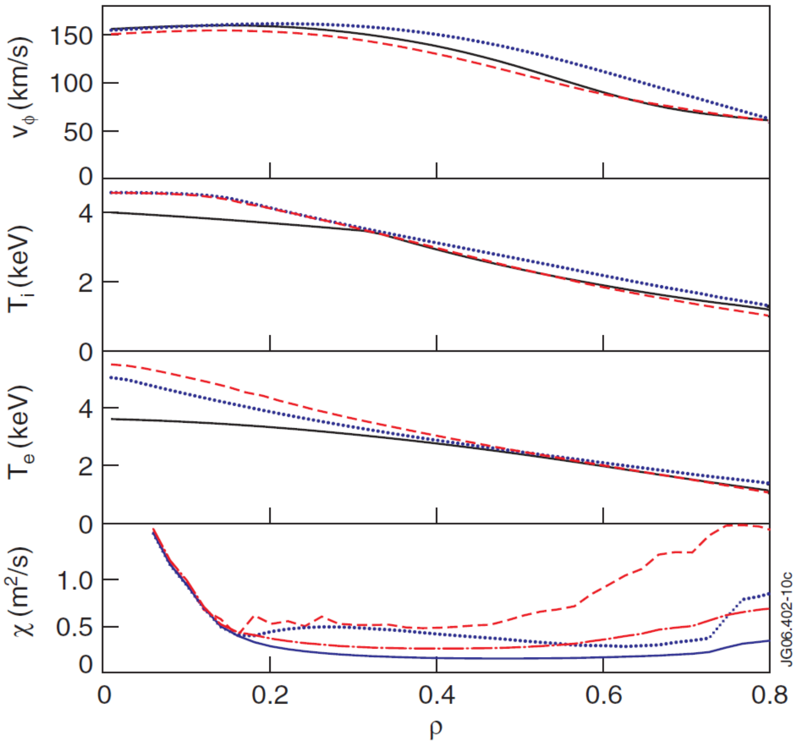
[1] Thyagaraja & McClements Phys. Plasmas **13**, 062502 (2006)

Temperature – density relation: experiment



Rotation, temperature & momentum/ion heat diffusivity profiles in JET #57865; solid black curves show experimental profiles[1]

- Spectroscopic measurements in JET plasmas with $T_i \approx T_e$ show that Ω_ζ & T_i have similar profiles, contradicting prediction based on (i) assumption of rigid body rotation & (ii) neglect of momentum sources/sinks
- Similar relation between profiles observed in MAST
    - In order to account for measured rotation & temperature profiles it is necessary to invoke **either** momentum sources/sinks **or** non-rigid rotation of flux surfaces

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[1] Tala *et al*. Nucl. Fusion **47**, 1012 (2007)

Ohm’s law for rotating tokamak plasma

When $F = F(\Psi)$ poloidal component of electron momentum balance yields

$$\frac{J}{neF}\frac{\partial p_e}{\partial\theta} = \frac{J}{F}\frac{\partial\Phi}{\partial\theta} - \frac{V_L}{2\pi R^2} - \eta\left(\frac{\partial p}{\partial\Psi} - \frac{1}{2}m_i n\Omega_\zeta^2\frac{\partial R^2}{\partial\Psi} - \frac{F^{'} B^2}{F}\right)$$

Flux surface average of this yields

$$FF^{'} = -\frac{\mu_0 F^2}{\left\langle B^2\right\rangle}\left[\frac{V_L}{2\pi\eta}\left\langle\frac{1}{R^2}\right\rangle + \left\langle p^{'}\right\rangle - \frac{1}{2}m_i\left\langle n\Omega_\zeta^2\frac{\partial R^2}{\partial\Psi}\right\rangle\right]$$

Substituting into Grad-Shafranov equation & integrating over (R,Z) → plasma current in terms of loop voltage & resistivity:

$$I_p = 2\pi\int_{\Psi_0}^{0}\frac{\left\langle B_\theta^2\right\rangle}{\left\langle B^2\right\rangle}\left(\frac{1}{2}m_i\left\langle n\Omega_\zeta^2\frac{\partial R^2}{\partial\Psi}\right\rangle - \left\langle p^{'}\right\rangle\right)\frac{d\Psi}{J} + V_L\int_{\Psi_0}^{0}\frac{\left\langle B_\zeta^2\right\rangle\left\langle 1/R^2\right\rangle}{\left\langle B^2\right\rangle}\frac{d\Psi}{\eta J}$$

- could be used e.g. in fluid turbulence codes to calculate loop voltage required to maintain specified current, given profiles of Ω_ζ, η, J, $\langle B^2\rangle$ etc.

Radial electric field

- Radial component of ion momentum balance equation yields

$$\frac{\partial\Phi}{\partial\Psi} = -\frac{1}{en}\frac{\partial p_i}{\partial\Psi} - \Omega_\zeta + \frac{m_i\Omega_\zeta^2}{2e}\frac{\partial R^2}{\partial\Psi} - \eta\mathbf{j}\cdot\frac{\nabla\theta\times\nabla\zeta}{J}$$

- Last term on right hand side negligible under typical tokamak conditions; averaging remaining terms with respect to θ we obtain simple expression for $\partial\langle\Phi\rangle/\partial\Psi$ - measure of flux surface-averaged radial electric field; shear of this is thought to play important role in transport barrier physics

- Radial electric fields fields can be determined from motional Stark effect measurements;[1] - equilibrium electric field given by above expression could thus be compared directly with experiment

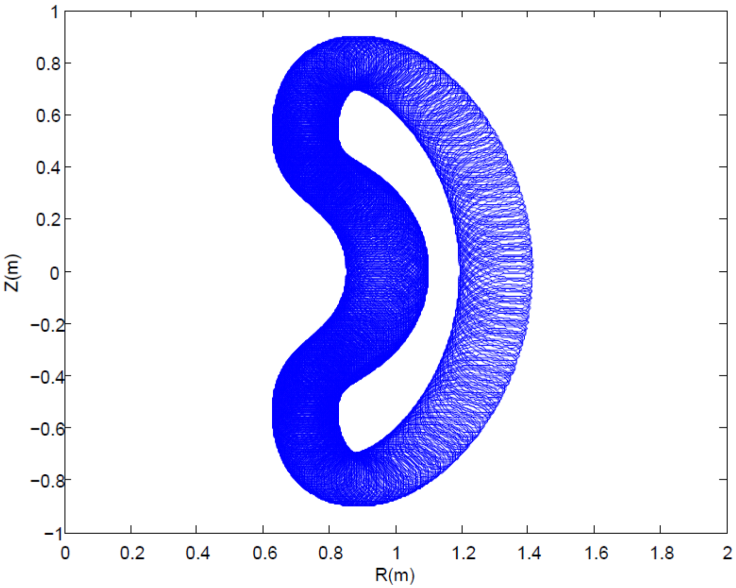
[1] Rice *et al.* Nucl. Fusion **37** 517 (1997)

Conclusions

- ☐ Previous analyses of axisymmetric plasma equilibria with toroidal flows extended to take into account continuous drive of such flows when dissipative effects are present (invariably the case in tokamaks)
- ☐ Ion & electron fluid momentum equations yield three coupled PDEs which indicate that flux surfaces do **not** rotate rigidly, as frequently assumed & required by ideal MHD in absence of momentum sources/sinks & poloidal flows
- ☐ For specific assumed relations between momentum drive & damping, expressions can be obtained for variation of density on flux surfaces - in principle can be tested experimentally
- ☐ **Either** momentum sources/sinks **or** non-rigid rotation of flux surfaces must be invoked to account for measured rotation & temperature profiles
- ☐ Relation derived between loop voltage & plasma current in tokamak plasma with toroidal flow - could be used to determine voltage required to maintain particular current in slowly-evolving discharge
- ☐ Simple expression obtained for equilibrium radial electric field - can be compared directly with experiment
- ☐ For further details see McClements & Thyagaraja Plasma Phys. Control. Fusion **53**, 045009 (2011)

Postscript: ripple transport in MAST (1)

- MAST has $R \approx 0.85$m, minor radius $a \approx 0.65$m, $B \approx 0.5$T, $I_p \approx 1$MA & is heated by 70keV deuterium neutral beams – plot shows trapped beam ion orbit in (R,Z) plane
- Due to presence of N = 12 toroidal field coils at R_{coil} = 2m, toroidal & radial field components are no longer axisymmetric:



$$\tilde{B}_\varphi = \frac{B_0 R_0}{R}\left(\frac{R}{R_{\text{coil}}}\right)^N \cos(N\varphi) \qquad \tilde{B}_R = \frac{B_0 R_0}{R}\left(\frac{R}{R_{\text{coil}}}\right)^N \sin(N\varphi)$$

- even in absence of prompt losses, collisions & turbulence, beam ions are no longer perfectly confined in plasma because

$$\dot{P}_\varphi = \frac{d}{dt}\left(mRv_\varphi + Ze\Psi\right) = -\frac{\partial H}{\partial \varphi} \neq 0$$

Postscript: ripple transport in MAST (2)

- ❑ Cyclotron resonance between particle motion & ripple field can produce additional transport & loss;[1] resonance condition for zero frequency field perturbation is $k_{||}v_{||} = \Omega_i$ where Ω_i is beam ion cyclotron frequency
- ➢ approximately satisfied by beam ions in MAST

- ❑ Significant anomalous beam ion transport has been reported in MAST[2] - ripple effects may be contributing to this

- ❑ Transport of fast ions due to ripple, or other non-axisymmetric perturbations, can be modelled using `CUEBIT` full orbit test-particle code

- ❑ Principal aim of my visit is to pursue this project in collaboration with MJH

[1] Putvinskii JETP Lett. **36** 397 (1982)

[2] Turnyanskiy *et al.* Nucl. Fusion **49** 065002 (2009)