Numerical approaches to kinetic MHD in stellarators

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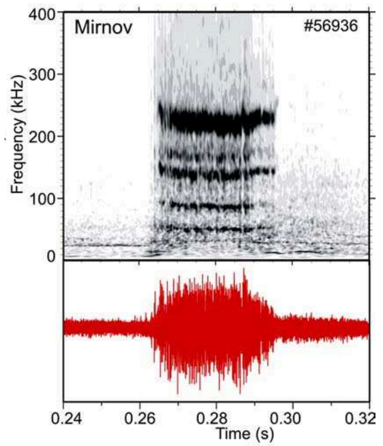
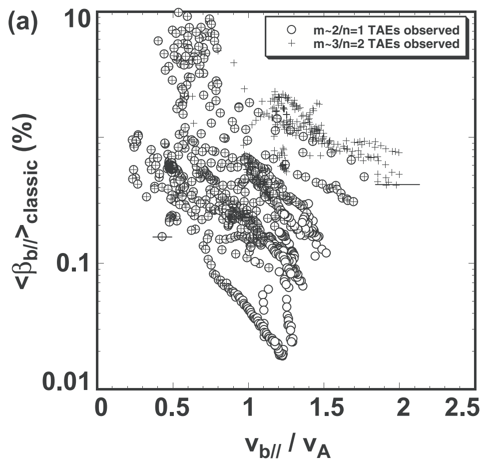
1. **introductory remarks on wave particle-interaction**
2. **numerical 3D kinetic MHD model (CAS3D-K)**
3. **application study (W7-AS/W7-X), limits and extentions of the model**
4. **PIC based linear kinetic MHD models**

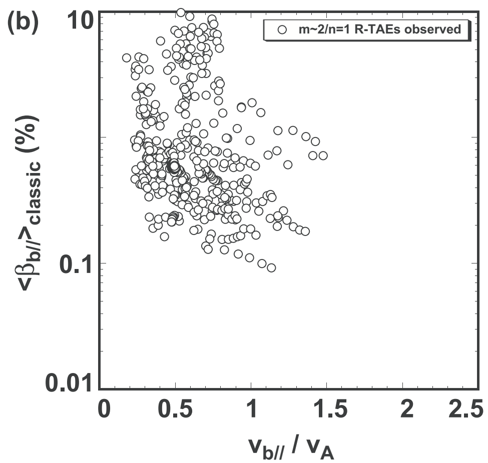
Fast particle - wave interaction

highly energetic particles may interact with ideal MHD modes:

- **destabilization of magneto-hydrodynamic (MHD) modes by resonant interaction with fast particles**
- **source of free energy: temperature/density gradient of fast particles**
- **kinetic description needed**
- **numerous experimental observations in tokamaks as well as in stellarators**







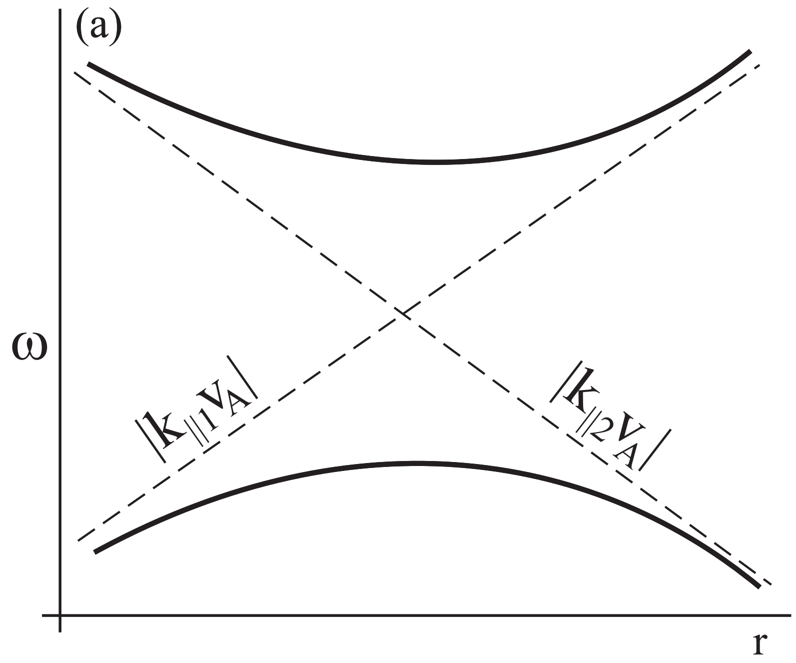
Questions to ask in energetic particle physics

(remembering the hole in wall observed in TFTR)

- **Under which scenarios fast particles get lost ?**
- **How many fast particles get lost ?**
- **Where the fast particles hit the wall ?**
- **study particle-wave interaction**
- **look for most "dangerous" mode**

note: difference to MHD instabilities

- looking for particle interaction with the stable part of MHD spectrum
- source of free energy density or temperature gradient of fast particles
- kinetic description needed
- stable MHD spectrum:
  analogy to Schrödinger equation in a solid state
  - $\omega^2 W_{kin}(\xi^*, \xi) = W_{mag}(\xi^*, \xi)$
  - slab/cylinder dispersion relation:
    $\omega^2 = k_{||}^2 v_A^2$
  - degeneracy removed by symmetry breaking terms $\Rightarrow$ gap
- MHD allows for global modes in the gap

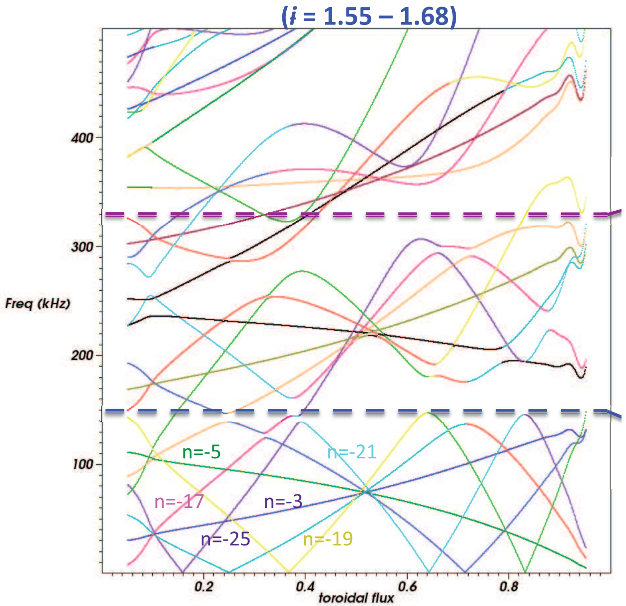


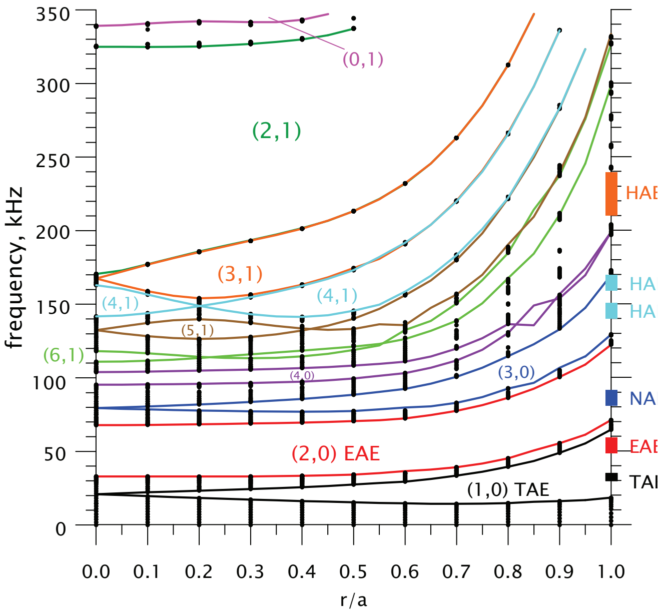
Possible approaches to wave-particle interaction

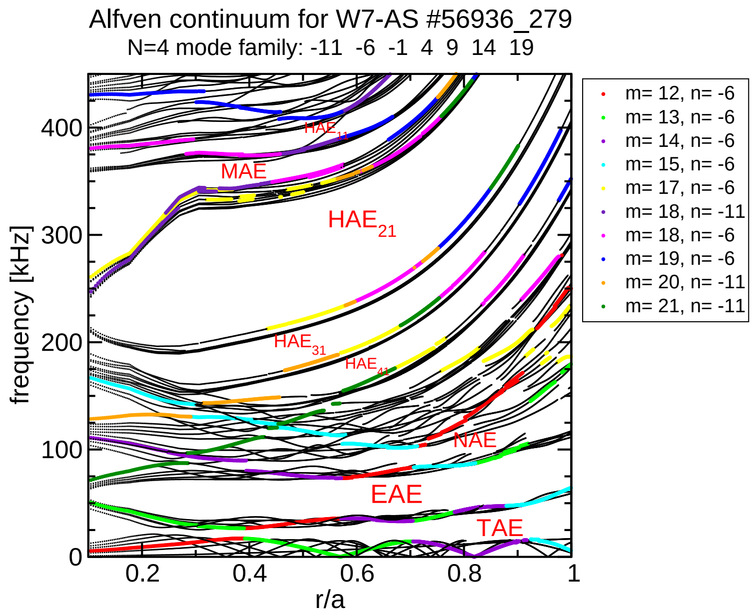
- **Linear ideal MHD**
- **Extended MHD (kinetic MHD, non-linear kinetic MHD)**
- **gyro-fluid models**
- **gyro-kinetic models**
  - **particle in cell codes**
  - **eigenvalue codes**
  - **turbulence Vlasov codes**
- **wave-equation with plasma response in dielectric function**

3D linear ideal MHD continuum

- STELLGAP (D. Spong)
- COBRAS (Kolesnichenko et al.)
- CONTI (A. Könies)







Linear ideal MHD

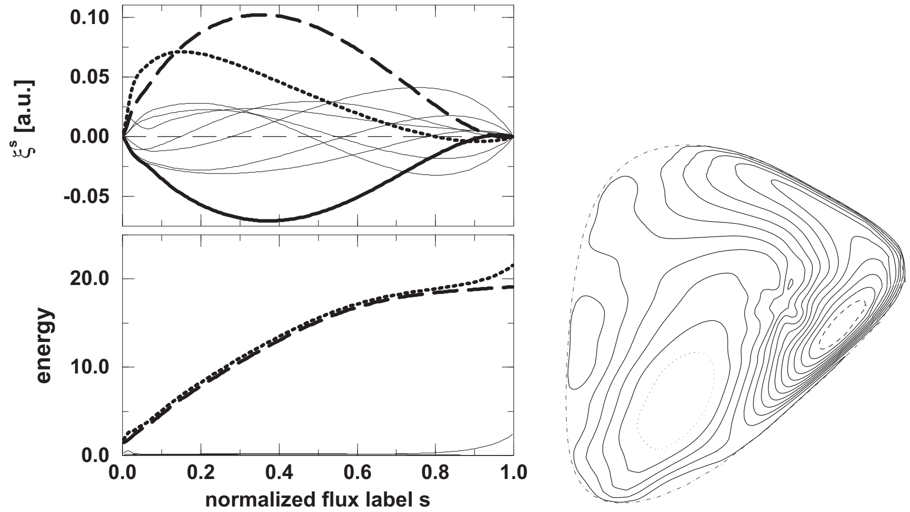
3D full MHD stability/ Alfvén and sound waves:

- **CAS3D (C. Nührenberg)**
- **TERPSICORE (W. A. Cooper et al.)**

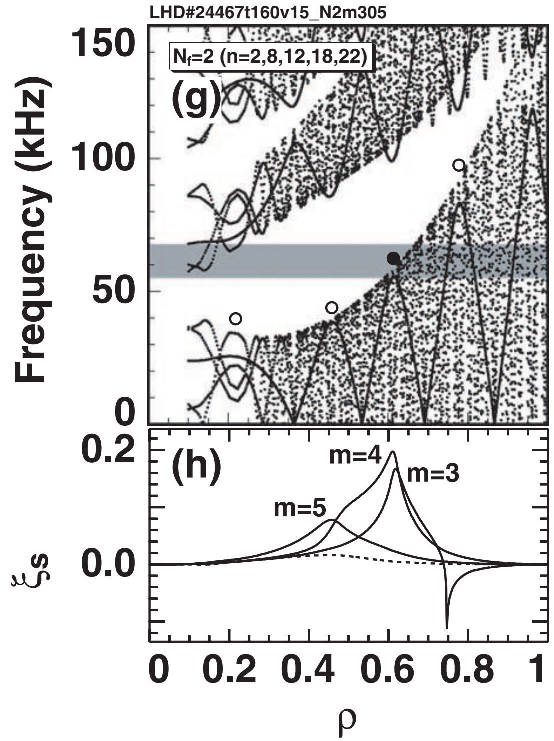
3D reduced MHD:

- **AE3D (D. Spong)**
- **CKA (A. Könies and T. Feher)**
- **BOA (Kolesnichenko et al.)**

Results from CAS3D for W7-AS and LHD



CAS3D3 results for an $N = 1$ mode family global mode for which $\nu_{\mathrm{CAS3D}} = 3.9$ MHz.



see C. Nührenberg, NF (1999), Yamamoto et al. NF (2005)

many applications for other stellarator configurations

What is non-ideal MHD? What is kinetic MHD?

example: reduced MHD equation

$$\vec{\nabla} \cdot \vec{\nabla}_\perp \left((\frac{3}{4}\rho_i + \rho_s)\frac{\omega^2}{v_A^2}\vec{\nabla} \cdot \vec{\nabla}_\perp \phi \right) + \vec{\nabla} \left(\frac{\omega^2}{v_A^2} \vec{\nabla}_\perp \phi \right) = -\vec{\nabla} \cdot \left(\vec{b}\vec{\nabla}^2 (\vec{b} \cdot \vec{\nabla})\phi \right)$$

$$-\vec{\nabla} \cdot \left(\vec{b}\vec{\nabla} \cdot \left(\frac{\mu_0 j_{||}}{B}\vec{b} \times \vec{\nabla}\phi \right) \right) + \nabla \left(\frac{\mu_0 i\omega p_\perp^{(1)}}{B^2}\vec{b} \times \vec{\nabla} B \right) + \nabla \left(\frac{\mu_0 i\omega p_{||}^{(1)}}{B}\vec{b} \times \vec{\kappa} \right)$$

fast particles can be coupled to the equation via the pressure or current pertubation

$$p_{||}^{(1)} = p_{||\mathrm{bulk}}^{(1)} + \int d^3v \, M v_{||}^2 f^{(1)} \qquad p_\perp^{(1)} = p_{\perp\mathrm{bulk}}^{(1)} + \int d^3v \, \mu B f^{(1)}$$

Kinetic MHD

1. **restriction to MHD-like perturbations**
   $\phi = 0$ **no electrostatic potential**
   $\vec{A}^{(1)} = \vec{\xi} \times \vec{B}$
   $\vec{B}^{(1)} = \vec{\nabla} \times \left(\vec{\xi} \times \vec{B}\right)$

2. **derivation of an energy functional from the MHD moment equation**

$$\vec{\nabla} \cdot \overleftrightarrow{P} = -\vec{B} \times \left(\vec{\nabla} \times \vec{B}\right)$$

3. **replace $\overleftrightarrow{P}$ with a kinetic expression, i.e. an expression involving integrals of the distribution function**

remark: this is equivalent to calculate growth/ damping rates considering the particle-wave energy transfer

Drift kinetic equation

Vlasov equation after transformation to guiding center variables and averaging over the gyro phase:
(e.g. Porcelli et al. 1994, Catto et al. 1980, Littlejohn 1983, cf. Hahm 1988)

$$\frac{\partial f}{\partial t} + \dot{\vec{R}} \cdot \frac{\partial f}{\partial \vec{R}} + \dot{v}_{\|} \frac{\partial f}{\partial v_{\|}} + \dot{y} \frac{\partial f}{\partial y} = 0$$

$y = \mu B$: perpendicular energy, $v_{\|}$: parallel velocity $\| \vec{B}$
$\vec{R}$: location of the guiding center

distribution function $f(\vec{R}, y, v_{\|}, t)$: distribution of guiding centers

correct up to first order in

$$\delta = \frac{\text{gyro radius}}{\text{system length}} \ll 1$$

Drift kinetic equation

Linearization: $f = F + f^{(1)}$

(equilibrium part: F + perturbation: $f^{(1)}$)

zero order:

$$\dot{\vec{R}}^{(0)} \cdot \left(\frac{\partial F(\epsilon, \mu, \vec{R})}{\partial \vec{R}} \right)_{\epsilon,\mu} = 0$$

- **regard this equation as being approximatively solved**
- **for times scales with negligible drifts:**

$$F = F(s, \epsilon, \mu, \sigma)$$

s**: flux label;** σ**: sign of** $v_{||}$

Drift kinetic equation

linearized 3D drift kinetic equation to first order:

- $f^{(1)}$ **splits into an adiabatic ...**

$$f^{(1)} = \vec{A}^{(1)} \cdot \frac{\vec{b} \times \vec{\nabla} F}{B} + Ze\phi^{(1)} \left(\frac{\partial F}{\partial \epsilon}\right)_{\vec{R},\mu} - \frac{B^{(1)}}{B} \left(\frac{\partial F}{\partial \mu}\right)_{\vec{R},\mu} + h^{(1)}$$

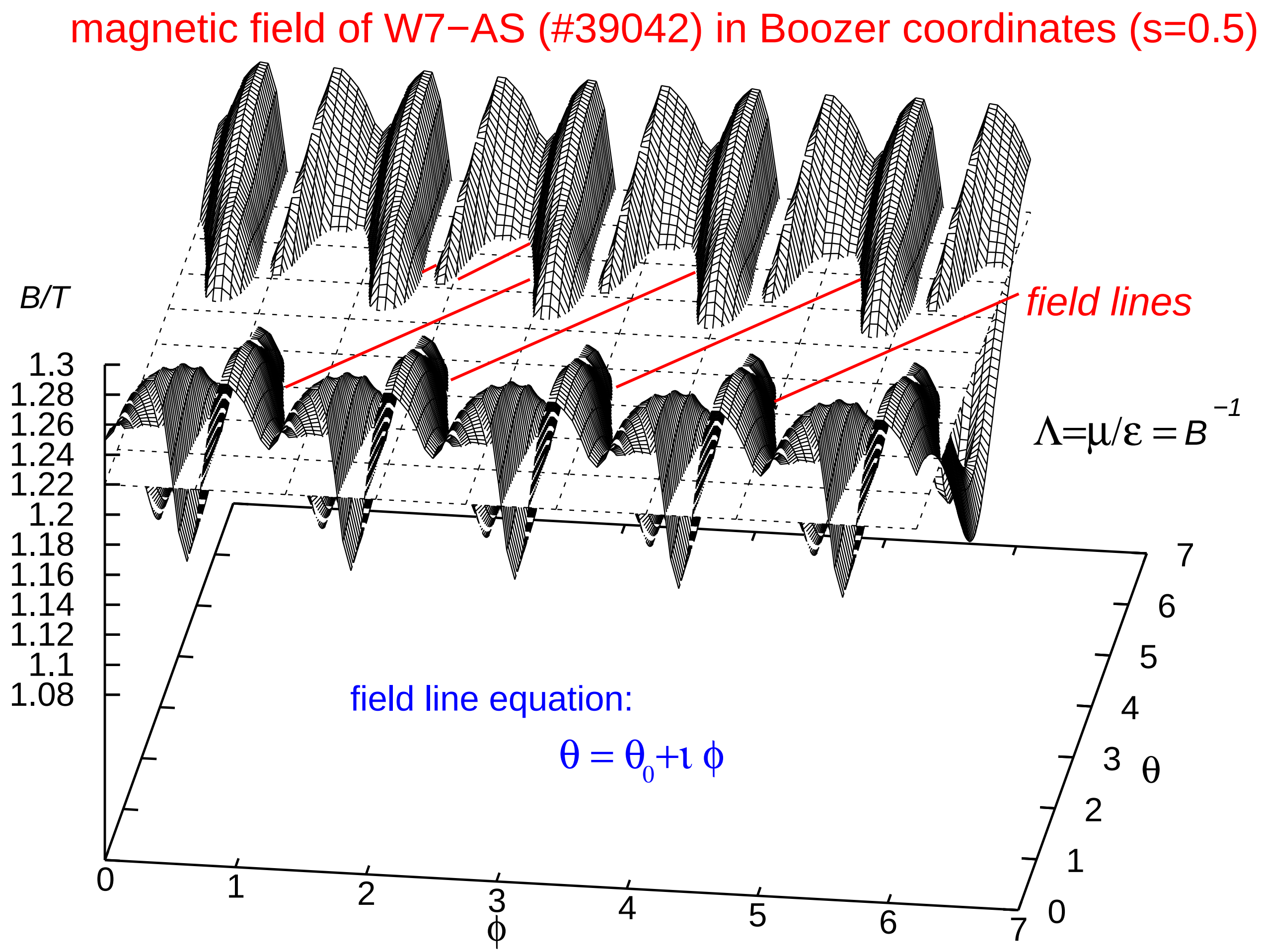
- **... and non-adiabatic part:**

$$\frac{d}{dt} h^{(1)} = \left[\left(\frac{\partial F}{\partial \vec{R}}\right) \cdot \frac{\vec{b} \times \vec{\nabla}}{M\Omega} + \left(\frac{\partial F}{\partial \epsilon}\right) \frac{\partial}{\partial t} \right] L^{(1)}$$

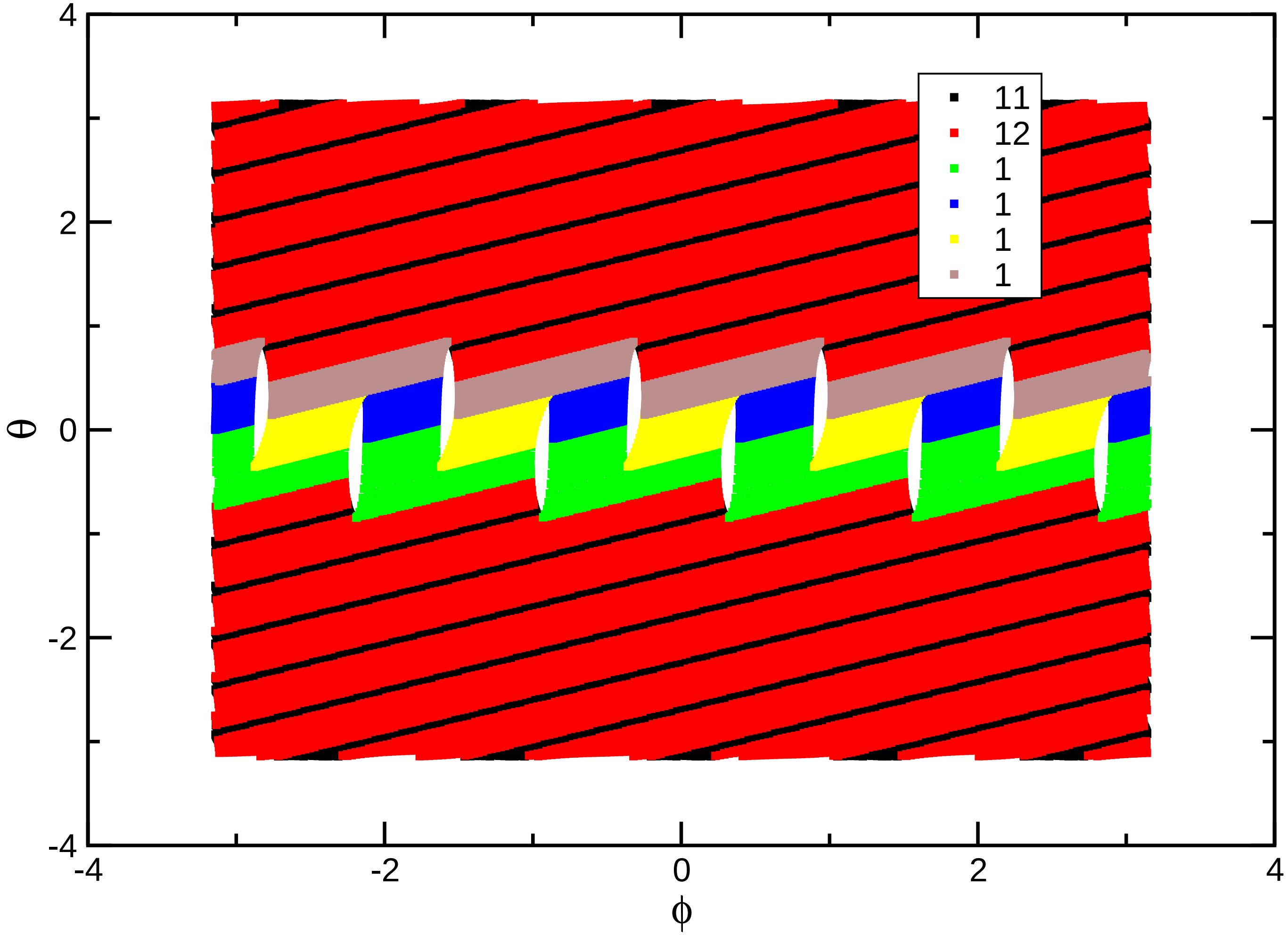
$L^{(1)}$**: perturbed Lagrangian** $L^{(1)} = [\dot{\vec{R}} \cdot \vec{A}^{*}]^{(1)} - \mu B^{(1)} - Ze\phi^{(1)}$

integration along field lines
bounce averaged drifts within flux surface considered
no radial drifts

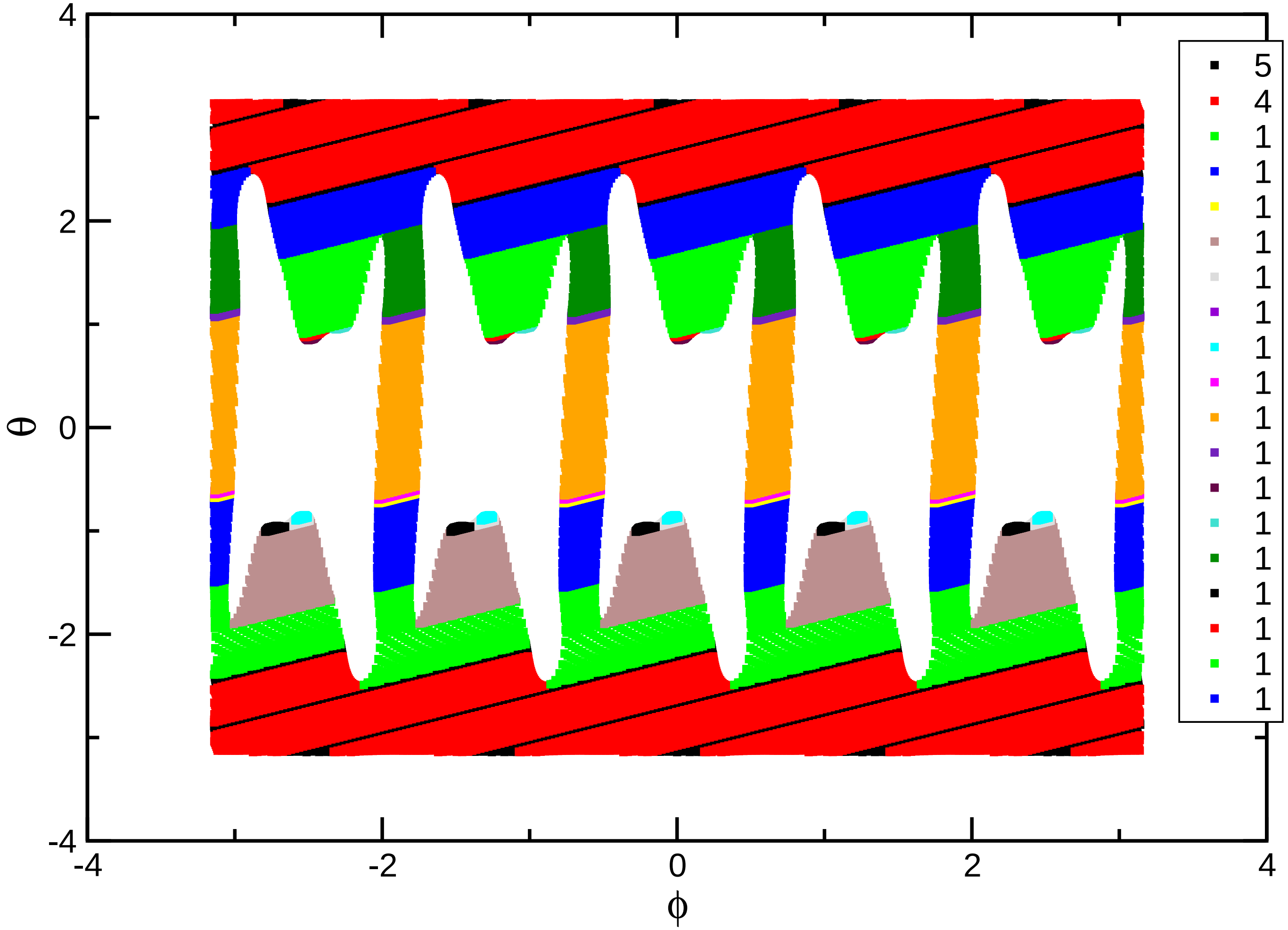
field line orbits (W7-AS)



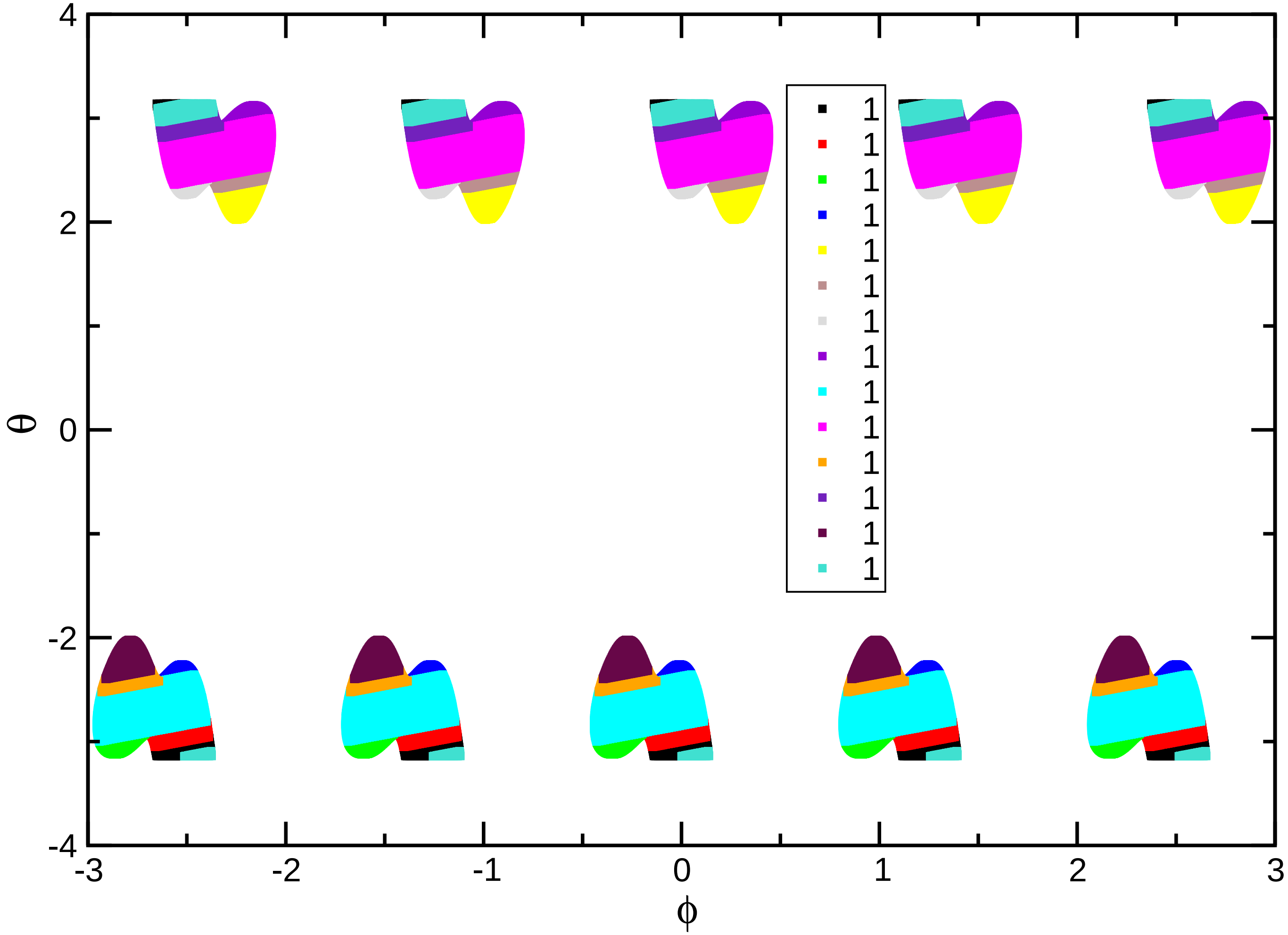
field line orbits (W7-AS)



field line orbits (W7-AS)



field line orbits (W7-AS)



Kinetic energy integral

there is an energy integral considering kinetic effects

$$\delta W_{\mathrm{kin}} = \omega^2 \frac{1}{2} \int d^3\vec{x} \left|\vec{\xi}_\perp\right|^2 \rho_M = \delta W_{\mathrm{mag}} + \sum_{s=\mathrm{i,e,fast}} \delta W_s(\omega)$$

(Kruskal/Oberman 1958 ... Antonsen/Lee 1984)

$$\delta W_{\mathrm{mag}} = \frac{1}{2} \int d^3\vec{x} \left\{ \left|B_\perp^{(1)}\right|^2 + \left|B_\parallel^{(1)}\right|^2 + \vec{j}_\parallel \cdot \left(\vec{\xi}_\perp \times \vec{B}_\perp^{(1)}\right) - \frac{B_\parallel^{(1)}}{B} \vec{\xi}_\perp^* \cdot \vec{\nabla} p + \left(\vec{\nabla} \cdot \vec{\xi}_\perp^*\right) \left(\vec{\xi}_\perp \cdot \vec{\nabla} p\right) \right\}$$

the non-adiabatic contributions from the hot and thermal component replace the MHD fluid compression term
the contributions from the thermal plasma ($\delta W_{\mathrm{i,e}}$) and the fast particles δW_{fast}) depend on the perturbed particle Lagrangian $L^{(1)}$

(A. Könies, PoP 2000)

Kinetic contribution

particle- wave- energy- exchange by resonant interaction

$$\delta W_s = \frac{\pi}{M_s^2} \left\{ \sum_\sigma \right\} \int ds \int d\varphi \int d\mu \, d\epsilon \left(- \int \frac{d\vartheta}{|v_{||}|} \sqrt{g} B \right) \sum_{\substack{n,m \\ n',m'}} \sum_{p=-\infty}^{\infty} e^{-i \frac{2\pi}{N_p}(n'-n)\varphi} \times$$

$$\times \left(\frac{\partial F_s}{\partial \epsilon} \right)_\mu \frac{\omega - 2\pi(\frac{n}{N_p} J - mI)\omega^*}{m \langle \omega_d^\vartheta \rangle + \frac{n}{N_p} \langle \omega_d^\varphi \rangle + \left\{ \begin{matrix} \sigma(p+nq) \\ p \end{matrix} \right\} \omega_{\left\{ \substack{t \\ b} \right\}} - \omega} L_{m'n'}^{(1)*} \mathcal{M}_{pn}^{m'n'*} L_{mn}^{(1)} \mathcal{M}_{pn}^{mn}$$

definition of $\mathcal{M}_{pn}^{m'n'}$:

for passing particles:

$$\mathcal{M}_{pn}^{m'n'} = \left\langle e^{i[2\pi(m'+n'q)\vartheta'' - (p+nq)\omega_t t'']} \right\rangle_{\vartheta''}$$

for reflected particles:

$$\mathcal{M}_{pn}^{m'n'} = \left\langle e^{2\pi i(m'+n'q)\vartheta''} \cos(p\omega_b t'') \right\rangle_{\vartheta''}$$

$\langle \ldots \rangle$ denotes the transit or bounce average

perturbed particle Lagrangian:

$$L^{(1)} = -(Mv_{||}^2 - \mu B)\vec{\xi}_\perp \cdot \vec{\kappa} + \mu B \vec{\nabla} \cdot \vec{\xi}_\perp$$

Realization of kinetic MHD in CAS3D-K

CAS3D-K: perturbative stability code based on a hybrid MHD-drift kinetic model

- **3-dimensional**
- **general mode structure and equilibrium**
- **particle drifts are approximated as bounce averaged drifts**
- **zero radial orbit width**
- **perturbative growth/damping rates from:**

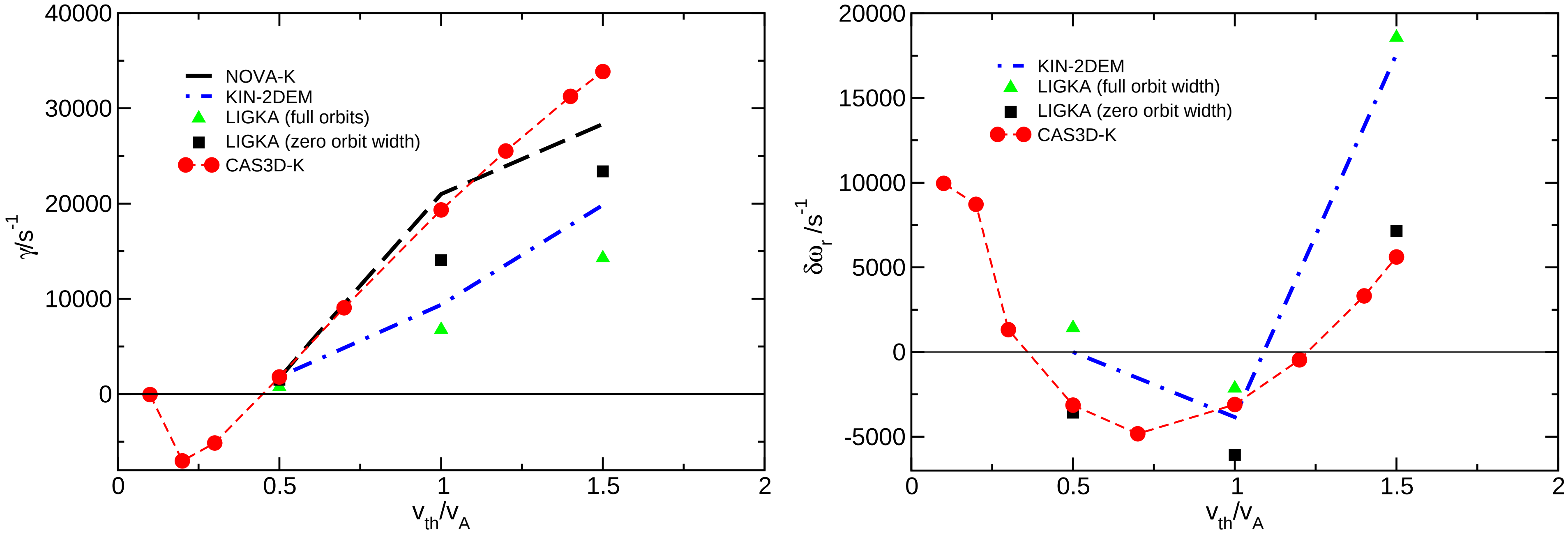
$$\Delta\omega_s + i\gamma_s \approx \frac{1}{2}\frac{\delta W_\mathrm{s}(\omega_0)}{\delta W_\mathrm{mag}}\omega_0$$

using the MHD eigenfunctions and the MHD frequency ω_0

- δW_mag **from the ideal MHD stability code $CAS3D$ (C. Nührenberg, 1996, 1998, 2000, . . .)**

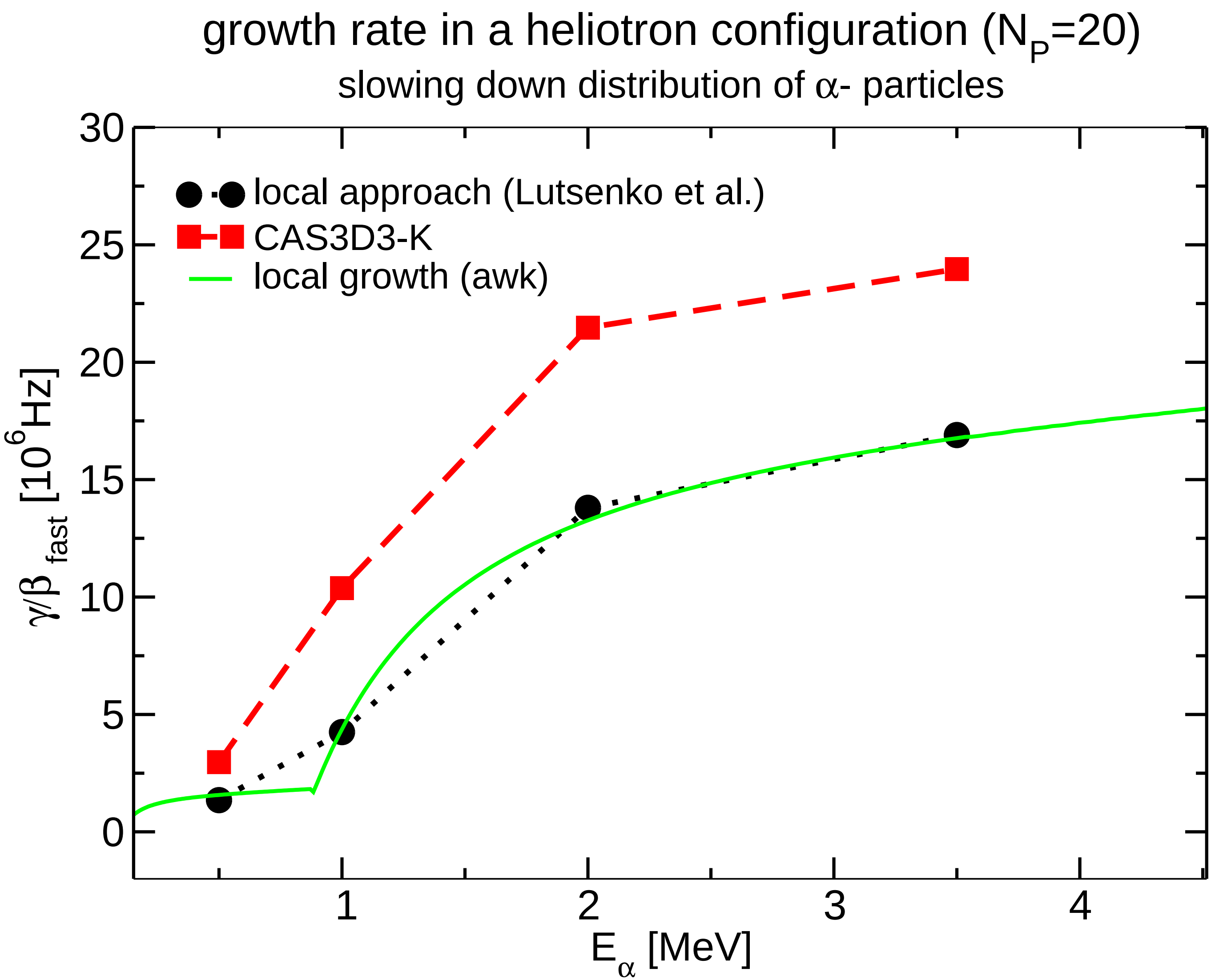
(3,-2)/(2,-2) TAE Benchmark with LIGKA

circular tokamak $A = 4$, Maxwellian distribution of fast hydrogen ions



LIGKA gyro-kinetic eigenvalue code (Ph. Lauber et al., J Comp. Phys. 2007)

3D benchmark with analytical theory



3D analytical theory - What can we learn?

(see Kolesnichenko et al. 2002)

- **proportionality to equilibrium quantities**

$$\frac{\gamma}{\omega_0} \propto A^2 \sum_{m'n'} |\epsilon^{\kappa}_{m'n'}|^2 \approx A^2 \sum_{m'n'} |\epsilon^{B}_{m'n'}|^2$$

- **coupling is approximately given by the structure of B**
  $\Rightarrow$ **investigate spectrum of B**

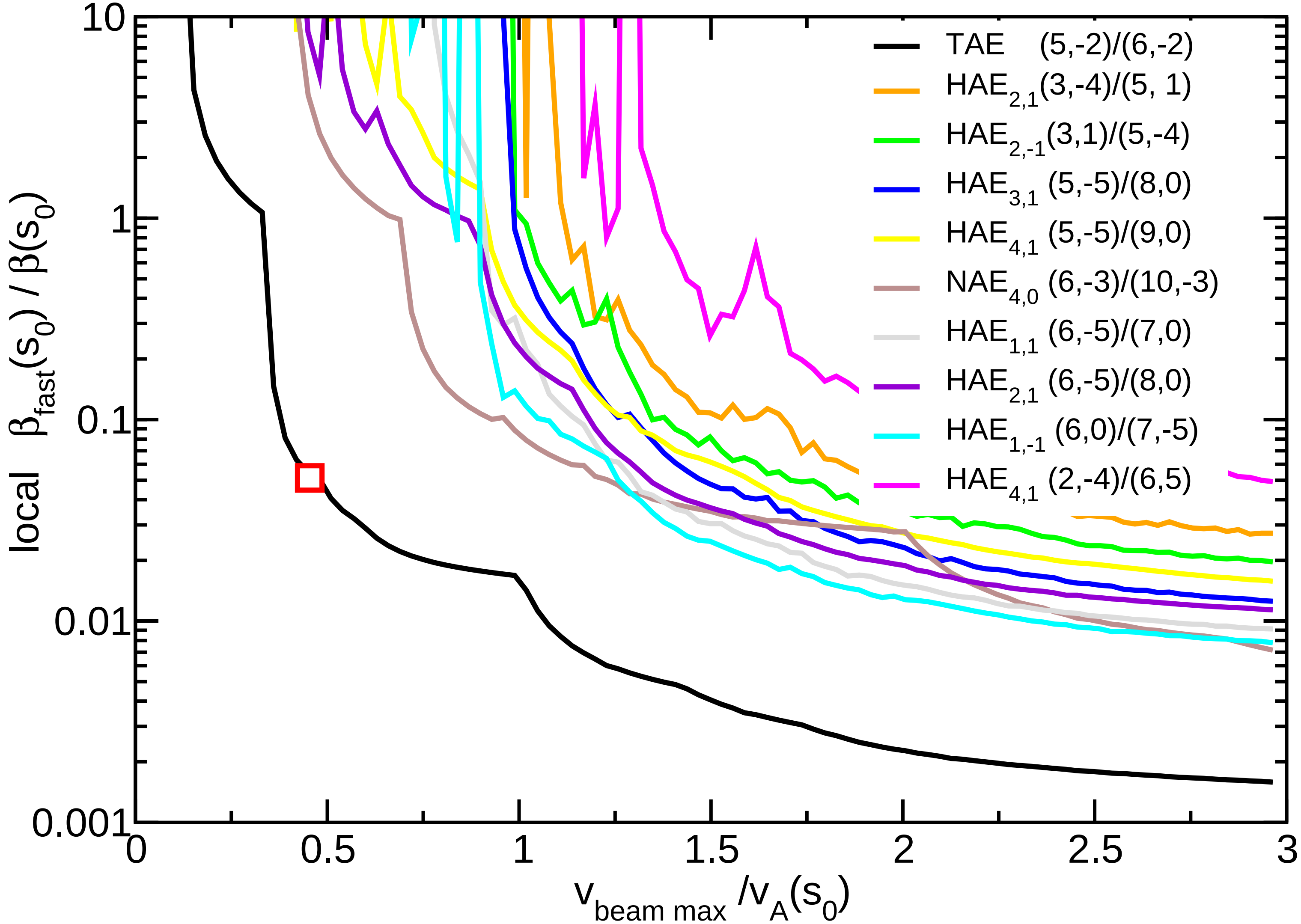
- **note, that for a TAE in a large aspect ratio tokamak: $\frac{\gamma}{\omega_0}$ is independent of the equilibrium**

- **the resonance condition $\omega - k_{||}v_{th} = 0$ determines**

$$v^{\mathrm{res}}_{m'n'} = v_A \left| 1 \pm \frac{m'\iota^* + n'N_p}{m\iota^* + n} \right|^{-1}$$

i.e. well known resonances at $v_0 = v_A$ and $v_0 = v_A/3$ for a Tokamak TAE

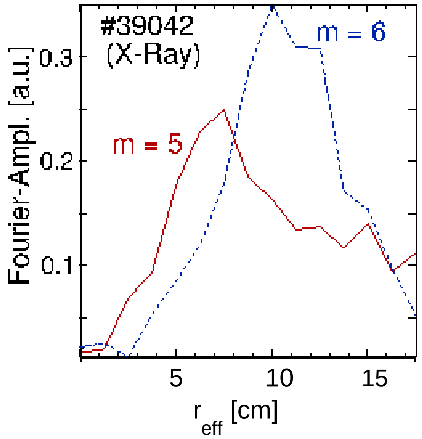
W7-AS: most unstable mode at given m (LGRO)

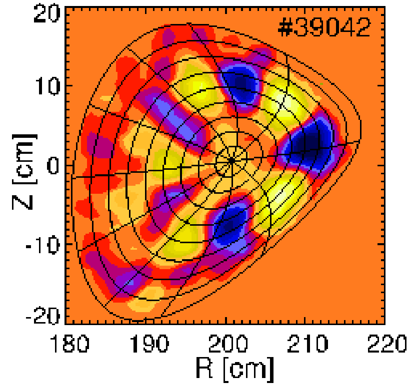


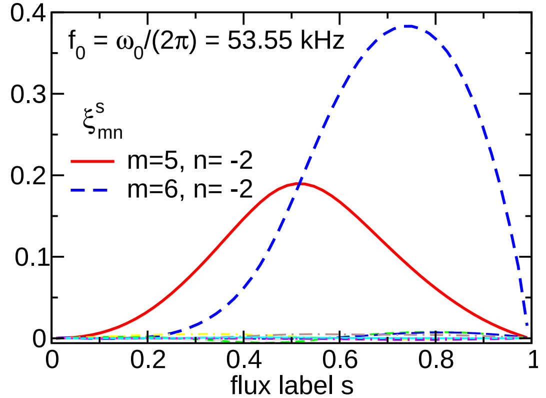
TAEs in W7-AS (#39042) and W7-X

W7-AS

A. Weller et al., Phys. Plasmas, **8**, 931 (2001):





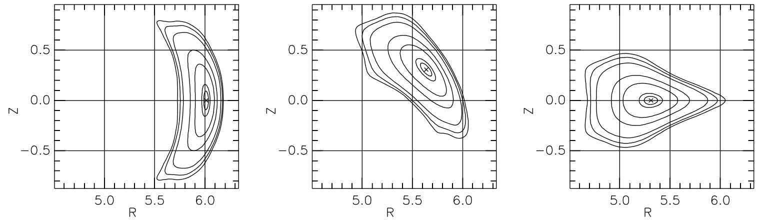


W7-X

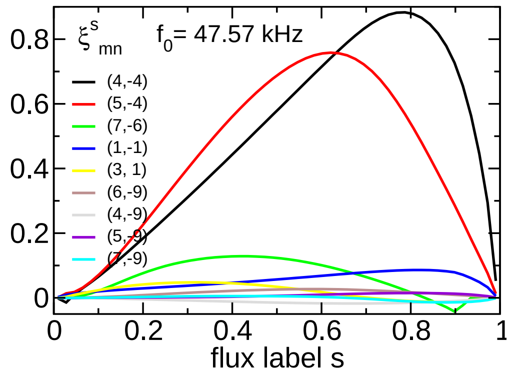
equilibrium:

M. Drevlak et al., Nucl. Fusion, 45, 731 (2005):

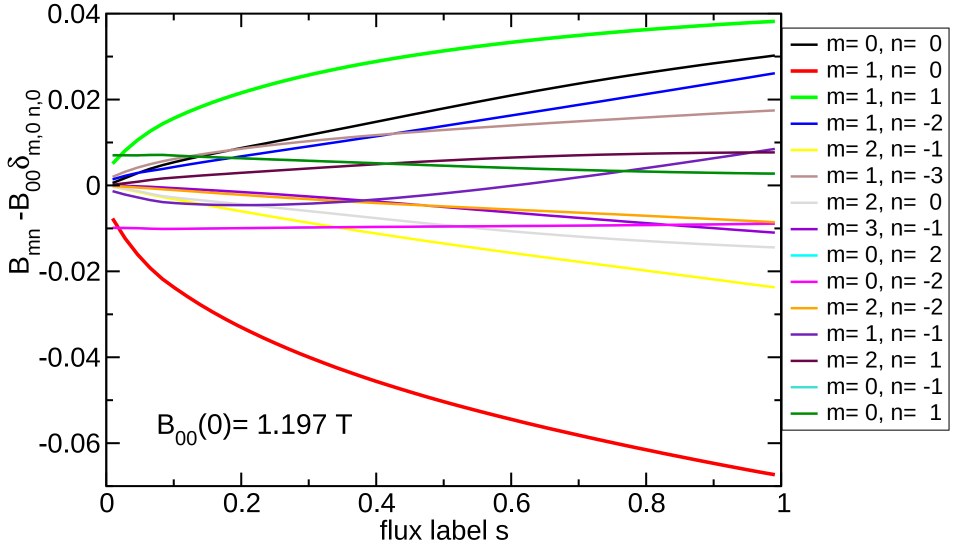
from PIES calculation: practically island free

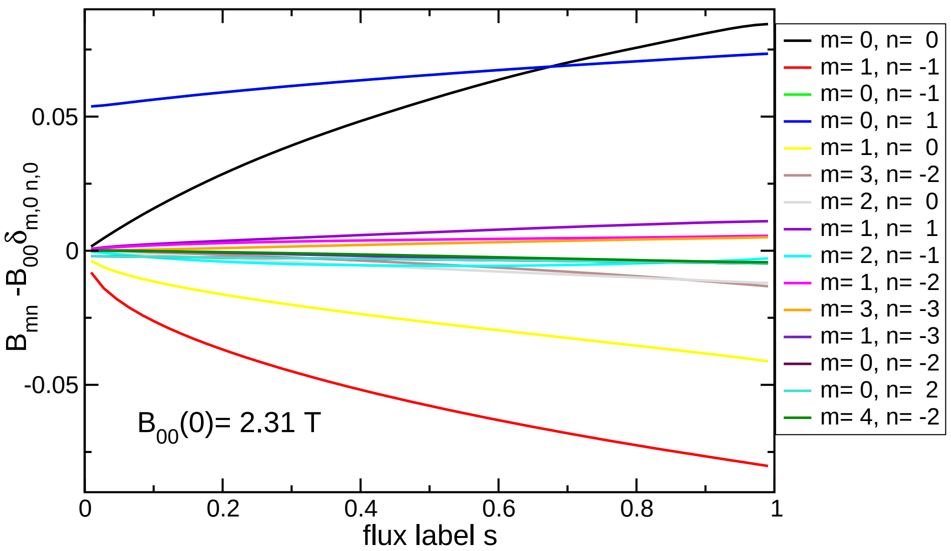


mode:



extract possible coupling from B spectrum



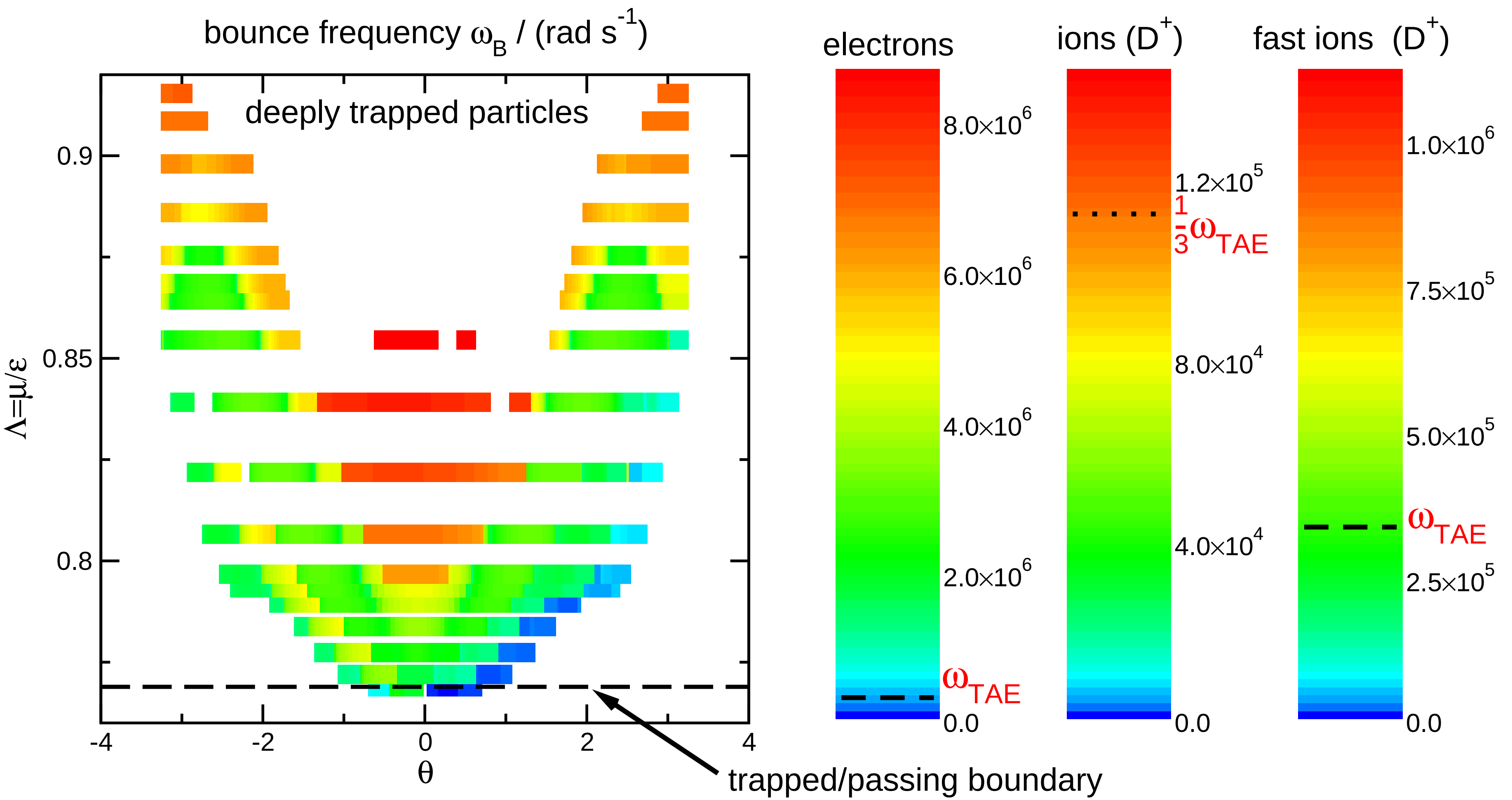


W7-AS **W7-X**

for both discharges:

NBI injection with a slowing down distribution

W7-AS: influence of reflected particles



reminder: Kinetic contribution

particle- wave- energy- exchange by resonant interaction

$$\delta W_s = \frac{\pi}{M_s^2}\left\{\sum_\sigma\right\}\int ds\int d\varphi\int d\mu\, d\epsilon\left(-\int\frac{d\vartheta}{|v_{||}|}\sqrt{g}B\right)\sum_{\substack{n,m\\n',m'}}\sum_{p=-\infty}^{\infty}e^{-i\frac{2\pi}{N_p}(n'-n)\varphi}\times$$

$$\times\left(\frac{\partial F_s}{\partial\epsilon}\right)_\mu\frac{\omega-2\pi(\frac{n}{N_p}J-mI)\omega^*}{m\left\langle\omega_d^\vartheta\right\rangle+\frac{n}{N_p}\left\langle\omega_d^\varphi\right\rangle+\left\{\begin{matrix}\sigma(p+nq)\\p\end{matrix}\right\}\omega_{\left\{\begin{smallmatrix}t\\b\end{smallmatrix}\right\}}-\omega}L^{(1)*}_{m'n'}\mathcal{M}^{m'n'*}_{pn}L^{(1)}_{mn}\mathcal{M}^{mn}_{pn}$$

definition of $\mathcal{M}^{m'n'}_{pn}$:

for passing particles:

$$\mathcal{M}^{m'n'}_{pn}=\left\langle e^{i[2\pi(m'+n'q)\vartheta''-(p+nq)\omega_t t'']}\right\rangle_{\vartheta''}$$

for reflected particles:

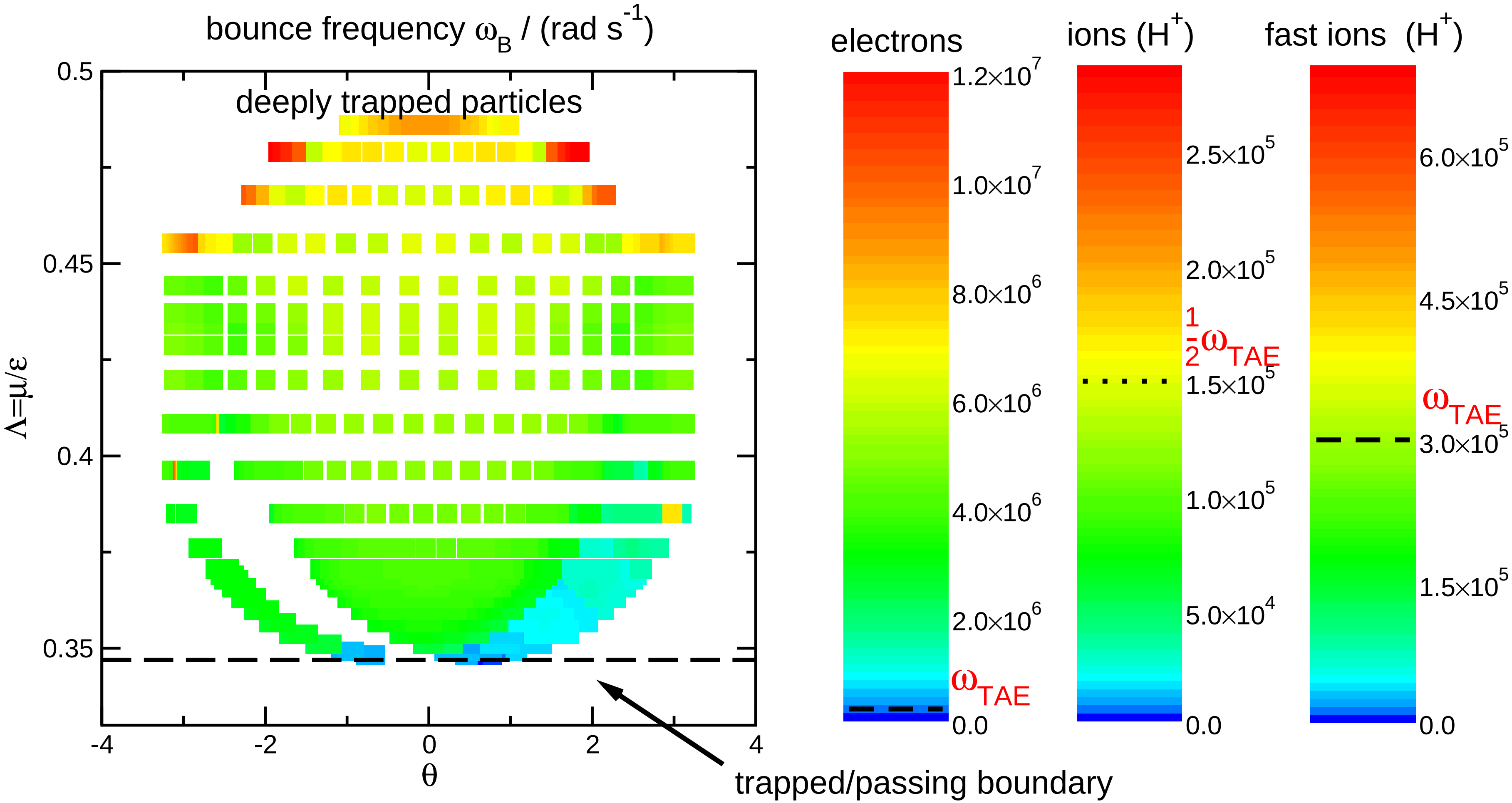
$$\mathcal{M}^{m'n'}_{pn}=\left\langle e^{2\pi i(m'+n'q)\vartheta''}\cos(p\omega_b t'')\right\rangle_{\vartheta''}$$

$\langle\ldots\rangle$ **denotes the transit or bounce average**

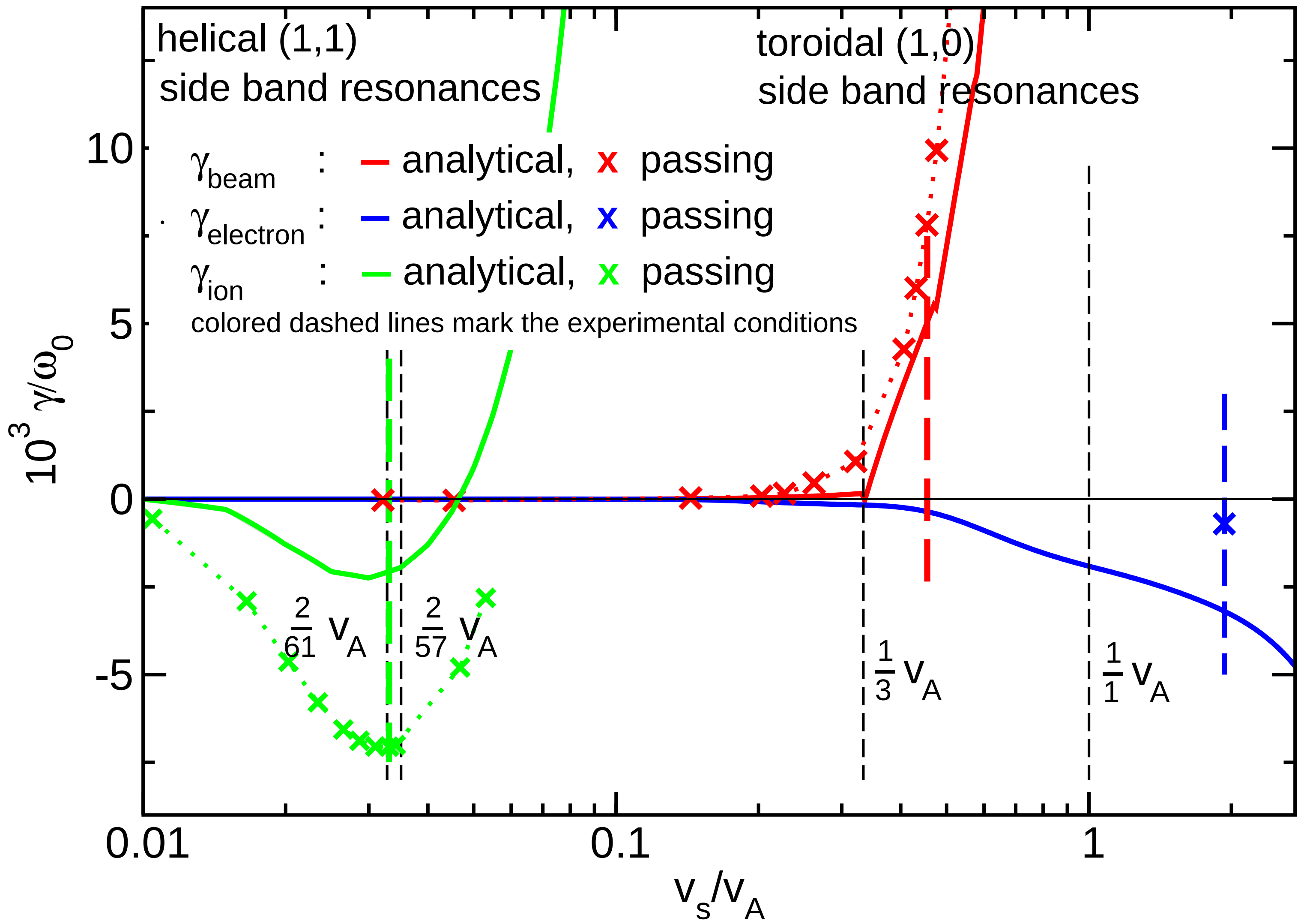
perturbed particle Lagrangian:

$$L^{(1)}=-(Mv_{||}^2-\mu B)\vec{\xi}_\perp\cdot\vec{\kappa}+\mu B\vec{\nabla}\cdot\vec{\xi}_\perp$$

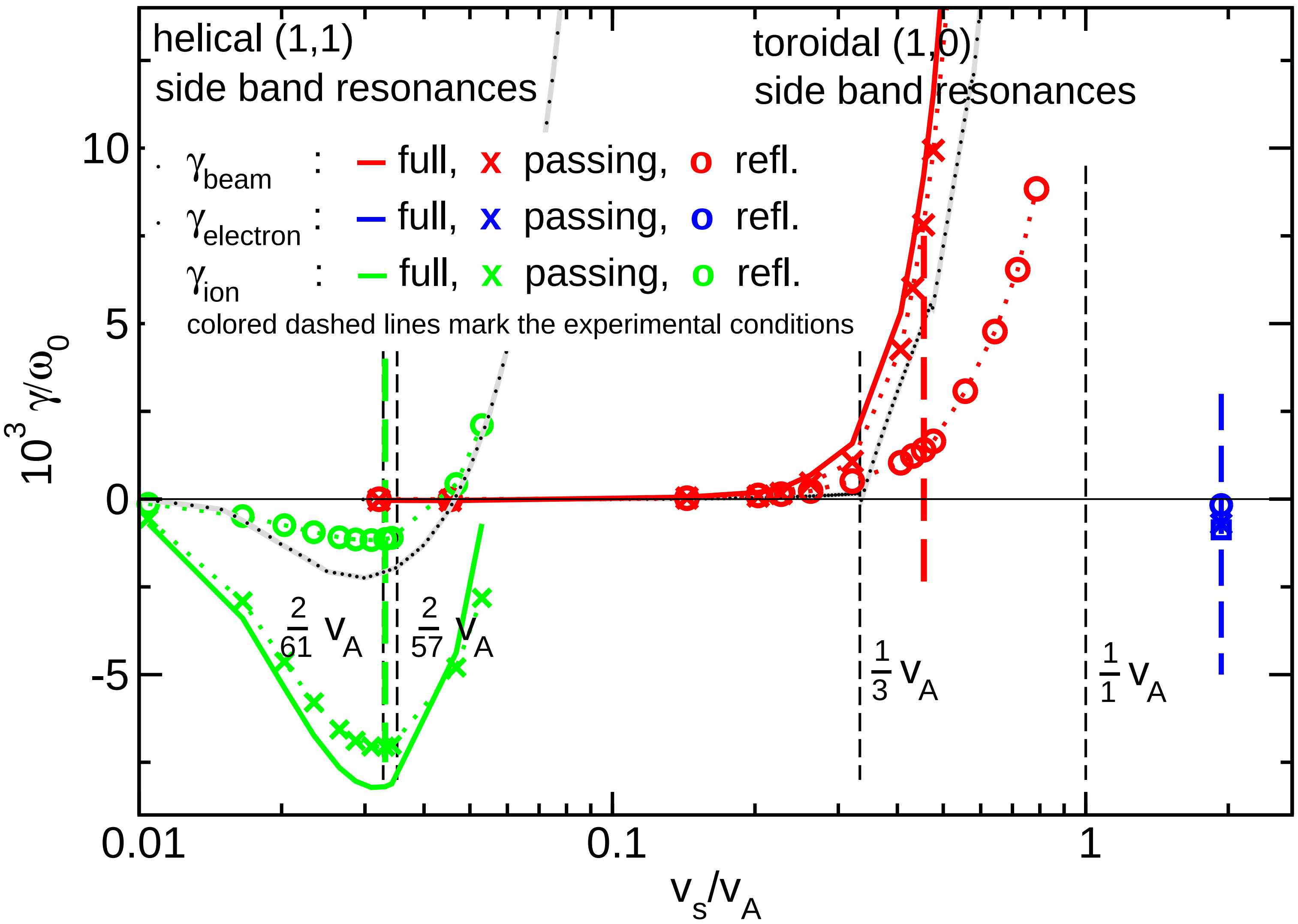
W7-X: influence of reflected particles



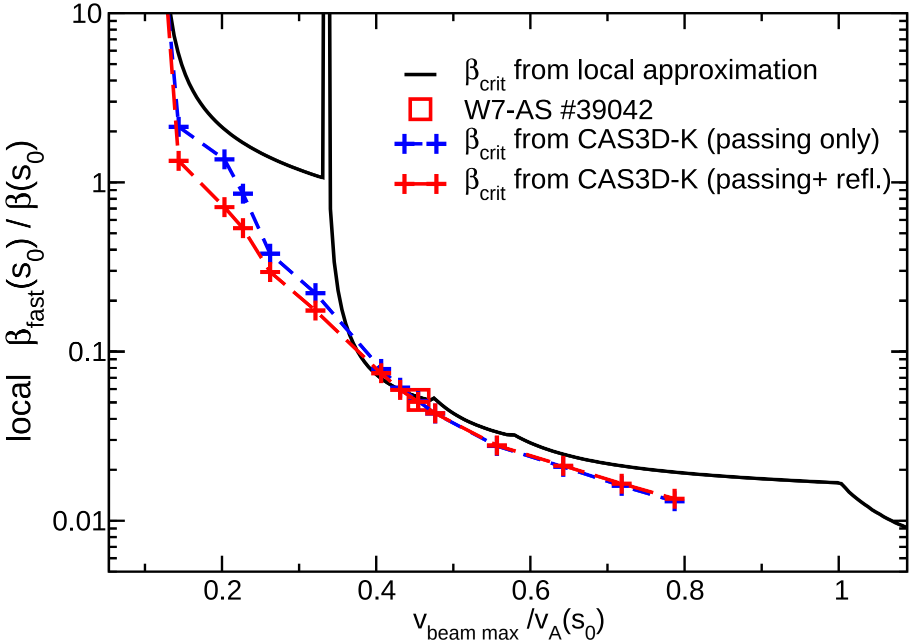
growth and damping rates for TAE in #39042



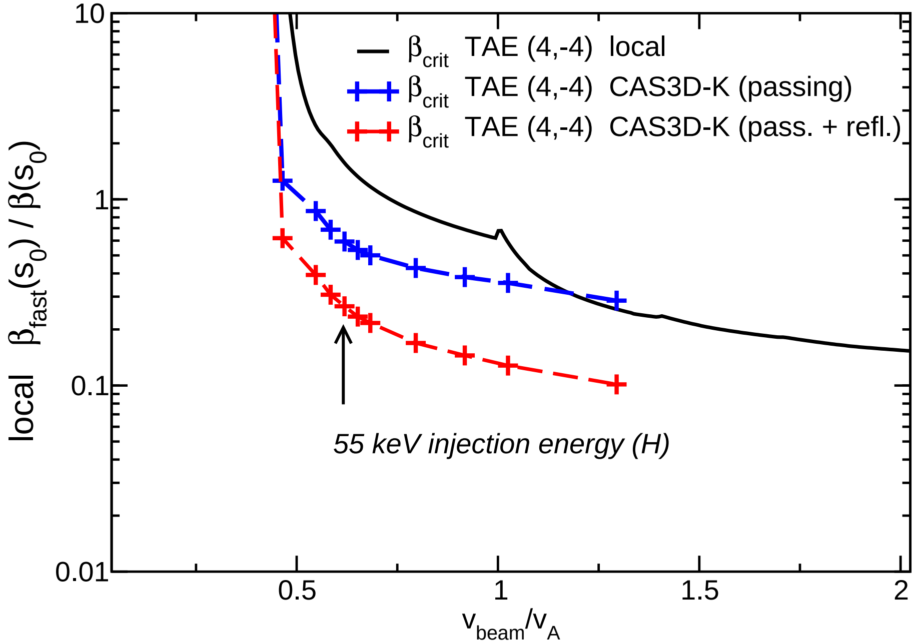
W7-AS: influence of reflected particles



stability diagrams/ critical β

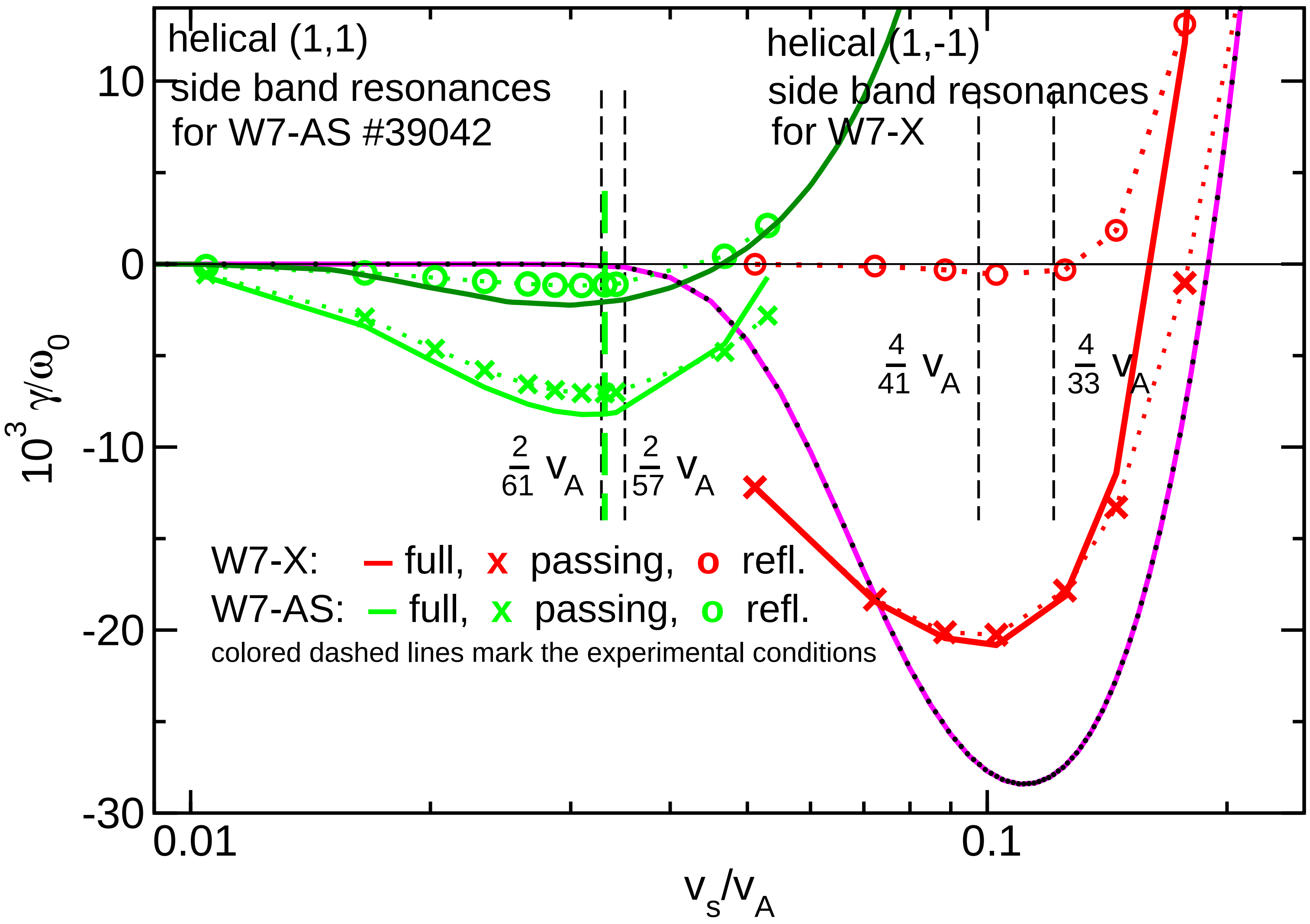


(5,-2),(6,-2) TAE in W7-AS



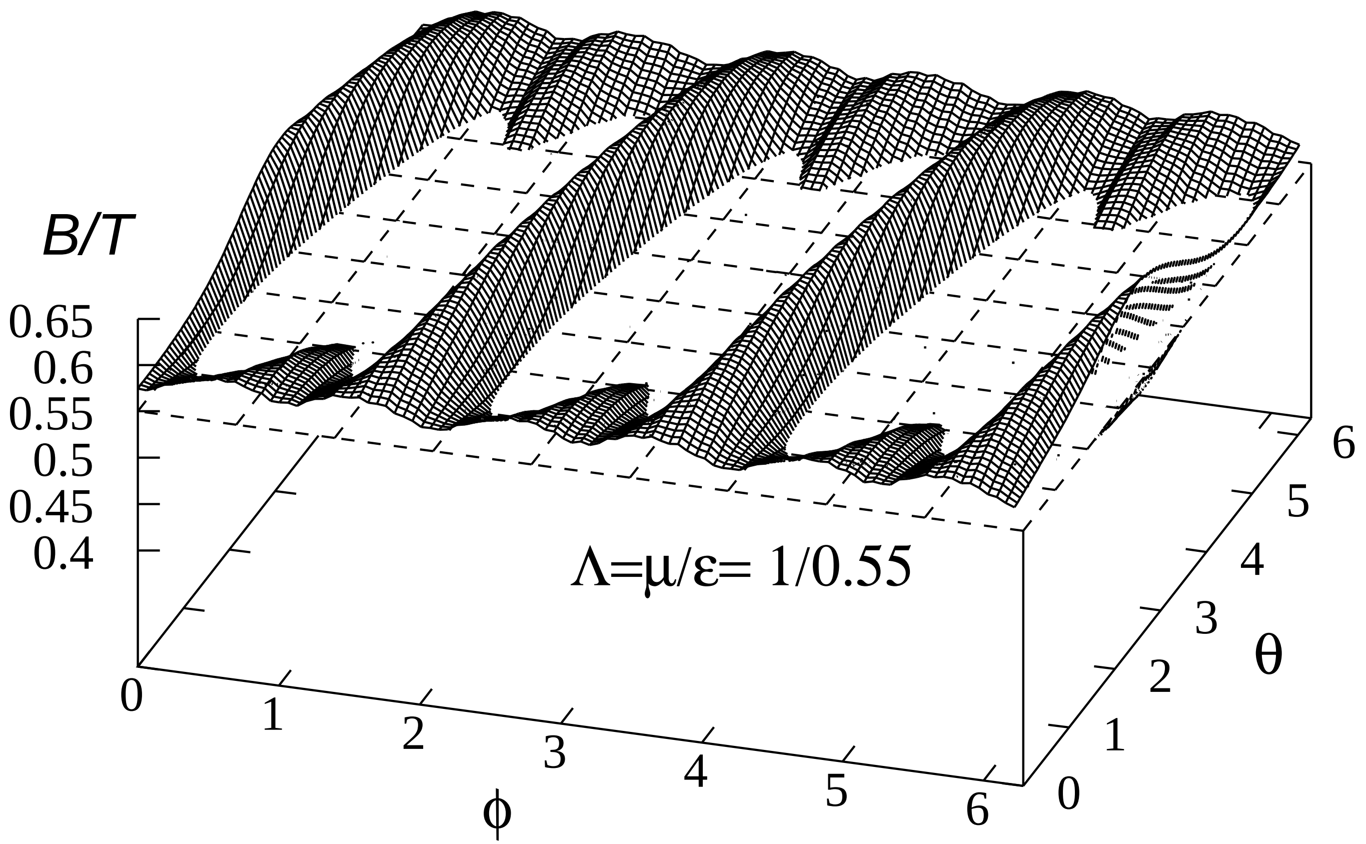
(4,-4),(5,-4) TAE in W7-X

damping by thermal ions

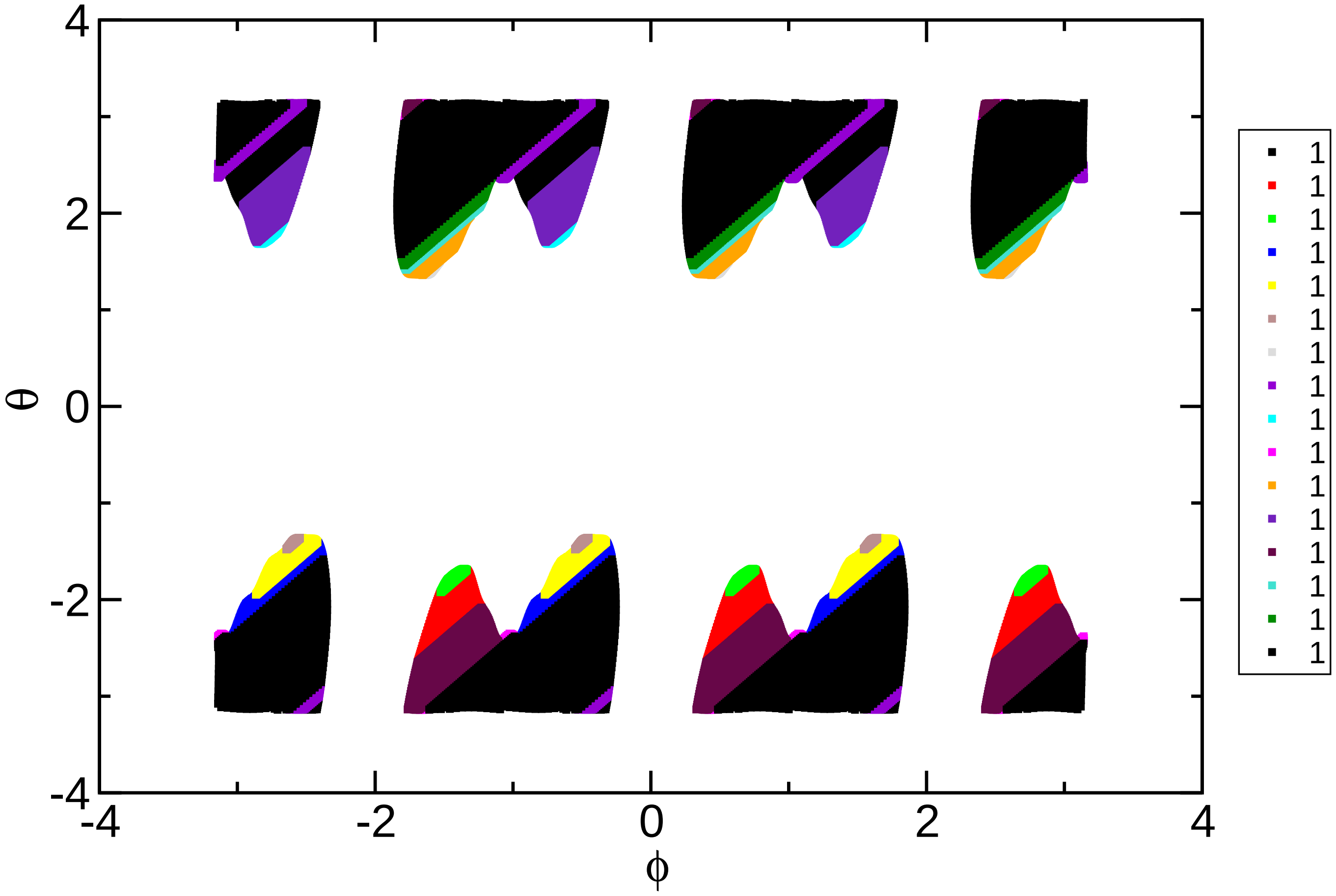


B field in H1

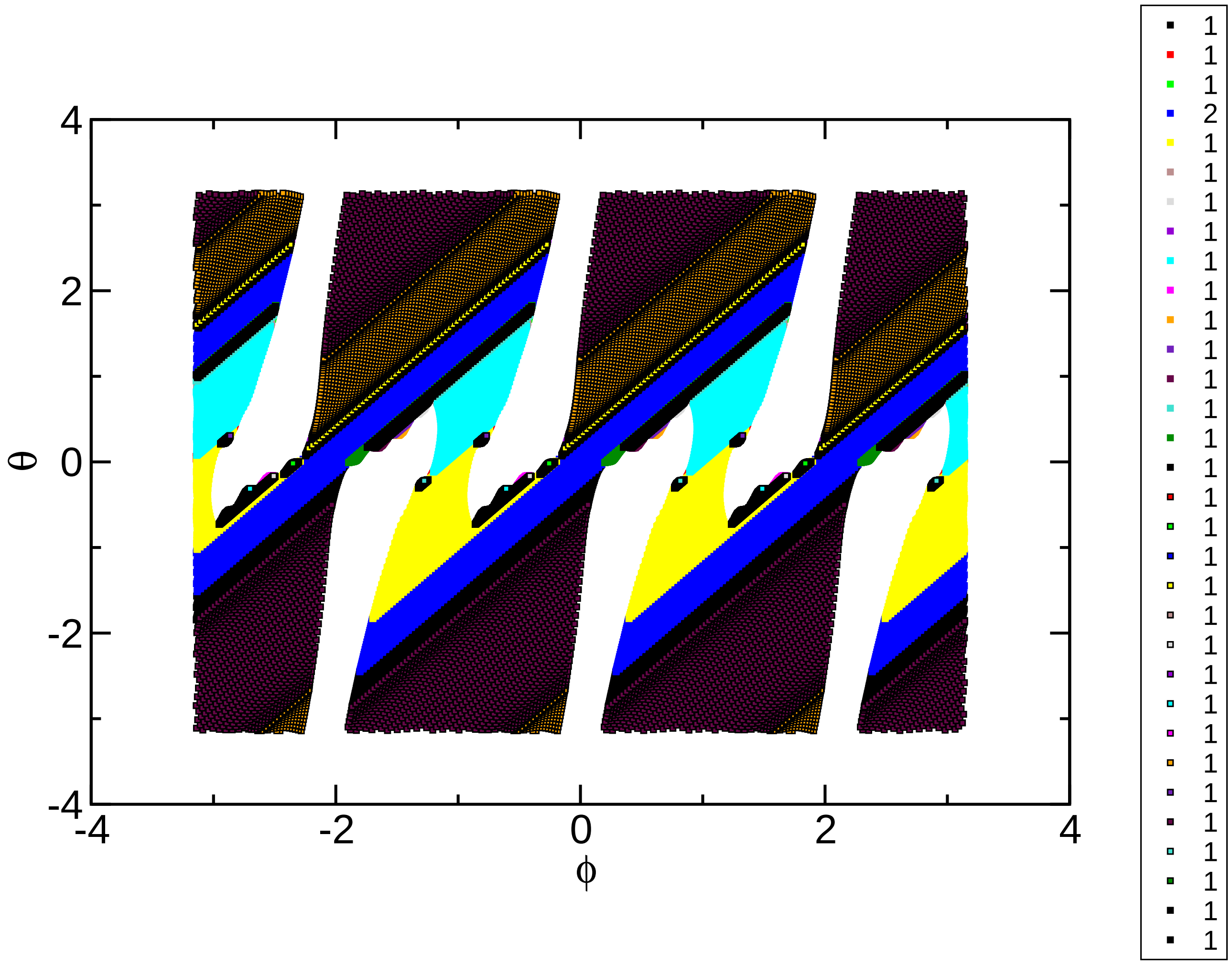
modulus of B for H1 with $\kappa = 0.03$



field line orbits in H1



field line orbits in H1



Linear kinetic MHD with PIC

- calculate Alfvén Eigenmode with ideal MHD
- move marker particles in the wave field and calculate wave-particle energy transfer

$$\frac{\partial \mathcal{E}_{kin}}{\partial t}$$

- growth rate is

$$\gamma = -\frac{1}{2}\frac{\frac{\partial \mathcal{E}_{kin}}{\partial t}}{\mathcal{E}_{field}}$$

code	developer	MHD	PIC	
VENUS-K	M. Isaev	CAS3D	VENUS	Kurtshatov+CRPP+IPP
AE3D-K	D. Spong	AE3D	PIC	ORNL+IPP
CKA-EUTERPE	T. Feher	CKA	EUTERPE	IPP

MHD description

- **Linearised reduced MHD equations**
- **Transformed to an eigenvalue problem:**

$$\omega^2\left[\nabla\cdot\left(\frac{1}{v_A^2}\nabla_\perp\Phi\right)+\frac{3}{4}\nabla\nabla_\perp\left(\rho_i^2\frac{1}{v_A^2}\nabla\cdot\nabla_\perp\Phi\right)\right]=-\nabla\cdot\left[\mathrm{b}\nabla^2(\mathrm{b}\nabla)\Phi\right]$$
$$-\nabla\cdot\left[\mathrm{b}\nabla\left(\mu_0\frac{j_{||}}{B}\mathrm{b}\times\nabla\Phi\right)\right]-\nabla\cdot\left[\frac{\mu_0\delta p_\perp}{B^2}\mathrm{b}\times\nabla B\right]-\nabla\cdot\left[\frac{\mu_0\delta p_{||}}{B^2}\mathrm{b}\times\kappa\right]$$

- **The CKA code is used to solve the MHD equations in 3D real magnetic geometry**
- **Determines the mode frequency ω and the mode structure $\Phi(r)$, $A_{||}(r)$**

$$E_{||}=-\nabla\Phi-\frac{\partial A_{||}}{\partial t}=0$$

Kinetic description

$$\frac{\partial \delta F}{\partial t} + \dot{\mathrm{R}}^{(0)} \frac{\partial \delta F}{\partial \mathrm{R}} + \dot{v_{\|}}^{(0)} \frac{\partial \delta F}{\partial v_{\|}} = -\dot{\mathrm{R}}^{(1)} \frac{\partial F_0}{\partial \mathrm{R}} - \dot{v_{\|}}^{(1)} \frac{\partial \delta F_0}{\partial v_{\|}} - \dot{v_{\perp}}^{(1)} \frac{\partial \delta F_0}{\partial v_{\perp}}$$

$$\frac{\mathrm{dR}}{\mathrm{d}t} = \frac{\mathrm{B}^*}{B_{\|}^*} \left(v_{\|} - \frac{q}{m} \langle A_{\|} \rangle \right) + \frac{1}{q B_{\|}^*} \mathrm{b} \times \left(\mu \nabla B + q \nabla \langle \Phi - v_{\|} A_{\|} \rangle \right)$$

$$\frac{\mathrm{d}v_{\|}}{\mathrm{d}t} = -\frac{1}{m} \frac{\mathrm{B}^*}{B_{\|}^*} (\mu \nabla B + q \nabla \langle \Phi - v_{\|} A_{\|} \rangle),$$

- **The EUTERPE code is used to integrate the equations**
- **PIC code, δf method**
- **Full particle orbits**
- **Currently the linearised version is used**
- **Possible to use multiple species**

completely gyro-kinetic PIC code

EUTERPE (R. Kleiber, R.Hatzky, et al.):

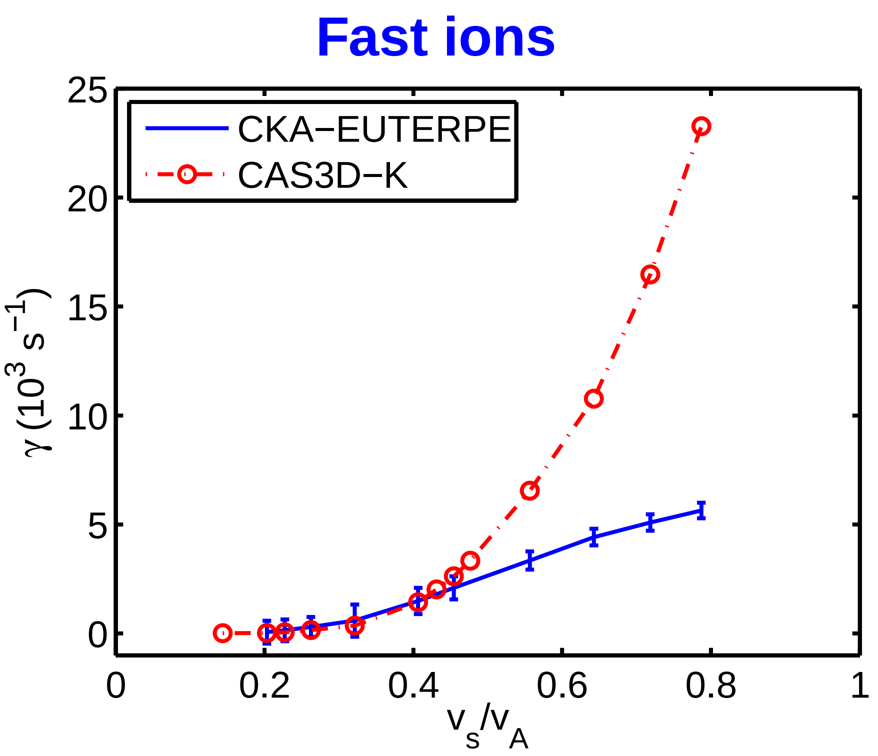
- Global linear 3D fully gyrokinetic δf code
- Slab, tokamak and stellarator geometries
- code solves linearized or non-linear gyrokinetic Vlasov-Maxwell system
- three kinetic species: ions, electrons and energetic particles
- The “Klimontovich” representation for the distribution function

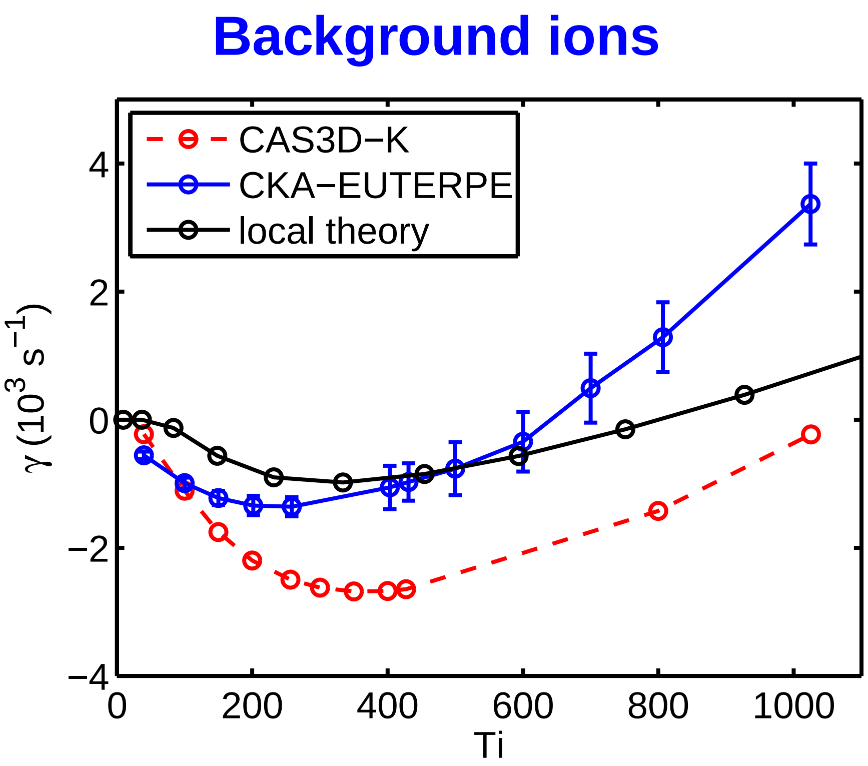
$$\delta f = e^{iS(\vec{x})} \sum_{\nu=1}^{N_p} w_\nu \delta(z - z_\nu)$$

- The “Ritz-Galerkin” representation for the fields (using B splines)

$$\phi(\vec{x}) = e^{iS(\vec{x})} \sum_{k=1}^{N_{\mathrm{FE}}} \phi_k \Lambda_k(\vec{x}) \ , \quad A_{\|}(\vec{x}) = e^{iS(\vec{x})} \sum_{k=1}^{N_{\mathrm{FE}}} a_k \Lambda_k(\vec{x}) \ ,$$

W7-AS #39024 Energy transfer





Summary: Energetic particle numerics in 3D devices

- **achievements:**
  - linear MHD physics (Alfvén modes, continuum, pressure driven modes)
  - zero orbit width perturbative kinetic MHD (analytical and numerical)
  - continuum damping in two mode model
  - direct fast particle losses (IPP: M. Drevlak, ANTS)
  - radiative/ continuum damping (CRPP: Mellet, LEMan)
- **under development:**
  - full orbit width kinetic MHD (IPP: T. Feher, ORNL, Kurtschatov)
  - reduced non-linear kinetic MHD model
  - linear fully gyro-kinetic model (IPP: R. Kleiber, EUTERPE)
  - non-ideal reduced MHD (IPP: A. Könies, CKA)
- **not yet . . .**
  - mode induced particle losses/ redistribution/ most unstable mode

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- A. Weller, S. Zegenhagen, A. Werner
- H. Leyh
- A. Pulss

MEGA: non-linear kinetic MHD

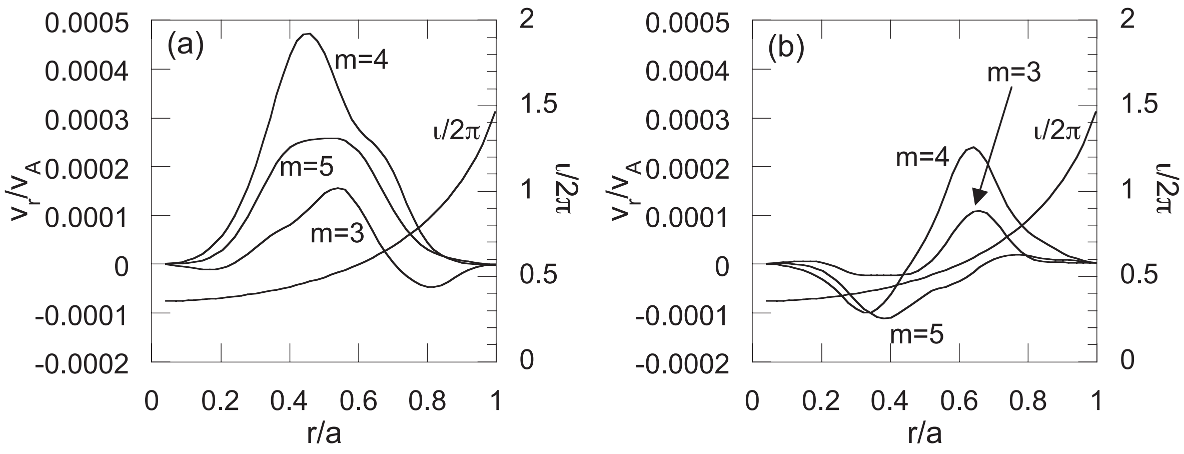
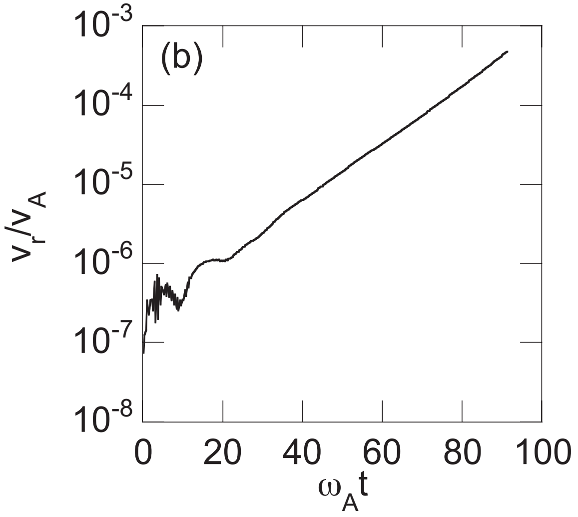


FIG. 6. Spatial profile of (a) cosine part and (b) sine part of radial velocity of the TAE in the LHD plasma with the toroidal mode number $n = 2$ and the rotational transform profile.



(Y. Todo, IAEA FEC 2004)

wave-equation eigenvalue codes

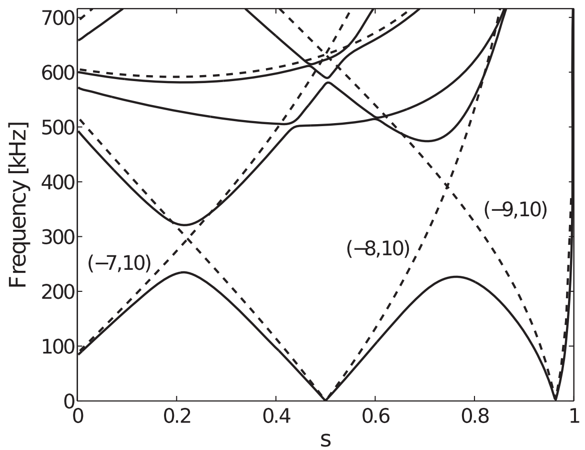
wave equation

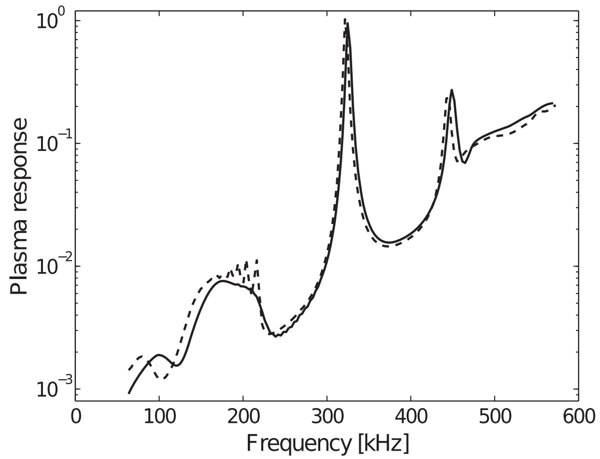
$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} - \frac{\omega^2}{c^2} \overleftrightarrow{\epsilon} \cdot \vec{E} = \frac{4\pi i}{c^2} \omega \vec{j}_{ext}$$

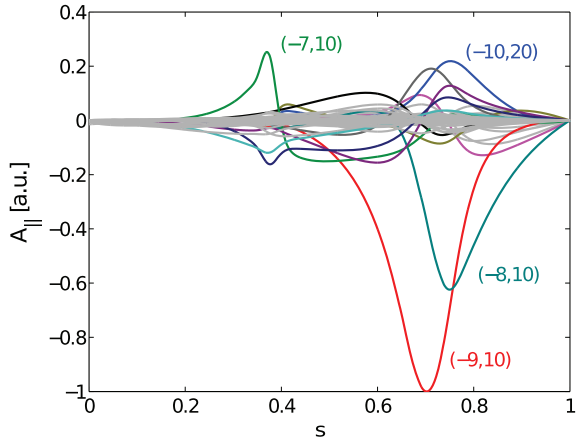
solved as eigenvalue problem:
$\vec{j}_{ext}$: **antenna current**
$\overleftrightarrow{\epsilon}$: **plasma response**

- **LEMan (Mellet, Popovitch et al.)**
  - **no fast ions**
  - **accurate $k_{||}$ representation confirms earlier results**
- **TASK (A. Fukuyama)**
  - **drift kinetic fast ions**
  - **accurate $k_{||}$ representation**

LEMan: TAE for recent LHD configuration







TAE frequency: $\omega = 326$ kHz, damping $\gamma/\omega = 0.5\%$

kinetic equation entering the dielectric tensor:

$$\frac{\partial}{\partial t}\tilde{f} + v_{||}\frac{\partial}{\partial v_{||}}\tilde{f} + \Omega\frac{\partial}{\partial \alpha}\tilde{f} = \frac{q}{m}\tilde{\vec{E}}\cdot\frac{\partial F_0}{\partial \vec{v}}$$

(from N. Mellet et al. CPC 182, 2011)