A computational approach to continuum damping of Alfvén waves in two and three-dimensional geometry

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• global Alfvén modes (TAE, EAE, HAE ...) reside within gaps of the continuous spectrum

• may be driven unstable by interaction with fast particles, but continuum damping if gaps are closed (depending on density and iota profiles)

• not accounted for in 3D computational MHD codes like CAS3D

• usually a 4th order operator with a very small imaginary part \( (i\delta) \) (artificial damping) is added to the system (e.g. 2D CASTOR code)

• alternative approach: integrating in complex plane adopted from shooting method for eigenvalue problems

  M. R. Scott, J. Comp. Phys. 12, 334 (1973)

  For tokamaks, a contour integration method has been applied in

ideal MHD equations in large aspect ratio limit

coupled system of equations:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( \frac{\omega^2}{v_A^2} - k_{||} \right) \frac{\partial}{\partial r} \phi \right] \quad \quad - \quad \quad \frac{\omega^2}{v_A^2} \phi + \left( k_{mn}^2 \right)^\prime \frac{\phi}{r} =
\]

\[-\sum_{\mu,\nu} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \hat{\epsilon} \frac{\omega^2}{v_A^2} \phi_{m+\mu,n+\nu} \right] - \left[ \frac{\partial}{\partial r} \frac{\omega^2}{v_A^2} \hat{\epsilon} \right] \frac{\phi_{m+\mu,n+\nu}}{r} \right. \quad \quad - \quad \quad \frac{\omega^2}{v_A^2} \hat{\epsilon} \frac{\phi_{m+\mu,n+\nu}}{r^2}
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{r \hat{\epsilon} g k_{||} k_{||} m+\mu,n+\nu \phi_{m+\mu,n+\nu}}{2} \right] - \left[ \frac{\partial}{\partial r} k_{||} k_{||} m+\mu,n+\nu \right] \frac{\epsilon_g}{2} \frac{\phi_{m+\mu,n+\nu}}{r}
\]

\[-k_{||} k_{||} m+\mu,n+\nu \frac{\epsilon_g \phi_{m+\mu,n+\nu}}{2} \frac{r^2}{r^2} \}

here: restriction to two mode model

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coupling of different Fourier harmonics determined by $\hat{\epsilon}$ parameter:

in large aspect ratio tokamak equilibrium:

$$\hat{\epsilon} = \frac{r}{R}$$

in more general equilibria:

$$\hat{\epsilon} = \frac{\epsilon_{g}^{(\mu,\nu)}}{2} - 2\epsilon_{B}^{\mu,\nu}$$

with

$$g^{ss} = (g^{ss})_{0} \left( 1 + \sum_{\mu,\nu} \epsilon_{g}^{\mu,\nu} \cos(\mu \theta + \nu N_{p} \varphi) \right)$$

$$B = \bar{B} \left( 1 + \sum_{\mu,\nu} \epsilon_{B}^{\mu,\nu} \cos(\mu \theta + \nu N_{p} \varphi) \right)$$
Solution with Riccati shooting method

- system of linear equations \( \frac{d\vec{u}}{dx} = A\vec{u} + B\vec{v}, \frac{d\vec{v}}{dx} = C\vec{u} + D\vec{v} \) is transformed into matrix Riccati equation \( \frac{dR}{dx} = -RCR - RD + AR + B \) with \( \vec{u} = R\vec{v} \).
- this equation is solved as initial value problem in the complex plane \( x \rightarrow z \) along an arbitrary path to avoid singularities.
- eigenvalues are zeros of \( |R_l(\omega) - R_r(\omega)| \) at a prescribed fitting point \( (R_l, R_r: \text{left/right solutions}) \).
- highly accurate adaptive integration method used.

\( \Rightarrow \) very robust method especially for stiff equations and higher order equations.
Finite element method using B-splines

- B splines (piecewise continuous polynomials) of arbitrary order
- huge sparse matrices
- parallel storage and solvers using PETSc (Portable, Extensible Toolkit for Scientific Computation)
- solve eigenvalue problem using SLEPc
- variety of solvers direct/iterative
• integration contour has to be deformed when the singularity moves away from the real axis

• deformation depends on sign of $\frac{\partial \omega}{\partial r}$

(see also M. S. Chu et al. Phys. Plasmas 1, 1214 (1994).)

• for the cases considered here, a contour in the lower half plane of the complex $r$-contour can be chosen
small damping $\gamma$: $\omega \rightarrow (\omega - i\gamma)$ with $\gamma > 0$

resonance of mode frequency $\omega$ with continuum frequency $\omega_C(r)$ at $r_S$

$\Rightarrow r_S = \omega_C^{-1}(\omega)$ position of the resonance on the real axis

continuation into complex plane $\omega \rightarrow \omega - i\gamma$

$\Rightarrow$ position of $r$ in complex plane:

$$r \approx r_S - i \left(\frac{\partial \omega_C}{\partial r}\right)^{-1} \gamma$$
Tokamak test case I

$\frac{(\omega/\omega_A)^2}{r/a}$

- Global mode
- TAE gap
- Continuum branches

$m=2$
$m=3$
Tokamak test case II

\[ \text{Im}\left(\frac{\omega}{\omega_A}\right)^2 \]

\[ \delta = -4 \times 10^{-7} \]
\[ \delta = -4 \times 10^{-8} \]
\[ \delta = -4 \times 10^{-9} \]
\[ \delta = -4 \times 10^{-10} \]

\[ \alpha = 0.025 \]
\[ \alpha = 0.04 \]
\[ \alpha = 0.05 \]
\[ \alpha = 0.06 \]
\[ \alpha = 0.08 \]
\[ \alpha = 0.09 \]

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M. Drevlak et al., Nucl. Fusion, 45, 731 (2005):
from PIES calculation: practically island free
Alfvén continuum of W7-X

The diagrams show the Alfvén continuum for different modes and flux labels. The plots illustrate the behavior of various modes such as HAE, EAE, and TAE, with labels for specific modes like $m=9, n=-6$, $m=11, n=-11$, and $m=12, n=-10$. The flux label $s$ and normalized toroidal flux are used as axes in the plots.
TAE modes with continuum damping

\[ \frac{\omega}{\omega_A} = 0.2604 \]
\[ \gamma/\omega_0 = -1.03 \times 10^{-4} \]
TAE modes with continuum damping

\begin{align*}
\omega_0/\omega_A &= 0.2283 \\
\gamma/\omega_0 &= -2.15 \times 10^{-4}
\end{align*}
TAE modes with continuum damping

\[ \omega / \omega_A = 0.1791 \quad \gamma / \omega_0 = -1.3 \times 10^{-3} \]
variation of density profile changes the behaviour of the branches of the continua and leads to a variation of the damping
HAE$_{2,-1}$ modes with continuum damping

HAE$_{2,-1}$ (m=9, n=-6)  
$s_h=0.8$

$\omega_0/\omega_A = 0.7096$  
$\gamma/\omega_0 = -5.03 \cdot 10^{-6}$

$\omega/\omega_A = 1.64$  
$\gamma/\omega = 6.84 \cdot 10^{-4}$
Change strength of damping

\[\frac{n}{n_0} \quad (r/a)^2\]

- \(s_h = 1.1\)
- \(s_h = 0.8\)
- \(s_h = 0.5\)
- \(s_h = 0.3\)

\[\gamma/\omega \times 10^3\]

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for comparison:

kinetic growth/damping rates from a local calculation

\[ \frac{\gamma}{\omega} = 6.18 \cdot 10^{-4} \]

(TAE: fast ions)

\[ \frac{\gamma}{\omega} = -1.77 \cdot 10^{-3} \]

(TAE: electron Landau damping)

\[ \frac{\gamma}{\omega} = 2.7 \cdot 10^{-5} \]

(HAE_{2,-1}: fast ions)

\[ \frac{\gamma}{\omega} = -1.8 \cdot 10^{-2} \]

(HAE_{2,-1}: electron Landau damping)

(Theory see Kolesnichenko et al. Phys. Plasmas 9, 517 (2002))
Summary

- good agreement between different numerical models
- contour integration has better numerical properties than \( i\delta \rightarrow i0 \)
- can be generalized to arbitrary dependence on geometry (analytic continuation of the equilibrium quantities)
- coupling with perturbative kinetic MHD codes
- open issues: modes with \( \phi \sim (a - r) \) as \( r \rightarrow a \)
- next step: generalization to multi mode calculation